Theoretical Foundations of Spatially-Variant Mathematical Morphology Part II: Gray-Level Images

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Abstract—In this paper, we develop a spatially-variant (SV) mathematical morphology theory for gray-level signals and images in the Euclidean space. The proposed theory preserves the geometrical concept of the structuring function, which provides the foundation of classical morphology and is essential in signal and image processing applications. We define the basic SV gray-level morphological operators (that is, SV gray-level erosion, dilation, opening, and closing) and investigate their properties. We demonstrate the ubiquity of SV gray-level morphological systems by deriving a kernel representation for a large class of systems, called V-systems, in terms of the basic SV gray-level morphological operators. A V-system is defined to be a gray-level operator, which is invariant under gray-level (vertical) translations. Particular attention is focused on the class of SV flat gray-level operators. The kernel representation for increasing V-systems is a generalization of Maragos' kernel representation for increasing and translation-invariant function-processing systems. A representation of V-systems in terms of their kernel elements is established for increasing and upper semicontinuous V-systems. This representation unifies a large class of spatially-variant-linear and nonlinear systems under the same mathematical framework. The theory is used for analyzing special cases of signal and image processing systems such as SV order rank filters and linear-time-varying systems. Finally, simulation results show the potential power of the general theory of gray-level SV mathematical morphology in several image analysis and computer vision applications.

Index Terms—Spatially variant mathematical morphology, gray-level morphology, upper semicontinuous functions, adaptive order-statistic filters, linear-time-varying systems.

1 Introduction

Originally, the mathematical morphology theory was developed for translation-invariant transformations of binary images (or two level), that is, operators, which are invariant under the Euclidean group of translations [1], [2]. The theory has subsequently been extended to translation-invariant transformations of gray-level (or multilevel) images by Sternberg [3], Serra [2], and Maragos [4], [5]. A translation-invariant gray-level transformation is defined to be invariant under horizontal (space or time in 1D) translations and vertical (gray-level or signal values) translations. In mathematical morphology, sets are used as mathematical representations of binary signals and images, whereas functions represent gray-level signals. This characterization induces a similar classification for systems[1] into either function-processing (FP) systems, which accept as inputs and produce as outputs multilevel signals, or set-processing (SP) systems, whose inputs and outputs are binary signals [5]. The extension of translation-invariant binary morphology to the gray-level case was first derived based on the set representation of functions. There are two different but equivalent approaches to represent a function by a set or an equivalent class of sets: the umbra approach and the threshold sets approach. The umbra approach, which was introduced by Sternberg [3], relies on the fact that the points on and below the graph of a function correspond to a set representation of the function in a higher dimensional space. The threshold sets method, introduced by Serra [2], represents a function by an equivalent class of sets, called cross sections or threshold sets, by thresholding it at successive levels. Although the umbra approach has a nice geometrical interpretation of the morphological gray-level operations in terms of their binary counterparts, it may be the source of many mistakes if it is not handled properly [6], [7]. The problem with the umbra approach is that, in general, the union of a collection of umbras is not an umbra and, consequently, the binary dilation of two umbras is not necessarily an umbra again [7]. However, these technical difficulties vanish if we restrict the space to the set of umbras of upper-semicontinuous functions.[2] Both Serra [8] and Maragos and Schafer [9] restrict their presentation of the extension of set operators to function operators to upper-semicontinuous functions. This restriction is not necessary if one wants to construct gray-level operators from set

1. In this paper, we will use interchangeably "operator" and "system" to denote processes that accept as inputs and produce as outputs multidimensional signals.

2. Upper-semicontinuity is a property of functions that is weaker than continuity. f is upper-semicontinuous if for all $x_0 \in \mathbf{E}$, $\limsup_{x \to x_0} f(x) \leq f(x_0)$.

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Manuscript received 20 July 2006; revised 21 June 2007; accepted 10 July 2007; published online 8 Aug. 2007.
Recommended for acceptance by D. Fleet.
For information on obtaining reprints of this article, please send e-mail to: tpami@computer.org, and reference IEEECS Log Number TPAMI-0537-0706.
Digital Object Identifier no. 10.1109/TPAMI.2007.70756.

operators by thresholding. However, the threshold sets method is limited to systems, which commute with thresholding (for example, flat structuring functions).

The extension of Matheron's kernel representation theorem for translation-invariant set-processing systems [1] to function-processing systems was carried out by Maragos [4], [5]. He showed that every increasing and translation-invariant function-processing system can be represented as the supremum (respectively, infimum) of function-processing erosions (respectively, dilations). Furthermore, Maragos showed that the kernel representation of function-processing systems is redundant in the sense that a smaller subset of the kernel is sufficient for the representation of the system. Subsequently, he provided sufficient conditions for translation-invariant function-processing systems to admit a basis representation [4], [5]. Heijmans [7], [10] relaxed the translation-invariance assumption by studying function-processing systems that are invariant under horizontal translations, the so-called H-operators. However, so far, no comprehensive mathematical framework has been presented to establish the foundations of spatially-variant (SV) gray-level mathematical morphology in the Euclidean space.

Following Serra's work in [2, Chapters 2, 3, and 4], we elaborated in [11] on the general theory of spatially-variant mathematical morphology in the Euclidean space for binary signals. This theory captures the geometrical interpretation of the structuring element (SE), which is crucial in signal and image processing applications. This paper extends the theory of spatially-variant mathematical morphology presented in [11] to the gray-level case. Specifically, we consider the class of V-systems, which are function-processing systems that are spatially-variant and spatially invariant under gray-level translations. In other words, the structuring function varies in space independently of the signal values. V-systems have been used extensively in adaptive filtering applications [12], [13], [14], [15], [16], [17], [18], [19], [20]. Moreover, morphological V-systems have an elegant geometric interpretation, which is consistent with their translation-invariant counterparts. In this paper, we define the basic spatially-variant function-processing (SVFP) morphological operators (that is, SVFP erosion, dilation, opening, and closing) and investigate their properties. We show that the basic properties of translation-invariant function-processing morphological systems [21] can be transposed to SVFP systems. Special focus is devoted to the class of V-systems, which commute with thresholding. The class of translation-invariant systems, which commute with thresholding, has been extensively studied in the literature, under many different names. Heijmans [7] calls them *flat operators*, Maragos [4], [5] refers to them as *function-set-processing systems*, and Wendt et al. [22] denote them by *stack filters*. In our presentation, we will refer to V-systems, which commute with thresholding, as spatially-variant function-set-processing (SVFSP) systems, as this nomenclature is more appealing to the signal and image processing community. We demonstrate the ubiquity of the basic SVFP and SVFSP morphological systems by providing a SV kernel representation for V-systems. Specifically, we prove that every increasing V-system can be represented as the supremum (respectively, infimum) of SVFP erosions (respectively, SVFP dilations). The latter kernel representation is a generalization of Maragos' kernel representation for increasing and translation-invariant function-processing systems [4], [5]. Furthermore, based on Maragos' development of the basis representation for translation-invariant FP systems [4], [5], we provide sufficient conditions for the existence of a basis representation for V-systems. Examples, which are provided throughout the paper, demonstrate that the proposed SV gray-level mathematical morphology unifies different methods in adaptive gray-level morphology such as adaptive neighborhood morphology [18], [19] and vertically invariant morphology [17]. Our goal is to provide a sound mathematical framework to unify current and future research in spatially-variant morphological signal processing and to provide the mathematical tools needed for the design and analysis of spatially-variant morphological filters in image analysis and computer vision applications.

