ADAPTIVE MATHEMATICAL MORPHOLOGY: A UNIFIED REPRESENTATION THEORY

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ABSTRACT

In this paper, we present a general theory of adaptive mathematical morphology (AMM) in the Euclidean space. The proposed theory preserves the notion of a structuring element, which is crucial in the design of geometrical signal and image processing applications. Moreover, we demonstrate the theoretical and practical distinctions between adaptive and spatially-variant mathematical morphology. We provide examples of the use of AMM in various image processing applications, and show the power of the proposed framework in image denoising and segmentation.

***Index Terms*—** adaptive mathematical morphology, kernel representation, basis representation.

1. INTRODUCTION

Mathematical morphology was initially devoted to increasing and translation-invariant operators [1], [2]. Increasing operators preserve order in the sense that they prohibit extraction of information from occluded regions. This property is akin to the limitations of the human visual system which have been investigated in the field of cognitive psychology. Specifically, the high-level vision models of gestalt psychology state that the perceptual processes underlying the visual interpretation of a scene are increasing operators [2]. The increasing condition is thus of fundamental importance in image processing applications. However, the translation-invariance assumption, which specifies that the structuring element (SE) remains fixed in the entire space, is not appropriate for many applications ranging from image restoration to pattern recognition [3–13]. When the SE changes are determined a priori, independently of the data, the SE's can be said to be space variant [7], [12], [13]. If the changes are made on the basis of local properties of the image (e.g., gray-level values), the SE's can be said to be data dependent or adaptive [4]. In [12] and [13], we laid down the theoretical foundations of spatially-variant mathematical morphology (SVMM) in the Euclidean space. The proposed theory is the most general framework of spatially-variant mathematical morphology that preserves the concept of structuring element, which is crucial in the design of geometrical signal and image processing applications. However, SVMM theory does not account for adaptive SE's. In this paper, we propose a unified framework for adaptive mathematical morphology (AMM), which retains the notion of a geometrical structuring element, and integrates the techniques proposed thus far [3–14] into a comprehensive mathematical framework.

In this paper, the proofs can be easily extended from the spatially-variant case presented in [12], [13], and thus will be omitted.

2. ADAPTIVE MATHEMATICAL MORPHOLOGY

In this paper, we consider the continuous or discrete Euclidean space $\mathbf{E} = \mathbb{R}^n$ or $\mathbb{Z}^n$ for some $n > 0$. The set $\mathcal{P}(\mathbf{E})$ denotes the set of all subsets of $\mathbf{E}$. We use $\mathcal{O} = \mathcal{P}(\mathbf{E})^{\mathcal{P}(\mathbf{E})}$ to denote the set of all operators mapping $\mathcal{P}(\mathbf{E})$ into itself. We shall restrict our attention to non-degenerate operators, i.e., $\psi(\mathbf{E}) = \mathbf{E}$ and $\psi(\emptyset) = \emptyset$, for every $\psi \in \mathcal{O}$.

2.1. Basic adaptive morphological operators

Consider the adaptive structuring element θ given by a mapping from $\mathbf{E} \times \mathcal{P}(\mathbf{E}) \mapsto \mathcal{P}(\mathbf{E})$. Defined this way, the adaptive SE can change its size, orientation, shape, etc, based on the pixel being probed and the (local) properties of the image. The transposed adaptive structuring element θ' is given by a mapping from $\mathbf{E} \times \mathcal{P}(\mathbf{E}) \mapsto \mathcal{P}(\mathbf{E})$ such that

$$\theta'(y, X) = \{z \in \mathbf{E} : y \in \theta(z, X)\}. \quad (1)$$

In the translation-invariant case, $\theta(z, X) = B + z,\ \forall z \in \mathbf{E}, \forall X \in \mathcal{P}(\mathbf{E})$, where B is the fixed SE used to probe the image. It is straightforward to verify that $\theta'(y, X) = \check{B} + y,\ \forall y \in \mathbf{E}, \forall X \in \mathcal{P}(\mathbf{E})$, where $\check{B} = -B$ is the reflected set of B.

