Individual

1. $f(x) = ke^{2x}$

2. 1 (since $4^3 \equiv 1 \bmod 9$)

3. Letting *a, b, c* be the respective class quiz per hour grading rates for the three graduate students, we have: $a+b=1/2,\ b+c=1/3,\ a+c=1/4$. We will find the answer to this problem without divulging the individual rates of the graduate students, as that might cause some embarrassment, and is in fact not allowed by the agreement with the graduate student union. So adding all three equations gives $2(a+b+c)=13/12$, and so $a+b+c=13/24$ class quizzes per hour. So working together they will take $24/13$ hours, assuming that three does not imply chaos.

5. Assume $5\cdot 2^k+1$ is prime. Then since 3 does not divide $5\cdot 2^k$, and 3 does not divide $5\cdot 2^k+1$, 3 must divide $5\cdot 2^k+2=2(5\cdot 2^{k-1}+1)$ so that 3 divides $5\cdot 2^{k-1}+1>3$.

7. First the other letters (M, I, I, I, P, P, I) can be arranged in $7!/4!2!=105$ distinguishable ways. Now we choose 4 of the 8 spaces adjacent to these letters to place the S's. The answer is $105\times C(8,4)=105\times 8!/4!4!=105\times 70=7350$.

8. Let I denote the value of this integral. Let $u=\pi/2-x$. Then $du=-dx$ and the integral becomes

$$\int_{\pi/2}^{0}\frac{\cos(\pi/2-u)}{\sin(\pi/2-u)+\cos(\pi/2-u)}(-du)=\int_0^{\pi/2}\frac{\sin(u)}{\cos(u)+\sin(u)}du = \text{I}.$$

Adding the two integrals gives 2 I = $\int_0^{\pi/2}1dx=\pi/2$, so I= $\pi/4$.

11. We hope that this series is periodic. Now $u_3=2^{-\varphi}$,

$u_4=\dfrac{1}{(2^{-\varphi})^{\varphi}\cdot 2}=\dfrac{1}{2^{1-\varphi^2}}=2^{\varphi^2-1}=2^{\varphi}$ (by the identity given),

$u_5=\dfrac{1}{(2^{\varphi})^{\varphi}\cdot 2^{-\varphi}}=\dfrac{1}{2^{\varphi^2-\varphi}}=\dfrac{1}{2^1}$ (by the identity given), $u_6=\dfrac{1}{(1/2)^{\varphi}2^{\varphi}}=1$,

$u_7=\dfrac{1}{1^{\varphi}\cdot(1/2)}=2$. The sequence is periodic with period 5. So $u_{2007}=u_2=2$. (The same periodicity results for any two positive initial values.)

Team

1. The invariant here is the three pounds of *anhydrous cucumber matter* (ACM). The three pounds of ACM is 2% of 150 pounds.

2. The longest pole is the minimum of the length of a segment containing A, P, B, where A is a point on the outer wall of the 20 feet wide corridor, P is the point of intersection of the inside walls of the two corridors, and B is a point on the outside wall of the 10 feet wide corridor. Let θ be the acute angle that such a segment makes with the outer wall of the 20 feet wide corridor. Using similar triangles, the length of this segment is $l(\theta) = \dfrac{20}{\sin\theta} + \dfrac{10}{\cos\theta}$. Solving $l'(\theta) = \dfrac{20\cos\theta}{\sin^2\theta} - \dfrac{10\sin\theta}{\cos^2\theta} = 0$, we find that $\theta = \arctan 2^{1/3}$. Thus the longest pole is

$$l(\arctan 2^{1/3}) = \frac{20}{2^{1/3}/\sqrt{1+2^{2/3}}} + \frac{10}{1/\sqrt{1+2^{2/3}}} \approx 41.619 \text{ feet.}$$

3. a. Notice that for $n \ge 1$ $x_{n+1} = x_n + (x_{n-1} - x_n)/2$, so $x_2 = 1 - 1/2,\ x_3 = 1 - 1/2 + 1/4,$ etc., and so the limit is the sum of the geometric series with ratio –1/2 and first term 1. Hence the answer is $\dfrac{1}{1-(-1/2)} = 2/3$.

b. Here $x_2 = 0 + 1/2$, $x_3 = 0 + 1/2 - 1/4$, etc. and so the answer is the sum of the geometric series with ratio –1/2 and first term 1/2, i.e., $\dfrac{1/2}{1-(-1/2)} = 1/3$. Why must the sum of the answers to parts a and b add to 1?

c. Solution one: The sequence in this part is $a \times$(sequence of part b) + $b \times$(sequence of part a). Hence the answer here is $a/3 + 2b/3$.