This paper is organized as follows: In Section 2, we define the basic SVFP and SVFSP morphological systems. Their properties are investigated in Appendix A. In Section 3, we establish a kernel representation for increasing V-systems and a basis representation for increasing and upper-semicontinuous V-systems. Section 4 illustrates the theory through the study of two adaptive systems: SV order-statistic filters and linear-time-varying (LTV) systems. Simulation results, in Section 5, show the power of the proposed theory of spatially-variant gray-level morphology in denoising, multiscale filtering, and segmentation. Finally, a summary of the results of this paper is provided in Section 6.

The proofs of all theoretical results that are new contributions in this paper have been inserted in the Appendix.

2 Spatially-Variant Function-Processing Basic Morphological Systems

2.1 Preliminaries

In this paper, we consider the space $\mathbf{E} = \mathbb{R}^m$ or $\mathbb{Z}^m$ for some $m \geq 1$. The power set of $\mathbf{E}$ will be denoted by $\mathcal{P}(\mathbf{E})$. A gray-level signal is a function from $\mathbf{E}$ to a gray-level space $\mathcal{T}$, where $\mathcal{T} = \mathbb{R}$, or $\mathbb{Z}$. The collection of such functions is denoted as $Func(\mathbf{E})$. The least and greatest elements of $Func(\mathbf{E})$ are denoted by $\mathcal{O}$ and $\mathcal{I}$: These are the functions that are identically equal to $-\infty$ and $+\infty$, respectively. An important subset of $Func(\mathbf{E})$ is the collection of upper-semicontinuous functions [23], [9], denoted by $USC(\mathbf{E})$. In this paper, we will only consider upper-semicontinuous functions. Elements of $USC(\mathbf{E})$ will be denoted by lowercase letters, for example, f and g. Set-processing systems will be denoted by lowercase Greek letters, for example, ψ and ϕ, whereas function-processing systems will be denoted by uppercase Greek letters, for example, Ψ and Φ. "$\Longrightarrow$, $\Longleftrightarrow$, $\forall$, $\exists$" denote respectively "implies," "if and only if," "for all," and "there exist(s)." The support of a function f is defined as $\mathrm{Spt}(f) = \{x \in \mathbf{E} : f(x) \neq -\infty\}$. The umbra $U[f]$ of a function f is defined by

$$U[f] = \{(x, y) \in \mathbf{E} \times \mathcal{T} : y \leq f(x)\}. \tag{1}$$

The threshold set of the function f at level t is given by

$$\mathcal{X}_t(f) = \{x \in \mathbf{E} : f(x) \geq t\}. \tag{2}$$

The reflected function $\check{f}$ of a function f is defined as $\check{f}(x) = f(-x)$, $\forall x \in \mathbf{E}$. The horizontal (or spatial) translate f_a of a function f by $a \in \mathbf{E}$ is defined as $f_a(x) = f(x - a)$, $\forall x \in \mathbf{E}$. The vertical translate $f + b$ of the function f by $b \in \mathcal{T}$ is defined by $(f + b)(x) = f(x) + b, \forall x \in \mathbf{E}$. The translation of a function f by the vector $(a,b) \in \mathbf{E} \times \mathcal{T}$ is defined as $f_{(a,b)}(x) = f(x-a)+b$, $\forall x \in \mathbf{E}$. The counterpart of the set complementation for functions is the function negation, defined by $f^*(x) = -f(x)$, $\forall x \in \mathbf{E}$. An order is imposed on $USC(\mathbf{E})$ by setting $f \leq g$ if and only if $f(x) \leq g(x)$, $\forall x \in \mathbf{E}$. The latter order induces an order on the class of FP systems by setting $\Psi_1 \leq \Psi_2$ if and only if $\Psi_1(f) \leq \Psi_2(f), \forall f \in USC(\mathbf{E})$. $\vee$ and $\wedge$ denote the supremum and infimum operations, respectively. The counterpart of the dual operator for an FP system Ψ is the negative function-processing system Ψ^*, defined by $\Psi^*(f) = -\Psi(-f), \forall f \in USC(\mathbf{E})$. In this paper, we consider only nondegenerate FP (respectively, SP) systems; that is, $\Psi(\mathcal{I}) = \mathcal{I}$ (respectively, $\psi(\mathbf{E}) = \mathbf{E}$), and $\Psi(\mathcal{O}) = \mathcal{O}$ (respectively, $\psi(\emptyset) = \emptyset$).

2.2 Spatially-Variant Function-Processing Morphological Systems

The spatially-variant structuring function Θ is a mapping from $\mathbf{E}$ to $USC(\mathbf{E})$, which associates to each point $x \in \mathbf{E}$ a upper-semicontinuous structuring function $\Theta(x)$. The transposed structuring function mapping is given by

$$[\Theta'(x)](u) = [\Theta(u)](x), \quad \forall x, u \in \mathbf{E}. \tag{3}$$

In the translation-invariant case, the structuring function mapping is the horizontal translation operator of a fixed structuring function g; that is, $\Theta(x) = g_x$, $\forall x \in \mathbf{E}$. Then, $[\Theta'(x)](u) = [\Theta(u)](x) = g_u(x) = g(x-u) = \check{g}(u-x) = \check{g}_x(u)$, $\forall x$, $u \in \mathbf{E}$. Thus, $\Theta'(x) = \check{g}_x$, $\forall x \in \mathbf{E}$. That is, the transposed structuring function mapping reduces to the translation of the reflected function $\check{g}$. Therefore, the definition of the SV structuring function is consistent with the translation-invariant case. This analogy might give the impression that the structuring function mappings Θ and Θ' are the same up to a symmetry. This is not true in general. For example, consider the structuring function mapping, which associates to each point in space a line through the origin, with varying slope. Then, the transposed structuring function mapping assigns a hyperbola function at each point in space.

An order on the mappings from $\mathbf{E}$ to $USC(\mathbf{E})$ is induced by the order on the space $USC(\mathbf{E})$; that is, $\Theta_1 \leq \Theta_2$ if and only if $\Theta_1(x) \leq \Theta_2(x)$ for every $x \in \mathbf{E}$. We say that the mapping Θ is continuous if for every convergent sequence $\{x_n\}_{n \in \mathbb{N}} \in \mathbf{E}$ with limit point $x \in \mathbf{E}$, the sequence of upper-semicontinuous functions $\{\Theta(x_n)\}_{n \in \mathbb{N}}$ converges toward the upper-semicontinuous function $\Theta(x)$ in the sense specified by Serra [2, Theorem XII-2, p. 429]. In the remainder of this paper, the structuring function mapping Θ is assumed to be continuous from $\mathbf{E}$ to $USC(\mathbf{E})$, and the support of $\Theta'(x)$ is assumed to be compact for every $x \in \mathbf{E}$.[3]

Definition 1. *The spatially-variant function-processing (SVFP) erosion is given by*

$$\mathcal{E}_\Theta(f)(x) = \bigwedge_{u \in \mathrm{Spt}(\Theta(x))} \{f(u) - [\Theta(x)](u)\} \tag{4}$$

$$= \vee\{v \in \mathcal{T} : \Theta(x) + v \leq f\}, \tag{5}$$

if $\mathrm{Spt}(\Theta(x)) \subseteq \mathrm{Spt}(f)$, *and* $-\infty$ *otherwise.*

Definition 2. *The spatially-variant function-processing dilation is given by*

$$\mathcal{D}_\Theta(f)(x) = \bigvee_{u \in \mathrm{Spt}(f) \cap \mathrm{Spt}(\Theta'(x))} \{f(u) + [\Theta'(x)](u)\} \tag{6}$$

$$= \wedge\{v \in \mathcal{T} : -\Theta'(x) + v \geq f\}, \tag{7}$$

$\forall f \in USC(\mathbf{E})$, $\forall x \in \mathbf{E}$.

