New Jersey Undergraduate Mathematics Contest
Spring 2005
Individual Part

No calculators. Justify all answers. 60 minutes. Place your team number and individual letter in the upper-left corner of all pages. Clearly indicate the question number for each of your solutions.

1. Find all functions $f(x)$ such that $\dfrac{d^n}{dx^n} f(x) = 2^n f(x)$ for all positive integers $n \geq 1$.

2. What is the remainder of 4^{300} when divided by 9?

3. Graduate students A, B, and C work as T.A.'s for a calculus course. Working together A and B can grade a quiz in 2 hours, B and C can grade a quiz in 3 hours, and A and C can grade a quiz in 4 hours. How long will it take A, B and C to grade a quiz if they work together?

4. Find the total number of positive integer solutions (a,b,c,d) for the following equation: $a+b+c+d=10$.

5. Suppose that $k \geq 1$ is a positive integer and $5 \cdot 2^k + 1$ is prime. Prove that $5 \cdot 2^{k-1} + 1$ is not prime.

6. Prove each of the following statements:
   a. For any positive integer k, $k^2 - k$ is an even integer.
   b. For any positive integer *k*, there exists a positive integer *n* such that
$$k = \frac{1+\sqrt{8n-7}}{2}$$

7. How many distinguishable arrangements of the letters in the word MISSISSIPPI have no S's next to each other?

8. Determine the numerical value of $\displaystyle\int_0^{\pi/2} \frac{\cos(x)}{\sin(x)+\cos(x)}dx$

9. Find all positive integers *k* where $\dfrac{1}{k}+\dfrac{2}{k}+\dfrac{3}{k}+\cdots+\dfrac{9}{k}$ is also a positive integer.

10. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (2n+2)}{(2n+3)!}$.
    a. Prove that this series converges.
    b. Find its exact value.

11. Suppose that a sequence is defined recursively by $u_1 = 1, u_2 = 2$, $u_{n+1} = \left(u_n^{\varphi} u_{n-1}\right)^{-1}$, where $\varphi = \dfrac{1+\sqrt{5}}{2}$ satisfies $\varphi^2 = \varphi + 1$. For example $u_3 = 2^{-\varphi}$. Find u_{2007}.

12. Let $f(x)$ be a differentiable function and G be the graph of $f(x)$. Let $p = (a, f(a))$ be the point on G closest to $(0,0)$. Prove that $a + f(a)\cdot f'(a) = 0$.

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Team Part

No calculators. Justify all answers. 90 minutes. Each question should be answered on a separate page, with your team number in the upper-left corner and the question number in the upper-right corner.

1. A 300 pound pile of cucumbers sitting in storage in a supermarket is determined to be 99% water by weight. This pile sits undisturbed by human, rodent, or insect for a period of time, after which it is determined that it is 98% water by weight. How much does it weigh at this point?

2. Two corridors in a building, one 20 feet wide and the other 10 feet wide, intersect at right angles. Determine the length of the longest pole that can be maneuvered around the corner formed by these corridors. Assume the pole is always parallel to floor.

3. Suppose the sequence x_n is defined by $x_0 = a, x_1 = b$, and $x_{n+1} = \frac{x_n + x_{n-1}}{2}$.

a. Find $\lim_{n\to\infty} x_n$ if $x_0 = 0$ and $x_1 = 1$.

b. Find $\lim_{n\to\infty} x_n$ if $x_0 = 1$ and $x_1 = 0$.

c. Find $\lim_{n\to\infty} x_n$ if $x_0 = a$ and $x_1 = b$.

4. Suppose the following system of linear equations has no solutions. Find k.

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= k \\ x + y + kz &= k^2 \end{aligned}$$

5.
Definition: A *partition* of a natural number N is an unordered *k*-tuple of natural numbers $(a_1, a_2, \ldots, a_k)$ with $a_1 + a_2 + \cdots + a_k = N$

Definition: If a partition $(a_1, a_2, \ldots, a_k)$ contains an even number of even numbers, then the partition is called *even*; otherwise, it is called *odd*.

Definition: If a partition $(a_1, a_2, \ldots, a_k)$ consists of distinct odd numbers, call it *distinct-odd* (DO); otherwise, call it *non-distinct-odd* (NDO).

Example: The number $N = 3$ has exactly three partitions: (1,1,1), (1,2), and (3). There is exactly one partition that is both even and distinct-odd. It is (3). Similarly, (1,1,1) is the only partition that is both even and non-distinct-odd and (1,2) is the only partition that is both odd and non-distinct-odd. We summarize these results in the following table, where the numbers indicate the number of partitions of $N = 3$ that meet both criteria:

	DO	NDO
EVEN	1	1
ODD	0	1

a. Find all partitions of $N = 5$, and for each partition, determine whether it is even or odd and whether it is distinct-odd or non-distinct-odd. Summarize your results by filling in the table below:

	DO	NDO
EVEN		
ODD		

b. In both of these tables, one of the entries is zero; prove that this is the case for every N.

c. In both of these tables, two of the entries are equal; prove that this is the case for every N.

6. Suppose that $A = \sum_{n=0}^{\infty} a_n$ and $B = \sum_{n=0}^{\infty} b_n$ are divergent series of positive real numbers.

a. Give an example of series *A* and *B* for which $\sum_{n=1}^{\infty} \min(a_n, b_n)$ converges.

b. Is it still possible to do so if in addition $a_1 \geq a_2 \geq a_3 \geq \cdots$ and $b_1 \geq b_2 \geq b_3 \cdots$? Justify your answer.