Solution two: The sequence here is

$a,\ b,\ b + (a-b)/2,\ b + (a-b)/2 - (a-b)/2^2, \cdots$ so the limit is

$b + (a-b)(1/2/(1-(-1/2)) = b + (a-b)/3.$

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Solutions for Problem-Solving Competition: Individual

4. Because $a, b, c, d \geq 1$ and $a+b+c+d = 10$, we know each positive integer d will determine the sum $a+b+c$ for the rest 3. Hence the number of positive solutions (denoted as $p.i.s$) for $a+b+c+d = 10$ can be found by summing up all the numbers of positive integer solutions for $a+b+c = k \quad 3 \leq k \leq 9,\ k \in \mathbb{Z}$. We claim that the number of $p.i.s.$ for $a+b+c = k$ with each $k \geq 3$ will be $C(k-1, 2)$. Therefore, the number of $p.i.s.$ for $a+b+c+d = 10$ will be $84 = C(9,3) = C(2,2) + C(3,2) + C(4,2) + \ldots + C(8, 2) = C(9, 3) =$

   Proof of the claim:

   Let $k \geq 3$ and $a+b+c = k$. The number of the $p.i.s$ for this equation will be

   $p.i.s(a+b=2) + p.i.s.(a+b=3) + p.i.s(a+b=4) + \ldots + p.i.s(a+b=k-1)$
   $= 1+2+3+4+\ldots+(k-2) = \frac{[1+(k-2)]\cdot(k-2)}{2} = C(k-1, 2)$

6. (a) Let $k \in \mathbb{N}$. Then

$$k \equiv 0 \ or\ 1 \ (mod\ 2) \Rightarrow k^2 \equiv k \Rightarrow k^2 \equiv k \Rightarrow k^2 - k \equiv 0 \ \Rightarrow\ k^2 - k \ is\ even$$

   (b) Let $k^2 - k = 2p \quad for\ some\ p \geq 0$. Then

$$4(k^2-k)+1 = 8p+1 \Rightarrow 4k^2-4k+1 = 8(p+1)-7 \Rightarrow (2k-1)^2 = 8n-7 \geq 0,\ n = p+1 \geq 1$$

   Taking square roots, $2k-1 = \sqrt{8n-7} \Rightarrow k = \frac{1+\sqrt{8n-7}}{2} \qquad n \geq 1$

9. Since $\sum_{n=1}^{9} \frac{n}{k} = \frac{1}{k} \cdot \sum_{n=1}^{9} n = \frac{45}{k}$, it follows $\sum_{n=1}^{9} \frac{n}{k} = \frac{45}{k} \in \mathbb{N} \Rightarrow k|45$. The number $45 = 3^2 \cdot 5$ has 6 positive integer divisors: $\{1, 3, 5, 9, 15, 45\}$. Hence

$$k \in \{1, 3, 5, 9, 15, 45\}$$

12. Define the (square of) distance function $D(x) = x^2 + [f(x)]^2$ between two points $(x, f(x))$ and $o = (0, 0)$. The distance function is differentiable and $D'(x) = 2x + 2f(x) \cdot f'(x)$. The fact that $\overline{op}$ is minimum implies $D'(a) = 0$. Therefore $2a + 2f(a) \cdot f'(a) = 0$ and Hence $a + f(a) \cdot f'(a) = 0$

Solutions for Problem-Solving Competition: Group

4. The linear system can also be written in the matrix form $AX = B$. Apply row reductions on the augment matrix we get:

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & k \\ 1 & 1 & k & k^2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & k & k^2 \\ 0 & k-1 & 1-k^2 & k-k^2 \\ 0 & 1-k & 1-k^2 & 1-k^3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & k+1 & k^2+k+1 \\ 0 & 1 & -1 & -k-1 \\ 0 & 0 & -k-2 & -k^2-2k-2 \end{array}\right]$$

Notice that if $k = 1$,the system becomes one equation $x + y + z = 1$ and hence has infinitely many solutions; so we may assume $k - 1 \neq 0$. For the last matrix to have no solution, $k = -2$ ($det(A) = k^3 - 3k + 2 = (k-1)^2 \cdot (k-2) = 0$, then A is NOT invertible.)

IP10. Solution:

(a) Apply, e.g., the [absolute] ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^n(2(n+1)+2)}{(2(n+1)+3)!} \right| \div \left| \frac{(-1)^n(2n+2)}{(2n+3)!} \right|$$
$$= \lim_{n\to\infty} \left| \frac{2n+4}{2n+2} \right| \cdot \left| \frac{(2n+3)!}{(2n+5)!} \right|$$
$$= \lim_{n\to\infty} \frac{2n+4}{2n+2} \cdot \frac{1}{(2n+4)(2n+5)}$$
$$= \lim_{n\to\infty} \frac{1}{(2n+2)(2n+5)} = 0 < 1, \text{ so the series converges [absolutely].}$$

(b) For all x, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!}$ (converging uniformly on $[0,1]$),

so $x\, \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+2}}{(2n+1)!}$;

therefore $\int_0^x x\, \sin x\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+3}}{(2n+1)!(2n+3)}$.