The SVFP erosion and dilation can be derived from their SVSP counterparts by characterizing the upper-semicontinuous functions in terms of their umbras [24].

The SVFP opening and closing are defined in the following obvious way:

Definition 3. *The spatially-variant function-processing morphological opening is given by*

$$\Gamma_\Theta(f) = \mathcal{D}_\Theta(\mathcal{E}_\Theta(f)) = \vee\{\Theta(u) + v \leq f; (u,v) \in \mathbf{E} \times \mathcal{T}\}. \tag{8}$$

The spatially-variant function-processing morphological closing is given by

$$\Phi_\Theta(f) = \mathcal{E}_\Theta(\mathcal{D}_\Theta(f)) = \wedge\{\Theta'(u) + v \geq f; (u,v) \in \mathbf{E} \times \mathcal{T}\}. \tag{9}$$

Observe that (8) and (9) have a geometric interpretation that is analogous to their translation-invariant counterparts [8].

The properties of the SVFP erosion, dilation, opening, and closing are investigated in Appendix A. In particular, we show that they satisfy the main properties of their translation-invariant counterparts [10], [21].

2.3 Spatially-Variant Function-Set-Processing Systems

Given a set $A \in \mathcal{P}(\mathbf{E})$, we denote by C_A the characteristic function of A; that is, $C_A(z) = 1$ if $z \in A$, and $C_A(z) = 0$ if $z \notin A$. To each function-processing system Φ, we associate its set-processing (SP) system ϕ, defined as $\Phi(C_A) = C_{\phi(A)}$. We say that Φ obeys the *threshold superposition* principle if [5]

$$[\Phi(f)](x) = \vee\{t \in \mathcal{T} : x \in \phi[\mathcal{X}_t(f)]\} \quad (f \in \mathrm{USC}(\mathbf{E})). \tag{10}$$

Thus, a function-processing system satisfying (10) transforms a function f by decomposing it into its cross sections and transforming each cross section by the corresponding SP system. Such a system is called a spatially-variant function-set-processing (FSP) system by Maragos [5]. A sufficient condition for an FSP system to obey the threshold superposition property is to commute with thresholding [5], that is,

$$\phi[\mathcal{X}_t(f)] = \mathcal{X}_t[\Phi(f)], \quad (t \in \mathcal{T}, f \in \mathrm{USC}(\mathbf{E})). \tag{11}$$

The above condition allows us to analyze an function-processing system by looking at it as a set-processing system, which is simpler to analyze.

The following proposition shows that if $\mathcal{E}_\theta(X)$ (respectively, $\mathcal{D}_\theta(X)$) is the SVSP erosion (respectively, dilation) of the set $X \in \mathcal{P}(\mathbf{E})$ by the SV SE mapping $\theta : \mathbf{E} \to \mathcal{P}(\mathbf{E})$ [11], and $f \in USC(\mathbf{E})$, then the sets $\mathcal{E}_\theta(\mathcal{X}_t(f))$ (respectively, $\mathcal{D}_\theta(\mathcal{X}_t(f))$) satisfy the conditions to be the threshold sets of a function $\mathcal{E}_\theta(f)$ (respectively, $\mathcal{D}_\theta(f)$), defined as the SVFSP erosion (respectively, dilation) of f by the SE mapping θ:

$$\mathcal{E}_\theta(f)(x) = \bigwedge_{u \in \theta(x)} f(u) \qquad (f \in \mathrm{USC}(\mathbf{E}), x \in \mathbf{E}), \tag{12}$$

and

$$\mathcal{D}_\theta(f)(x) = \bigvee_{u \in \mathrm{Spt}(f) \cap \theta'(x)} f(u) \qquad (f \in \mathrm{USC}(\mathbf{E}), x \in \mathbf{E}). \tag{13}$$

3. If $\mathrm{Spt}(\Theta)$ is compact, and $\mathrm{Spt}(\Theta(x))$ is compact for all $x \in \mathbf{E}$, then $\mathrm{Spt}(\Theta'(x))$ is compact for all $x \in \mathbf{E}$.

Proposition 1. *We have*

$$\mathcal{E}_\theta(\mathcal{X}_t(f)) = \mathcal{X}_t[\mathcal{E}_\theta(f)] \qquad (f \in \mathrm{USC}(\mathbf{E})) \tag{14}$$

and

$$\mathcal{D}_\theta(\mathcal{X}_t(f)) = \mathcal{X}_t[\mathcal{D}_\theta(f)] \qquad (f \in \mathrm{USC}(\mathbf{E})). \tag{15}$$

Notice that if the mappings θ and θ' have finite range (that is, $|\theta(x)| < \infty$, and $|\theta'(x)| < \infty$, $\forall x \in \mathbf{E}$, where $|X|$ denotes the cardinality of the set X), then the SVFSP erosion and dilation, as defined in (12) and (13), respectively, correspond to the adaptive minimum and maximum operators.

The SVFSP erosion and dilation of f by the SV structuring element mapping θ are special cases of the SVFP erosion and dilation, as defined in (4) and (6), corresponding to the choices of the structuring element mapping $[\Theta(x)](u) = 0$, $\forall u \in \mathrm{Spt}(\Theta(x))$, $\forall x \in \mathbf{E}$. In this case, we say that the structuring function mapping Θ is *flat*. Observe that a flat structuring function mapping Θ is represented by its region of support $\theta(x) = \mathrm{Spt}(\Theta(x))$, $\forall x \in \mathbf{E}$.

2.4 Examples

2.4.1 Translation-Invariant Gray-Level Morphology [21], [25]

Consider a function $g \in USC(\mathbf{E})$. We showed in Section 2.2 that the translation-invariant gray-level morphology corresponds to a structuring function mapping $\Theta(x) = g_x$ and $\Theta'(x) = \check{g}_x$, $\forall x \in \mathbf{E}$. In particular, $\mathrm{Spt}(\Theta'(x))$ is compact if and only if $\mathrm{Spt}(g)$ is compact [9]. The SVFP erosion and dilation, as defined in (4) and (5), and (6) and (7), respectively, reduce to

$$f \ominus g(x) = \bigwedge_{u \in \mathrm{Spt}(g)+x} \{f(u) - g(u-x)\} \tag{16}$$

$$= \vee\{v \in \mathcal{T} : g_x + v \leq f\}, \tag{17}$$

and

$$f \oplus g(x) = \bigvee_{u \in [\mathrm{Spt}(f) \cap \mathrm{Spt}(\check{g})+x]} \{f(u) + g(x-u)\} \tag{18}$$

$$= \wedge\{v \in \mathcal{T} : -(\check{g})_x + v \geq f\}. \tag{19}$$

Equations (16) and (17), and (18) and (19) are the well-known translation-invariant gray-level erosion and dilation, respectively [9], [10]. A similar derivation can be used for showing that the SVFP opening and closing, as defined in (8) and (9), also reduce to their translation-invariant counterparts. Therefore, the translation-invariant gray-level morphology is a special case of the proposed spatially-variant gray-level morphology.

