More Odd Abundant Sequences

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An article by the first author (see reference 1) contained a discussion of the arithmetic sequence $945 + 630n$. This sequence appeared to be both curious and fascinating in the sense that odd abundant numbers were obtained for each of the initial 52 whole number inputs. During a colloquium presentation by the first author, the second author discovered that the sequence $3465 + 2310n$ generates odd abundant number outputs for the initial 193 whole number inputs. Energized by these discoveries, we pondered as to whether such sequences were rare at all. These two sequences turn out to be but the first of an infinite family of sequences that produce long initial odd abundant values. The goal of this article is to examine the existence of such arithmetic sequences.

Recall that an *abundant number* is one in which the sum of all its divisors, including itself, exceeds twice the number in question. The smallest odd abundant number is 945. A good introduction to odd abundant numbers is given in reference 1. A computer program in conjunction with MATHEMATICA® (see reference 2) enabled us to produce values for sequences with initial value less than three billion, see table 1. These sequences are obtained from so-called *seed values* a. These seed values are odd deficient numbers that are very close to being perfect. For example, if $\sigma(a)$ denotes the sum of the divisors of a, then

$$\begin{aligned}\sigma(1155) &= \sigma(3 \times 5 \times 7 \times 11)\\ &= \sigma(3) \times \sigma(5) \times \sigma(7) \times \sigma(11)\\ &= (3+1) \times (5+1) \times (7+1) \times (11+1)\\ &= 2304\\ &< 2310\\ &= 2 \times 1155.\end{aligned}$$

We recall that if $\sigma(a)/a > 2$ then n is abundant, while if $\sigma(a)/a = 2$ then a is perfect, and if $\sigma(a)/a < 2$ then a is deficient. Now

$$\frac{\sigma(1155)}{1155} = \frac{2304}{1155} = 1.994\,81,$$

which is just under 2. In order to get abundant numbers from the deficient seed values, we note that $\sigma(ka) > \sigma(a)$, whenever $k > 1$. So we choose our sequence to be $f(n) = (3+2n)a = 3a + 2an$, starting with $n = 0$. In general, the closer a is to being abundant the more likely the initial terms will be abundant. The *first failure point* for such a sequence is the smallest positive integer n for which the sequence $f(n) = 3a + 2an$ yields a deficient output. For example, if we start with seed value $a = 1155$, we get the sequence $f(n) = 3465 + 2310n$. Then

$$f(193) = 449\,295, \qquad \frac{\sigma(449\,295)}{449\,295} = 1.999\,93,$$

and so $f(193)$ is deficient, while for all smaller values of n, $f(n)$ is abundant. Since $f(n) = (3+2n)a$ and any multiple of an abundant number is itself abundant, the first failure point can

Table 1 Record holders.

Seed value a	Factorization of seed value	First failure point	Arithmetic sequence $3a + 2an$
315	$3^2 \times 5 \times 7$	52	$945 + 630n$
1 155	$3 \times 5 \times 7 \times 11$	193	$3465 + 2310n$
40 365	$3^3 \times 5 \times 13 \times 23$	452	$121\,095 + 80\,730n$
55 335	$3 \times 5 \times 7 \times 17 \times 31$	710	$166\,005 + 110\,670n$
106 425	$3^2 \times 5^2 \times 11 \times 43$	1 613	$319\,275 + 212\,850n$
629 145	$3^2 \times 5 \times 11 \times 31 \times 41$	2 062	$1\,887\,435 + 1\,258\,290n$
702 405	$3^3 \times 5 \times 11^2 \times 43$	2 128	$2\,107\,215 + 1\,404\,810n$
730 125	$3^2 \times 5^3 \times 11 \times 59$	8 113	$2\,190\,375 + 1\,460\,250n$
1 805 475	$3 \times 5^2 \times 7 \times 19 \times 181$	25 795	$5\,416\,425 + 3\,610\,950n$
13 800 465	$3^2 \times 5 \times 7 \times 193 \times 227$	85 190	$41\,401\,395 + 27\,600\,930n$
16 029 405	$3^2 \times 5 \times 7 \times 151 \times 337$	86 185	$48\,088\,215 + 32\,058\,810n$
16 286 445	$3^2 \times 5 \times 7 \times 149 \times 347$	180 962	$48\,859\,335 + 32\,572\,890n$
21 003 885	$3^2 \times 5 \times 7 \times 131 \times 509$	233 387	$63\,011\,655 + 42\,007\,770n$
32 062 485	$3 \times 5 \times 7 \times 13 \times 83 \times 283$	763 402	$96\,187\,455 + 64\,124\,970n$
132 701 205	$3 \times 5 \times 7 \times 13 \times 67 \times 1451$	3 159 554	$398\,103\,615 + 265\,402\,410n$
594 397 485	$3^2 \times 5 \times 11 \times 29 \times 47 \times 881$	6 604 424	$1\,783\,192\,455 + 1\,188\,794\,970n$
815 634 435	$3 \times 5 \times 7 \times 11 \times 547 \times 1291$	135 939 073	$2\,446\,903\,305 + 1\,631\,268\,870n$

only occur when $3+2n$ is prime. Indeed, in our example, $3+2\times 193 = 389$ is prime. Tabl lists those seed values a along with their sequences $f(n) = 3a + 2an$ whose first failure po is greater than that of any such sequence with a smaller seed value.

References

1 J. L. Schiffman, Odd abundant numbers, *Math. Spectrum* **37** (2004/2005), pp. 73–75.

2 S. Wolfram, *The Mathematica Book*, 5th edn. (Wolfram Research, Champaign, IL, 2003).

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Sums of squares and cubes

$$3^2 + 4^2 = 5^2 \quad \text{and} \quad 3^3 + 4^3 + 5^3 = 6^3.$$

Are there any other triples (x, y, z) of natural numbers such that $x^2 + y^2 = z^2$ and $x^3 + y^3 + z^3$ is a perfect cube, apart from multiples of $(3, 4, 5)$?

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