Definition 1 *The adaptive erosion $\mathcal{E}_\theta \in \mathcal{O}$ is defined for every $X \in \mathcal{P}(\mathbf{E})$ as*

$$\mathcal{E}_\theta(X) = \{z \in \mathbf{E} : \theta(z, X) \subseteq X\} = \bigcap_{y \in X^c} \theta'^c(y, X). \quad (2)$$

Definition 2 *The adaptive dilation $\mathcal{D}_\theta \in \mathcal{O}$ is defined for every $X \in \mathcal{P}(\mathbf{E})$ as*

$$\mathcal{D}_\theta(X) = \{z \in \mathbf{E} : \theta'(z, X) \cap X \neq \emptyset\} = \bigcup_{y \in X} \theta(y, X). \quad (3)$$

One can easily verify that the adaptive erosion and dilation are increasing, dual operators (i.e., $\mathcal{E}_\theta^* = \mathcal{D}_\theta$), and form an adjunction (i.e., $\mathcal{D}_\theta(X) \subseteq Y \Leftrightarrow X \subseteq \mathcal{E}_\theta(Y),\ \forall\ X, Y \in \mathcal{P}(\mathbf{E})$). Moreover, if $z \in \theta(z, X),\ \forall X \in \mathcal{P}(\mathbf{E})$, then the adaptive erosion is anti-extensive, and the adaptive dilation is extensive.

The adaptive opening, γ_θ, is given by

$$\gamma_\theta(X) = \mathcal{D}_\theta(\mathcal{E}_\theta(X)) = \bigcup\{\theta(y, X) : \theta(y, X) \subseteq X; y \in \mathbf{E}\}, \tag{4}$$

and the adaptive closing, ϕ_θ, is given by

$$\begin{aligned} \phi_\theta(X) &= \mathcal{E}_\theta(\mathcal{D}_\theta(X)) \qquad (5)\\ &= \{z : \theta(y, X) \cap X \neq \emptyset,\ \forall\ \theta(y, X) : z \in \theta(y, X)\}, \end{aligned}$$

for every $X \in \mathcal{P}(\mathbf{E})$.

The adaptive opening and closing are morphological filters, i.e., they are increasing and idempotent. Moreover, the adaptive opening is anti-extensive and the adaptive closing is extensive.

2.2. Examples

2.2.1. Spatially-variant mathematical morphology [12], [13]

If the adaptive structuring element element θ depends only on the current position being probed, and is independent of the signal (i.e., $\theta(z, X) = \theta(z)$, for all $z \in \mathbf{E}$ and all $X \in \mathcal{P}(\mathbf{E})$), then the proposed adaptive mathematical morphology reduces to the spatially-variant (SV) mathematical morphology developed in [12] and [13].

2.2.2. Adaptive neighborhood morphology [9], [10]

Consider $\mathbf{E} = \mathbb{R}^2$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a criterion mapping such as luminance or contrast. Let $m > 0$. For each $z \in \mathbf{E}$, define the connected set $V_m^h(z, X)$ by $V_m^h(z, X) = \{y : |h(y) - h(z)| \leq m\}$. Choose the SE mapping θ as follows:

$$\theta(z, X) = \bigcup_{y \in \mathbf{E}} \{V_m^h(y, X) : z \in V_m^h(y, X)\}. \tag{6}$$

The SV erosion and dilation defined in Eqs. (2) and (3), respectively, reduce to

$$\mathcal{E}_\theta(X) = \{z : \exists y \in \mathbf{E} \text{ such that } z \in V_m^h(y, X) \text{ and } V_m^h(y, X) \subseteq X\}. \tag{7}$$

and

$$\mathcal{D}_\theta(X) = \bigcup_{x \in X} \bigcup_{z \in \mathbf{E}} \{V_m^h(z, X) : x \in V_m^h(z, X)\}. \tag{8}$$

Equations (7) and (8) are, respectively, the adaptive neighborhood erosion and dilation presented in [9], [10]. Thus, adaptive neighborhood morphology is a special case of the adaptive mathematical morphology theory.

3. ADAPTIVE KERNEL REPRESENTATION

We define the kernel, *Ker*(ψ), of an adaptive operator $\psi \in \mathcal{O}$ as follows

$$\mathit{Ker}\ (\psi) = \{\theta : z \in \bigcap_{X \in \mathcal{P}(\mathbf{E})} \psi(\theta(z, X)),\ \text{for every } z \in \mathbf{E}\}. \tag{9}$$

The adaptive kernel reduces to Matheron's kernel for translation-invariant (TI) operators. In the translation invariant case, $\theta(z, X) = B + z,\ \forall z \in \mathbf{E}, \forall X \in \mathcal{P}(\mathbf{E})$, where $B \in \mathcal{P}(\mathbf{E})$. Let ψ be a TI operator. Then,

$$\begin{aligned} \mathit{Ker}\ (\psi) &= \{\theta : z \in \psi(\theta(z, X)),\ \forall z \in \mathbf{E}, \forall X \in \mathcal{P}(\mathbf{E})\} \\ &= \{B : z \in \psi(B + z),\ \forall z \in \mathbf{E}\} \\ &= \{B : 0 \in \psi(B)\}. \end{aligned}$$

The mapping that associates to each operator $\psi \in \mathcal{O}$ its kernel is an isomorphism.