2.4.2 Gray-Level Adaptive Neighborhood Morphology [18], [19]

Consider $\mathbf{E} = \mathbb{R}^2$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a criterion mapping such as luminance, contrast, and thickness. Let $m > 0$. For each $x \in \mathbf{E}$, define the connected set $V_m^h(x)$ by $V_m^h(x) = \{y : |h(y) - h(x)| \leq m\}$. Choose the flat structuring function mapping Θ with the following region of support θ:

$$\theta(x) = \bigcup_{z \in \mathbf{E}} \{V_m^h(z) : x \in V_m^h(z)\}. \tag{20}$$

One can easily verify that the transposed region of support $\theta' = \theta$ (that is, $\theta'(x) = \theta(x)$, $\forall x \in \mathbf{E}$). Then, the SVFSP erosion and dilation in (12) and (13) become

$$\mathcal{E}_\theta(f) = \bigwedge_{u \in \theta(x)} f(u), \qquad (f \in \mathrm{USC}(\mathbf{E})), \tag{21}$$

and

$$\mathcal{D}_\theta(f) = \bigvee_{u \in \theta(x)} f(u), \qquad (f \in \mathrm{USC}(\mathbf{E})). \tag{22}$$

Equations (21) and (22) are, respectively, the adaptive neighborhood erosion and dilation presented in [18] and [19]. Therefore, the adaptive neighborhood gray-level morphology framework is a special case of the proposed spatially-variant gray-level mathematical morphology theory.

2.4.3 Vertically-Invariant Morphology [17]

Given a structuring function mapping $K : \mathbf{E} \to USC(\mathbf{E})$, choose the structuring function mapping Θ such that $[\Theta(x)](u) = [K(x)](u-x)$, $\forall x, u \in \mathbf{E}$. Observe that the local structuring function $\Theta(x)$ at point x, which is evaluated at x, is equal to the local structuring function $K(x)$ at point x, which is evaluated at the origin; that is, $[\Theta(x)](x) = [K(x)](0)$. For instance, consider the following structuring function mapping K [17]:

$$[K(x)](u) = \begin{cases} \sqrt{r^2 - u^2}, & \text{if } |u| \leq r; \\ -\infty, & \text{if } |u| > r. \end{cases} \tag{23}$$

Then,

$$[\Theta(x)](u) = \begin{cases} \sqrt{r^2 - (u-x)^2}, & \text{if } |u-x| \leq r; \\ -\infty, & \text{if } |u-x| > r. \end{cases} \tag{24}$$

The local structuring function $K(x)$ at the point x is a circle with radius r centered at the origin, whereas the local structuring function $\Theta(x)$ at the point x is a circle with the same radius r but centered at the point x. This property of having the local structuring function $\Theta(x)$, which is centered at x, may be desired in some practical applications such as adaptive signal smoothing [17]. In this case, the SVFP erosion and dilation, as defined in (4) and (6), reduce, respectively, to

$$\mathcal{E}_K[f](x) = \bigwedge_{z \in \mathrm{Spt}(K(x))} \{f(x+z) - [K(x)](z)\}, \tag{25}$$

and

$$\mathcal{D}_K[f](x) = \bigvee_{z \in [\mathrm{Spt}(\check{f})+x] \cap [\mathrm{Spt}(\check{K}'(z))+x]} \{f(x-z) + [K(x-z)](z)\}, \tag{26}$$

$\forall f \in USC(\mathbf{E})$, $\forall x \in \mathbf{E}$.

Equations (25) and (26) coincide with the vertically invariant erosion and dilation defined in [17] and are used for adaptive signal smoothing. Therefore, the vertically invariant mathematical morphology provides another special case of the proposed spatially-variant gray-level morphology.

3 Spatially-Variant Kernel and Basis Representations

3.1 SV Kernel Representation

Definition 4. *A function-processing system $\Psi : USC(\mathbf{E}) \to USC(\mathbf{E})$ is called a V-system if $\Psi(f + y) = \Psi(f) + y$ for all $f \in USC(\mathbf{E})$ and $y \in \mathcal{T}$.*

In particular, a V-system is invariant with respect to DC biases. Examples of V-systems are given by the SVFP erosion, dilation, opening, and closing, as defined in Section 2.2, the adaptive neighborhood morphological systems defined in (21) and (22), the adaptive amoeba morphological systems presented in [20], and the vertically invariant morphological systems defined in (25) and (26). Moreover, one can easily verify that the class of V-systems is closed under duality; that is, if Ψ is a V-system, then its dual Ψ^* is also a V-system.

We extend Maragos' kernel representation theorem to V-systems as follows: Let Ψ be a V-system and consider its SV umbra processing system ψ_u, defined as $\psi_u(U[f]) = U[\Psi(f)]$, for every $f \in USC(\mathbf{E})$. From the definition of the kernel of SVSP systems in [11], the proof of the following proposition derives a one-to-one correspondence between the kernel of a V-system and the kernel of its umbra processing system.

Proposition 2. *The kernel of a V-system Ψ, $\mathcal{K}(\Psi)$, is given by the following collection of mappings:*

$$\mathcal{K}(\Psi) = \{\Theta : \mathbf{E} \to \mathrm{USC}(\mathbf{E}) : \Psi[\Theta(x)](x) \geq 0, \forall x \in \mathbf{E}\}. \quad (27)$$

In this paper, we use $\mathcal{K}$ to denote the kernel of an SVFP system, and Ker to denote the kernel of an SVSP system [11]. The one-to-one correspondence between the kernel of a V-system and the kernel of its SV umbra SP system will allow us to transpose the results of the kernel representation obtained for SVSP systems in [11] to V-systems. In particular, since the kernel of SVSP systems is nontrivial and unique [11, Propositions 1 and 2], it follows that the kernel of V-systems is also nontrivial and unique.

Using (2), (10), and the definition of the kernel of SVSP systems in [11], we obtain the kernel of a SVFSP system Φ from the kernel of its SP system ϕ as follows:

$$\mathcal{K}(\Phi) = \{\Theta : \mathcal{X}_0[\Theta(x)] \in \mathrm{Ker}(\phi), \text{ for all } x \in \mathbf{E}\}. \quad (28)$$

The following establishes the kernel representation for increasing V-systems.

Theorem 1. *A spatially-variant function-processing system $\Psi : USC(\mathbf{E}) \to USC(\mathbf{E})$ is an increasing V-system if and only if Ψ can be represented as the supremum of spatially-variant function-processing erosions by mappings in its kernel or, equivalently, as the infimum of spatially-variant function-processing dilations by the transposed mappings in the kernel of its dual, that is,*

$$\Psi(f) = \bigvee_{\Theta \in \mathcal{K}(\Psi)} \mathcal{E}_{\Theta}(f) = \bigwedge_{\Theta \in \mathcal{K}(\Psi^*)} \mathcal{D}_{\Theta'}(f) \quad (f \in \mathrm{USC}(\mathbf{E})). \quad (29)$$

Corollary 1. *A spatially-variant function-set-processing system $\Phi : USC(\mathbf{E}) \to USC(\mathbf{E})$ that commutes with thresholding is an increasing V-system if and only if it can be represented as the supremum of spatially-variant function-set-processing erosions by mappings in the kernel of its set-processing system ϕ or, equivalently, as the infimum of spatially-variant function-set-processing dilations by the transposed mappings in the kernel of ϕ^**

$$\Phi(f) = \bigvee_{\theta \in \mathrm{Ker}(\phi)} \mathcal{E}_{\theta}(f) = \bigwedge_{\theta \in \mathrm{Ker}(\phi^*)} \mathcal{D}_{\theta'}(f) \quad (f \in \mathrm{USC}(\mathbf{E})). \quad (30)$$