Setting $x = 1$, we obtain $\int_0^1 x\, \sin x\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+3)} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+2)}{(2n+3)!}$.

Computing this integral by parts ($u = x$, $dv = \sin x\, dx$), we obtain the desired value:

$$\int_0^1 x\, \sin x\, dx = [-x\, \cos x + \sin x]_0^1 = -\cos 1 + \sin 1 = \sin 1 - \cos 1.$$

TP5. Solution:

(a) The partitions are:

5 [E,DO]
$4+1$ [O,NDO]
$3+2$ [O,NDO]
$3+1+1$ [E,NDO]
$2+2+1$ [E,NDO]
$2+1+1+1$ [O,NDO]
$1+1+1+1+1$ [E,NDO]

	DO	NDO
Even	1	3
Odd	0	3

(b) The entry in the lower-left must be zero: any "DO" partition consists solely of distinct odd numbers, thus it has no even numbers, so it contains an even number of even numbers.

(c) The NDO partitions are equally split between even and odd – i.e., the two rightmost entries are equal.

To justify this, we can pair each Even NDO partition with an Odd NDO partition (and vice-versa), as follows. An NDO partition, by definition, does not consist of distinct odd numbers, so it must be the case that a number is repeated or an even number appears (or both). We will construct each NDO partition's twin by either (1) combining an equal pair of summands $[P + P \to 2P]$ or (2) splitting an even summand $[E \to \frac{E}{2} + \frac{E}{2}]$; note that these operations either increment or decrement the count of even numbers appearing, and thus interchange Odd and Even permutations. On each NDO partition, we perform the operation that results in (or starts with, respectively) the largest number, or operation (2) in case of a tie.

We can easily check that this operation is an involution. If P is a partition to which we applied (1), then the resulting partition will have a (unique) new even summand strictly larger than any other even summand and at least as large as any pair of equal summands, so reapplying the operation will split that summand to return us to P. Conversely, if P was a partition to which we applied (2), then the splitting will produce a pair of equal summands with sum at least as large as any other pair of equal summands and strictly larger than any even number appearing, so we will rejoin that pair and return to P. In summary, this operation pairs each Odd NDO partition with an Even NDO partition and vice-versa, so the counts of such partitions must be equal.

TP6. Solution:

(a) Take, e.g., $\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} + 1 + \frac{1}{4} + 1 + \frac{1}{8} + 1 + \frac{1}{16} + \cdots$
and $\sum_{n=1}^{\infty} b_n = \frac{1}{2} + 1 + \frac{1}{4} + 1 + \frac{1}{8} + 1 + \frac{1}{16} + 1 + \cdots$
These series clearly both diverge (their terms to not approach zero), but the termwise minimum is:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

[The $0 + 1 + 0 + 1 + 0 + \cdots$ version would be technically not quite correct, as the terms are not positive.]

(b) Yes; we can construct terms for a pair $\sum a_n$,$\sum b_n$ of divergent nonincreasing positive series whose termwise minimum converges to, e.g., 1, iteratively as follows:

Let the first term of $\sum a_n$ be 1. $\sum a_n$ now has $n_1 = 1$ more term defined than does $\sum b_n$.

Now let the first term of $\sum b_n$ be $\frac{1}{2}$, so that the minimum of the two terms so far adds to $\frac{1}{2}$, and add another $\frac{1}{2}$ so that we obtain a sum of 1 for the terms of $\sum b_n$ thus far defined. $\sum b_n$ now has $n_2 = 1$ more term defined than does $\sum a_n$.

Back to $\sum a_n$, let the next term be $\frac{1}{4}$, so that the termwise minimum so far adds up to $\frac{1}{2} + \frac{1}{4}$, then add 3 more of the same value, so that the total of the new terms in $\sum a_n$ adds to 1. $\sum a_n$ now has $n_3 = 3$ more terms defined than does $\sum b_n$.

Proceed, in general taking the next n_k terms of the shorter series equal to $\frac{1}{2^k n_k}$ (adding to $\frac{1}{2^k}$ in the termwise minimum) and continue the series for $n_{k+1} = 2^k n_k - n_k$ more terms of the same value, so that in sum they add to 1 in the relevant series.

We then have both series $\sum a_n$ and $\sum b_n$ diverging as $1 + 1 + 1 + \cdots$ and the termwise sum converging as $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ to 1.