Proposition 1 *[12] Given two operators $\psi_1, \psi_2 \in \mathcal{O}$, we have $\psi_1 \subseteq \psi_2$ if and only if ker $(\psi_1) \subseteq$ ker (ψ_2).*

A straightforward corollary of the above proposition is that the kernel of a non-degenerate operator is non-empty. We can now obtain the kernel representation of increasing operators based on adaptive erosions and adaptive dilations.

Theorem 1 *An operator $\psi \in \mathcal{O}$ is increasing if and only if ψ can be exactly represented as union of adaptive erosions by mappings in its kernel or equivalently as intersection of adaptive dilations by the transposed mappings in the kernel of its dual ψ^*, i.e.,*

$$\psi(X) = \bigcup_{\theta \in \mathit{Ker}(\psi)} \mathcal{E}_\theta(X) = \bigcap_{\theta \in \mathit{Ker}(\psi^*)} \mathcal{D}_{\theta'}(X), \quad (X \in \mathcal{P}(\mathbf{E})). \tag{10}$$

Spatially-variant mathematical morphology (SVMM) and adaptive mathematical morphology can be viewed as equivalent: increasing operators have a kernel representation in terms of spatially-variant erosions and dilations [12], [13], and in terms of adaptive erosions and dilations. The main value of adaptive morphology is that it can provide a much more succinct and simple kernel (and minimal basis) representation than those provided by SVMM. In effect, many SV erosions and SV dilations would be combined to form adaptive erosions and adaptive dilations, respectively. From a practical point of view, it could be much simpler to implement fewer adaptive operations rather than many SV operations. This can potentially result in much faster implementation. However, adaptive morphological operators are generally more complex to implement than their SV counterparts [8], since the latter are data independent and thus can be determined a priori for each point $z \in \mathbf{E}$.

4. ADAPTIVE BASIS REPRESENTATION

The adaptive kernel representation, given in Theorem 1, is redundant: Let $\psi \in \mathcal{O}$ be a non-degenerate increasing operator, and consider a mapping $\theta_0 \in Ker(\psi)$. Since ψ is increasing, we observe that every mapping θ that satisfies $\theta_0 \leq \theta$ is also in the kernel of ψ. Moreover, $\mathcal{E}_\theta \subseteq \mathcal{E}_{\theta_0}$, leading to an "infinitely redundant" kernel representation. We, thus, extend the notion of the minimal basis of the kernel, which was first introduced by Maragos for translation-invariant operators [15] as follows:

Definition 3 *Let $\psi \in \mathcal{O}$ be an increasing operator. The basis $\mathcal{B}_\psi$ of Ker (ψ) is the collection of minimal kernel mappings, i.e.,*

$$\mathcal{B}_\psi = \{\theta_M \in Ker\,(\psi) : \theta \in Ker\,(\psi)\ and\ \theta \leq \theta_M \Longrightarrow \theta = \theta_M\}.$$

If the minimal basis of an increasing operator exists, then the kernel representation of the operator in Theorem 1 reduces to an equivalent representation by the elements of the minimal basis, which allows a drastic reduction in the number of adaptive basic morphological operators needed to implement the increasing operator.

Let us denote by $\mathcal{F}$ (resp. $\mathcal{G}$) the set of all closed (resp. open) subsets of $\mathbf{E}$. Matheron defined a topology on $\mathcal{F}$ called the *hit-or-miss topology* [1]. We denote by $\mathcal{O}^{'}$ the set of all operators mapping $\mathcal{F}$ into itself. From now on, we consider only mappings in $\mathcal{O}^{'}$. In particular, the adaptive structuring element is now a mapping from $\mathbf{E}\times\mathcal{P}(\mathbf{E})$ to $\mathcal{F}$. A mapping ψ in $\mathcal{O}^{'}$ is *upper-semi-continuous* (u.s.c) if and only if for every sequence $\{X_n : n \in \mathbb{N}\}$ of elements of $\mathcal{F}$ such that $X_n \downarrow X$ in $\mathcal{F}$ (i.e., $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq \cdots$ and $X = \bigcap_{n\geq 1} X_n$), we have $\psi(X_n) \downarrow \psi(X)$ in $\mathcal{F}$ [1]. Observe that continuity implies upper-semi-continuity but the converse is not true in general. The following theorem establishes that increasing and upper-semi-continuous operators in $\mathcal{O}$' have a minimal basis representation.