3.2 Basis Representation

The kernel representations in Theorem 1 and Corollary 1 are powerful theoretical results, as they demonstrate the ubiquity of the SVFP and the SVFSP erosion and dilation and, hence, establish the spatially-variant mathematical morphology for gray-level signals as the general mathematical framework for the study of linear and nonlinear increasing systems in signal and image processing. However, the kernel representation theorems have no direct relevance for a practical implementation of V-systems because of the infinite cardinality of the kernel. To see this, consider an increasing V-system Ψ. From (27), we observe that if $\Theta \in \mathcal{K}(\Psi)$, then every structuring function mapping $\Lambda \geq \Theta$ is also in the kernel of Ψ. Therefore, we are led to the investigation of the existence of the minimal kernel elements for the representation of V-systems. Following Maragos' approach in defining the basis of translation-invariant set-processing and function-processing systems [5], we define the basis $\boldsymbol{B}_\Psi$ of the kernel of a V-system Ψ as the collection of the minimal elements of the kernel of Ψ. Formally,

$$\boldsymbol{B}_\Psi = \{\Theta \in \mathcal{K}(\Psi) : [\Lambda \in \mathcal{K}(\Psi) \text{ and } \Lambda \leq \Theta] \Longrightarrow \Lambda = \Theta\}. \quad (31)$$

In this paper, we use $\boldsymbol{B}_\Psi$ to denote the basis of the SVFP Ψ, and $\mathcal{B}_\psi$ to denote the basis of the SVSP system ψ defined in [11]. In the development of the basis representation for SVSP systems in [11], we had to restrict ourselves to the class of all closed subsets of $\mathbf{E}$. The equivalent class of functions is the class of upper-semicontinuous functions. In fact, a function f is upper-semicontinuous if and only if its umbra $U[f]$ is closed or, equivalently, if and only if its threshold sets $\mathcal{X}_t(f)$ are closed for all $t \in \mathcal{T}$. Let $f_n \downarrow f$ be a sequence of upper-semicontinuous functions that decrease monotonically to $f = \wedge_n f_n$. An increasing FP system Ψ is said to be upper-semicontinuous if and only if $f_n \downarrow f \Longrightarrow \Psi(f_n) \downarrow \Psi(f)$.

In the following, we prove that every increasing upper-semicontinuous V-system has a minimal element in its kernel.

Theorem 2. *Let $\Psi : USC(\mathbf{E}) \to USC(\mathbf{E})$ be an increasing upper-semicontinuous V-system. Then, the kernel of Ψ has a minimal element.*

Before we prove the potential of minimal elements for the exact representation of these systems, we need the following result:

Theorem 3. *Let Ψ be an increasing upper-semicontinuous V-system. Then, for every $\Theta \in \mathcal{K}(\Psi)$, there exists $\Theta_M \in \boldsymbol{B}_\Psi$ such that $\Theta_M \leq \Theta$.*

We can now establish the basis representation of upper-semicontinuous V-systems in terms of SVFP erosions.

Theorem 4. *Let $\Psi : USC(\mathbf{E}) \to USC(\mathbf{E})$ be an increasing upper-semicontinuous V-system. Then, Ψ can be represented as the supremum of spatially-variant function-processing erosions by mappings in its basis $\boldsymbol{B}_\Psi$:*

$$\Psi(f) = \bigvee_{\Theta \in \boldsymbol{B}_\Psi} \mathcal{E}_{\Theta}(f) \qquad (f \in \mathrm{USC}(\mathbf{E})). \quad (32)$$

To find a dual representation in terms of SVFP dilations, Theorem 4 has to apply to the dual V-system Ψ^*. In particular Ψ^* has to be upper-semicontinuous on $USC(\mathbf{E})$. Consequently, the class $USC(\mathbf{E})$ has to be invariant under function complementation; that is, if f is upper-semicontinuous, then $(-f)$ is also upper-semicontinuous This is, in particular, true for functions defined on $\mathbb{Z}^m$. In this case, both Ψ and Ψ^* are defined on $USC(\mathbb{Z}^m)$.

Corollary 2. *Let $\Psi : USC(\mathbb{Z}^m) \to USC(\mathbb{Z}^m)$ be an increasing and upper-semicontinuous V-system. If the dual system Ψ^* is also upper-semicontinuous, then Ψ can be represented as the supremum of erosions by mappings in its basis or, equivalently,* *as the infimum of dilations by the transposed mappings in the basis of its dual, that is,*

$$\Psi(f) = \bigvee_{\Theta \in \boldsymbol{B}_\Psi} \mathcal{E}_\Theta(f) = \bigwedge_{\Theta \in \boldsymbol{B}_{\Psi^*}} \mathcal{D}_{\Theta'}(f) \quad (f \in \text{USC}\,(\mathbb{Z}^m). \quad (33)$$

Corollary 3.

a. *Let* $\Phi : USC(\mathbf{E}) \to USC(\mathbf{E})$ *be a spatially-variant function-set-processing system that commutes with thresholding. Then,* Φ *can be represented as the supremum of erosions by mappings in the basis of its set-processing system* ϕ

$$\Phi(f) = \bigvee_{\theta \in \mathcal{B}_\phi} \mathcal{E}_\theta(f) \qquad (f \in \text{USC}(\mathbf{E})). \qquad (34)$$

b. *Let* $\Phi : USC(\mathbb{Z}^m) \to USC(\mathbb{Z}^m)$ *be a spatially-variant function-set-processing system that commutes with thresholding and consider its set-processing system* ϕ. *If the dual set-processing system* ϕ^* *is upper-semicontinuous, then* Φ *can be represented as the supremum of erosions by mappings in the basis of its set-processing system* ϕ *or, equivalently, as the infimum of dilations by the reflected mappings in the basis of* ϕ^*, *that is*

$$\Phi(f) = \bigvee_{\theta \in \mathcal{B}_\phi} \mathcal{E}_\theta(f) = \bigwedge_{\theta \in \mathcal{B}_{\phi^*}} \mathcal{D}_{\theta'}(f) \quad (f \in \text{USC}\,(\mathbb{Z}^m)). \qquad (35)$$

The perspectives of the basis theory are at least twofold. First, the redundancy of the kernel is infinitely reduced. Second, if the basis is finite, the corresponding V-system can be represented as the maximum of SVFP erosions or as the minimum of SVFP dilations. This can tremendously simplify the analysis and the implementation of these systems. However, the basis representation theorem is not constructive in the sense that it does not provide an algorithm for finding the basis elements for each increasing and upper-semicontinuous V-system. It is merely an existence theorem. In the following section, we present examples of practical V-systems that are used in signal and image processing applications, and we show how their basis can be obtained.

4 Examples

4.1 Order-Statistic Filters

In this example, we study the properties of the SVFSP order-statistic filters and show their relation to the SVFSP basic morphological systems.

Consider $\mathbf{E} \subseteq \mathbb{Z}^2$. Let B be a mapping from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ such that $y \in B(y)$ and $|B(y)| = \text{cardinality of } B(y) = n$ for every $y \in \mathbf{E}$. The $r\text{th}$ SVFSP order-statistic filter is defined by

$$[\Phi_r(f, B)](x) = r\text{th largest value of } \{f(y) : y \in B(x)\}, \quad (36)$$

where $r = 1, 2, \cdots, n$. Observe that the first SVFSP order statistic is the SVFSP dilation by the transposed mapping $\check{B} = \{-b : b \in B\}$, and the nth SVFP order statistic is the SVFSP erosion by the mapping B.

Proposition 3. *The* $r\text{th}$ *SVFSP order-statistic filter is an increasing V-system, which commutes with thresholding. Moreover, its dual is the* $(n - r + 1)\text{th}$ *SVFSP order-statistic filter.*

Thus, from Corollary 3b, the $r\text{th}$ SVFSP order-statistic filter can be represented as the supremum of SVFSP erosions by mappings in the kernel of its SP system or, equivalently, as the infimum of SVFSP dilations by the mappings in the kernel of the dual SP system. However, this representation is not very useful in practice because of the redundancy of the kernel. In [11], we proved that the SVSP median filter admits a basis representation. The same proof can be easily carried over to SVFSP order-statistic filters to obtain the following basis representation

$$[\Phi_r(f, B)](x) = \bigvee_{\theta \subseteq B, |\theta| = r} [\bigwedge_{u \in \theta(x)} f(u)] \qquad (37)$$

$$= \bigwedge_{\theta \subseteq B, |\theta| = n-r+1} [\bigvee_{u \in \theta'(x)} f(u)]. \qquad (38)$$

The importance of (37) and (38) stems from the fact that a SVFSP order-statistic filter can be exactly implemented by using max-min operations, which require much less computations than sorting, for quite small adaptive window sizes [5].

4.2 Linear Time-Varying Systems

In this example, we generalize the study of linear time invariant systems in [9] to linear time-varying (LTV) systems. This example will show the power of the proposed spatially-variant gray-level mathematical morphology theory to study not only SV nonlinear systems but also SV linear systems.

The output of a continuous linear time-varying (LTV) system is given by

$$[\Psi(f)](t) = \int_{\mathbb{R}} f(\tau)[h(\tau)](t)d\tau, \qquad (39)$$

where $h(\tau)$ is the response of the system to an impulse applied at time τ. Therefore, the impulse response mapping of an linear time-varying system can be viewed as a mapping from $\mathbb{R}$ to $Func(\mathbb{R})$ such that for each impulse applied at time $\tau \in \mathbb{R}$ corresponds an impulse response $h(\tau) \in Func(\mathbb{R})$. Without loss of generality, we consider systems such that their DC gain is equal to unity; that is, $\int_{\mathbb{R}}[h(\tau)](t)d\tau = 1$ for all $t \in \mathbb{R}$. Observe that this scaling condition ensures that the linear time-varying system Ψ, as defined in (39), is a V-system. In the following, we give a necessary and sufficient condition for Ψ to be increasing.

Proposition 4. *A linear time-varying system is increasing if and only if its impulse response mapping is nonnegative; that is, for each* τ, $h(\tau)$ *is a nonnegative function.*

The kernel of the linear time-varying system Ψ, as defined in (39), is given by

$$\mathcal{K}(\Psi) = \left\{ \Theta : \int_{\mathbb{R}} [\Theta(x)](\tau)[h'(x)](\tau)d\tau \geq 0, \; \forall x \in \mathbb{R} \right\}, \quad (40)$$

where h' is the transpose mapping of h (see (3)). Thus, from Theorem 1, we obtain a kernel representation of the linear time-varying system as follows:

Proposition 5. *Let* Ψ *be a linear time-varying system with unity gain and a nonnegative impulse response mapping. Then,*

$$\begin{aligned} [\Psi(f)](x) &= \int_{\mathbb{R}} f(\tau)[h(\tau)](x)d\tau \\ &= \bigvee_{\Theta \in \mathcal{K}(\Psi)} [\bigwedge_u \{f(u) - [\Theta(x)](u)\}]. \end{aligned} \qquad (41)$$

Proposition 5 gives a closed-form expression of the output of an linear time-varying system having unity gain and a nonnegative impulse response mapping in terms of sup-inf operations only. The drawback of this expression is that there are an infinite number of such operations, since the kernel of the system is infinite.

In what follows, we will investigate the existence of a basis representation for discrete linear time-varying systems. A sufficient condition for a discrete LTV system with a nonnegative impulse response mapping to be upper-semi-continuous is that the transposed mapping $h'(n)$ has a finite support for all n, where the support of the function $h(k)$, $\mathrm{Spt}(h(k))$, is defined as $\mathrm{Spt}(h(k)) = \{n \in \mathbb{N} : [h(k)](n) \neq 0\}$. One can easily verify that in this case, $f_n \downarrow f \Rightarrow \Psi(f_n) \downarrow \Psi(f)$.

Proposition 6. *Let Ψ be a discrete linear time-varying system with unity gain and a nonnegative impulse response mapping having a transposed mapping with finite support. Then, the basis of Ψ is given by*

$$\boldsymbol{B}_\Psi = \left\{ \Theta : \sum_{k=1}^{N} [h^{'}(n)](k)[\Theta(n)](k) = 0 \right.$$
$$\left. \text{and } \mathrm{Spt}(\Theta(n)) = \mathrm{Spt}(h'(n)), \forall n \in \mathbb{N} \right\}.$$

If we represent the finite extent function $h'(n)$ in a vector form, then we see that the basis elements $\Theta(n)$ belong to the hyperplane perpendicular to the vector $h'(n)$ for all $n \in \mathbb{N}$. The basis mappings are solutions of the linear system $\sum_{k=1}^{N}[h(k)](n)[\Theta(n)](k) = 0$, $\forall n \in \mathbb{N}$ subject to three constraints: 1) $h(k)$ is a nonnegative function for $k = 1, \cdots, N$, 2) $\sum_k [h(k)](n) = 1$, $\forall n \in \mathrm{Spt}(h(k))$, and 3) $\mathrm{Spt}[\Theta(n)] = \mathrm{Spt}[h'(n)]$ is finite $\forall n$. Consequently, we have the following basis representation:

$$\begin{aligned}[\Psi(f)](n) &= \sum_{k=1}^{N} f(k)[h(k)](n) \\ &= \bigvee_{\Theta \in \boldsymbol{B}_\Psi} \left\{ \bigwedge_{k \in \mathrm{Spt}(\Theta(n))} \{f(k) - [\Theta(n)](k)\} \right\}.\end{aligned} \tag{42}$$

Equation (42) relates the nonlinear SVFP morphological erosions to LTV systems. Moreover, it gives a closed-form expression to a large class of linear time-varying discrete systems.

Example: adaptive mean. Consider a normalized 1D signal $x(n)$ and its corrupted version by an impulse noise $z(n)$, that is,

$$z(n) = x(n) + \sum_{k \in I} (-1)^k \delta(n-k), \tag{43}$$

where $I \subset \mathbb{N}$ is the set of corrupted samples. We propose to adaptively denoise the signal $z(n)$ by using an LTV system with the following causal impulse response mapping:

$$[h(k)](n) = \begin{cases} \delta(n-k), & \text{if } n \notin I; \\ \frac{1}{2}[\delta(n-k) + \delta(n-1-k)], & \text{if } n \in I. \end{cases} \tag{44}$$

Here, $k \in \{n-1, n\}$, and $h(k) = 0$ if $k \notin \{n-1, n\}$. The output of this filter is given by

$$\begin{aligned} y(n) &= \sum_{k=n-1}^{n} z(k)[h(k)](n) \\ &= \begin{cases} x(n), & \text{if } n \notin I; \\ \frac{1}{2}[z(n-1) + z(n)], & \text{if } n \in I. \end{cases}\end{aligned} \tag{45}$$

From (45), we see that this LTV filter is an adaptive mean, which averages the last two samples of the signal if the current sample is noisy and leaves the current sample unchanged if it is not noisy. This simple strategy, as illustrated in Fig. 1, denoises the signal without oversmoothing it.