Theorem 2 *Let $\psi \in \mathcal{O}^{'}$ be an increasing upper-semi-continuous operator. Then, ψ is exactly represented as a union of adaptive erosions by mappings in its basis $\mathcal{B}_\psi$, i.e.,*

$$\psi(X) = \bigcup_{\theta\in\mathcal{B}_\psi} \mathcal{E}_\theta(X) \qquad (X \in \mathcal{F}). \tag{11}$$

A minimal representation of an increasing upper-semi-continuous operator as an intersection of adaptive dilations is obtained by duality as follows:

Corollary 1 *If ψ is increasing from $\mathcal{G}$ to $\mathcal{G}$ and has an upper-semi-continuous dual ψ^* from $\mathcal{F}$ to $\mathcal{F}$, then ψ can be exactly represented as an intersection of adaptive dilations by the transposed mappings in the basis of its dual, i.e.,*

$$\psi(X) = \bigcap_{\theta\in\mathcal{B}_{\psi^*}} \mathcal{D}_{\theta'}(X) \qquad (X \in \mathcal{G}). \tag{12}$$

In the discrete Euclidean space $\mathbb{Z}^n$, the set of open sets and closed sets are equivalent to the power set $\mathcal{P}(\mathbb{Z}^n)$. Therefore, every mapping ψ from $\mathcal{F}$ to $\mathcal{F}$ has a dual mapping ψ^* from $\mathcal{F}$ to $\mathcal{F}$. Hence, if ψ (resp., ψ^*) is increasing and u.s.c., then the basis representation as union of adaptive erosions (resp. intersection of adaptive dilations) exists. In particular, it can be shown that the adaptive median filter, with odd cardinality n, can be exactly represented in terms of union and intersection of $\binom{n}{\frac{n+1}{2}}$ specified sets. In particular, no sorting is required, which can be computationally efficient for large window sizes n [15].

5. SIMULATIONS

We illustrate the power of the proposed adaptive mathematical morphology framework in denoising and segmentation using the morphological opening by reconstruction technique. Consider the the binary image of blobs in Fig. 1(a) and its corrupted version by a germ-grain noise model [2] in Fig. 1(b). We want to segment the noisy image in order to recover the original blobs. We adopt the following adaptive structuring element

1. At each pixel z of the image, decide, by exploring its neighborhood, whether it belongs to a noise-grain or not (the germ-grain noise model is assumed to be known, *a priori*). The detection of the presence of a noise-grain $C(z)$ centered at the pixel z is determined by selecting the largest possible grain C which is present or absent in the degraded image Y. The SE mapping of the SV erosion is then selected as follows:

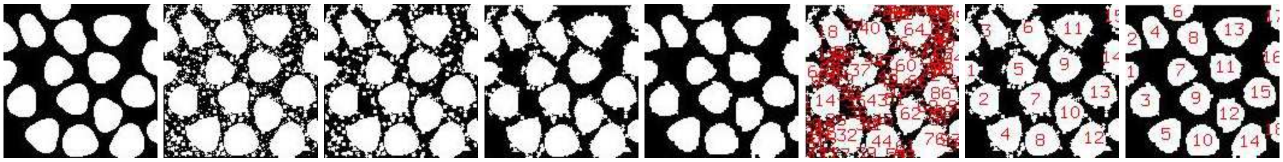
$$\theta(z) = \begin{cases} C(z) \oplus S, & \text{if } z \text{ is detected as a noisy pixel;} \\ S, & \text{otherwise,} \end{cases} \tag{13}$$

   where S denotes the rhombus structuring element. This choice of the SE mapping ensures that all noise-grains are removed completely (since the local SE is larger than the size of the noise-grain), while preserving the small main blobs in the image (which have size bigger than the rhombus).

The result of the translation-invariant and adaptive opening by reconstruction and segmentations are displayed in Fig. 1. A persisting noise in the reconstructed image has deleterious consequences for segmentation as it is either classified as main blobs (see Fig. 1(f)) or merges originally disconnected blobs (see Fig. 1(g)) and, in both cases, results in erroneous segmentation and blob detection. In medical imaging, prior anatomical knowledge could be used to specify the adaptive structuring element appropriately.

6. CONCLUSION

In this paper, we presented a general theory of adaptive morphology for binary images in the Euclidean space. This the-



(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 1. Translation-invariant and adaptive opening by reconstruction and segmentation (a) Original image; (b) Corrupted image by a germ-grain noise model; (c) Opening by reconstruction using the rhombus SE; (d) Opening by reconstruction using the rhombus SE dilated 3 times; (e) adaptive opening by reconstruction; (f), (g), (h) Segmentation of the reconstructed images in (c), (d) and (e), respectively.

ory provides a unified mathematical framework of numerous adaptive morphological schemes that have been proposed for various image processing applications, such as range imagery and adaptive filtering and segmentation [3–13]. Moreover, The adaptive morphology theory captures the geometrical interpretation of the structuring element, which is crucial in signal and image processing applications. The extension of the adaptive binary morphology to the gray-level case is straightforward and would follow the same steps presented in [13].

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