The impulse response mapping, as defined in (44), is nonnegative and has a unity DC gain, and the support of its transpose mapping is finite. Therefore, from Proposition 6, the basis functions are given by

$$\Theta(n) = \begin{cases} 0, & \text{if } n \notin I; \\ \begin{pmatrix} (\Theta(n))(n) \\ (\Theta(n))(n-1) \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}, & \text{if } n \in I, \end{cases} \tag{46}$$

where $\alpha \in \mathbb{R}$. Thus, from (42), the adaptive mean filter can be represented as the suprema of minima:

$$y(n) = \begin{cases} x(n), & \text{if } n \notin I; \\ \vee_{\alpha \in \mathbb{R}} \min\{x(n) - \alpha, x(n-1) + \alpha\}, & \text{if } n \in I. \end{cases} \tag{47}$$

5 Simulations

5.1 Adaptive Denoising

Image restoration is an important problem in image processing and analysis applications. It requires the development of an efficient filtering procedure, which restores an image from its noisy version while preserving the important features of the noise-free image. This is an important requirement, since many algorithms for pattern analysis, which process noisy data, critically depend on accurate geometrical and topological image description [2]. The traditional approach to solving this problem is by means of linear filtering techniques. Although this is a mathematically and practically simple approach, it usually results in a distortion of many important image characteristics. The alternative solution is by means of more powerful nonlinear filtering techniques and, specifically, by employing the class of morphological filters [9], [26]. However, it is known that there is an inherent trade-off in translation-invariant morphological filters between the noise removal capability of the filter and the feature preservation of the noise-free image [16], [17], [27]. This trade-off is due to the use of a fixed SE while morphologically filtering the signal (or image). The important structures of the signal that are smaller than the SE used will be removed or oversmoothed. One solution to the denoising problem, then, is to vary the SE according to the local characteristics of the image. In this example, we will show the power of linear and nonlinear SV denoising by considering the SV mean filter, the SV alternating filter [28], [2], and the SV median filter.

Consider the corrupted Lena image by a 10 percent salt (gray-level 0) and pepper (gray-level 255) noise, as shown in Fig. 1a. Let B be a fixed SE and consider a flat structuring function mapping with a region of support mapping θ given by

Fig. 1. Translation-invariant and spatially-variant denoising. (a) Noisy image. (b) Translation-invariant mean filter using a 3×3 window. (c) Spatially-variant mean filter using a 3×3 window for noisy pixels only. (d) Translation-invariant mean filter using a 7×7 window. (e) Spatially-variant mean filter using a 7×7 window for noisy pixels only. (f) Translation-invariant alternating filter using the 3×3 square SE. (g) Spatially-variant alternating filter. (h) Translation-invariant median filter using the 5×5 square SE. (i) spatially-variant median filter.

$$\theta(x) = \begin{cases} B, & \text{if } x \text{ is a salt or pepper pixel;} \\ \emptyset, & \text{otherwise.} \end{cases}$$

That is, only noisy pixels are filtered. A noisy pixel is detected as an isolated 0 or 255 gray-level pixel. This SV denoising scheme will significantly preserve the edges while effectively removing the noise. In our simulations, B is a square window of a predetermined size.

SV mean filter. The usual mean filter is a linear, simple, and easy to implement system, which is often used for image smoothing and denoising. There is an inherent trade-off in the choice of the window of the mean filter: a small window preserves, to an extent, the edges of the image but is sensitive to outliers, whereas a large window reduces the effect of outliers but significantly blurs the edges of the image. This trade-off is illustrated in Figs. 1b and 1d. We adaptively mean filter the noisy image using a 2D version of the impulse response mapping given in (44). The power of the adaptive mean filter in denoising is illustrated in Figs. 1c and 1e.

SV alternating filter. The alternating filter is a composition of closing and opening by the same SE. Maragos and Schafer [26] have demonstrated a strong relationship between the alternating filter and the median filter. The alternating filter has been experimentally demonstrated for its smoothing and noise removal capability in binary and gray-scale images [2], [28], [29]. Fig. 1f shows the translation-invariant alternating filter output, with a 3×3 square SE. Some salt noise remains, and the image is overly smoothed. The SV alternating filter, as displayed in Fig. 1g, removes all the noise and preserves the edges of the noise-free image.

SV median filter. The median filter is a self-dual rank-order filter (see Section 4). The translation-invariant median filter is more robust than the translation-invariant mean filter in removing the noise. However, it is also relatively expensive and complex to compute, as it requires sorting algorithms. Moreover, the translation-invariant median filter removes the noise at the expense of oversmoothing the image, as shown in Fig. 1h. On the other hand, the SV median filter preserves the noise-free image features, as can be seen in Fig. 1i.

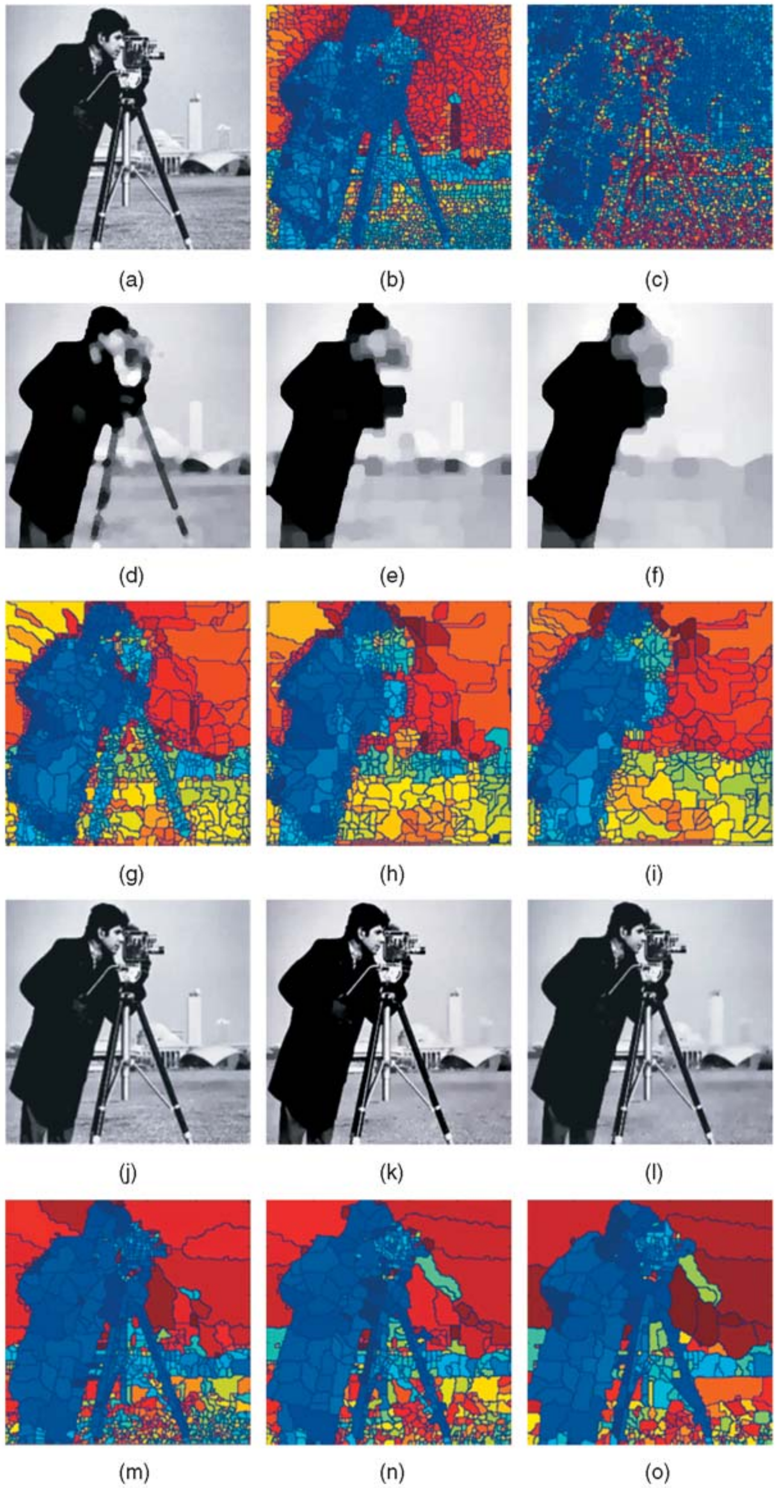

Fig. 2. Translation-invariant and spatially-variant multiscale decomposition using Alternating Sequential Filters (ASF), and segmentation using the watershed transformation. (a) The original cameraman image. (b) Segmentation of the original image. (c) Segmentation of the gradient image. (d), (e), and (f) Translation-invariant ASF. (g), (h), and (i) Segmentation of the results in (d), (e), and (f), respectively. (j), (k), and (l) spatially-variant ASF with a homogeneity tolerance of $m = 10$. (m), (n), and (o) Segmentation of the results in (j), (k), and (l), respectively. (d) TIASF_4, (e) TIASF_7, (f) TIASF_{10}, (g) Seg(TIASF_4), (h) Seg(TIASF_7), (i) Seg(TIASF_{10}), (j) SVASF_4^7, (k) SVASF_7^7, (l) SVASF_{10}^7, (m) Seg(SVASF_4^7), (n) Seg(SVASF_7^7), and (o) Seg(SVASF_{10}^7).

5.2 SV Multiscale Filtering and Segmentation

In this application, we will show the power of SV gray-level mathematical morphology in multiscale filtering and segmentation by presenting a multiscale representation of the cameraman image using the Alternating Sequential Filters (ASFs) and segmenting the filtered images using the watershed transformation. An alternating sequential filter is a composition of openings and closings by structuring elements of increasing sizes. The alternation of openings and closings is essentially a multiresolution technique, which introduces less distortion than individual openings and closings. Schonfeld and Goutsias showed that alternating sequential filters are the best filters in preserving crucial structures in the "least difference" sense [30].

The watershed transformation is a powerful tool for image segmentation [8], [31]. The intuitive idea underlying this method is that of a landscape or topographic relief, which is flooded by water, with watersheds being the dividing lines of the domains of attraction of rain falling over the region. An alternative approach is to imagine the landscape being immersed in a lake, with holes pierced in the local minima. The water entering through the holes floods the surface. When two or more floods coming from different minima may merge, dams are built. At the end of the process, only the dams emerge. These dams define the watershed lines. In order to produce a meaningful segmentation, the input image is generally transformed, and then, the watershed is applied. The gradient image is often used in the watershed transformation because the main criterion of the segmentation is the homogeneity of the gray values of the objects present in the image. However, the gradient image generally creates an oversegmentation, which is due to the presence of spurious minima. In this simulation, we show that applying the watershed transformation to a spatially-variant ASF (SVASF) produces better segmentations than applying it to the gradient image or to the translation-invariant counterpart filter (TIASF).

In our implementation, we use balls SEs of increasing radius for the TIASF, and the flat structuring function mapping represented by its region of support given in (20) for the SVASF. We write TIASF_p to denote the TIASF of order p, and SVASF_p^m to denote the SVASF of order p, with homogeneity m and criterion mapping given by the luminance. Fig. 2 shows the decomposition and segmentation results of the different filters. The translation-invariant ASF rapidly oversmooths the image, altering the transitions between the different objects and losing the original topology of the image. Even though it results in a less mosaic segmented image than the gradient, it loses the oversmoothed objects (see Figs. 2g and 2h, where the camera stand is not represented) and still results in an oversegmentation of the background. The SVASF, however, results in a simplified version of the image while conserving the topology and contours of its different objects and, at the same time, producing flat zones of the image, which lead to a much better segmentation than its translation-invariant counterpart.

6 Summary

We have proposed a spatially-variant gray-level mathematical morphology theory in the Euclidean space, which preserves the geometrical notion of the structuring function inherent in the classical translation-invariant morphology. We defined the basic spatially-variant gray-level morphological operations, that is, spatially-variant erosion, dilation, opening, and closing, and investigated their properties. We have demonstrated the ubiquity of spatially-variant function-processing (SVFP) erosions and dilations by showing that every increasing V-system, i.e., an increasing system that is invariant under vertical translations, can be represented as the supremum of SVFP erosions or, equivalently, as the infimum of SVFP dilations. Furthermore, we established a basis representation for the subclass of upper-semicontinuous increasing V-systems. If the basis of a system is finite, then it can be represented as the maximum of SVFP erosions or the minimum of SVFP dilations. In particular, we showed that adaptive order-statistic filters have a basis representation in terms of SVFP erosions and SVFP dilations. We have also related LTV systems to the basic nonlinear SVFP morphological operators. In particular, we established a closed-form expression for LTV systems in terms of the supremum and infimum of functions. Simulation results showed the enormous potential of the theory of SV gray-level mathematical morphology in image denoising and multiscale representation.

Appendix A

Properties of the Basic SVFP Morphological Operators

A.1 Properties of SVFP Erosion and Dilation

The following properties are valid for all functions $f \in USC(\mathbf{E})$.

Adjunction. For every structuring function mapping Θ, the pair $(\mathcal{E}_\Theta, \mathcal{D}_\Theta)$ is an adjunction, that is,

$$\mathcal{D}_\Theta(f) \leq g \Longleftrightarrow f \leq \mathcal{E}_\Theta(g). \tag{48}$$

Proof. We have

$$\begin{aligned}\mathcal{D}_\Theta(f) \leq g &\iff \forall x \in \mathbf{E}, \vee_{u\in\mathbf{E}}\{f(u) + [\Theta(u)](x)\} \leq g(x)\\ &\iff \forall x, u \in \mathbf{E}, f(u) + [\Theta(u)](x) \leq g(x)\\ &\iff \forall x, u \in \mathbf{E}, f(u) \leq g(x) - [\Theta(u)](x)\\ &\iff \forall u \in \mathbf{E}, f(u) \leq \wedge_{x\in\mathbf{E}}\{g(x) - [\Theta(u)](x)\}\\ &\iff \forall u \in \mathbf{E}, f(u) \leq \mathcal{E}_\Theta(g)(u)\\ &\iff f \leq \mathcal{E}_\Theta.\end{aligned}$$

□

Duality. For every structuring function mapping Θ, the SVFP systems $\mathcal{E}_\Theta$ and $\mathcal{D}_\Theta$ are dual; that is, $\mathcal{E}^*_\Theta = \mathcal{D}_{\Theta'}$.

Proof. We have

$$\begin{aligned}\mathcal{E}^*_\Theta(f) &= -\mathcal{E}_\Theta(-f) = -\vee\{v \in \mathcal{T} : \Theta(x) + v \leq -f\}\\ &= \wedge\{-v \in \mathcal{T} : f \leq -v - \Theta(x)\}\\ &= \wedge\{v \in \mathcal{T} : f \leq v - \Theta(x)\} = \mathcal{D}_{\Theta'}(f).\end{aligned}$$

□

Increasing. For every structuring function mapping Θ, the SVFP systems $\mathcal{E}_\Theta$ and $\mathcal{D}_\Theta$ are increasing systems.

Proof. The proof follows immediately from (4) and (6). □

Extensivity and antiextensivity. If $[\Theta(x)](x) \geq 0$, $\forall x \in \mathbf{E}$, then

$$\mathcal{E}_\Theta(f) \leq f \quad \text{and} \quad \mathcal{D}_\Theta(f) \geq f. \tag{49}$$

Proof.

$$\mathcal{E}_\Theta(f)(x) = \underset{u\in\mathrm{Spt}(\Theta(x))}{\vee}\{f(u) - [\Theta(x)](u)\} \\ \le f(x) - [\Theta(x)](x) \le f(x),$$

where the last inequality follows from the fact that $[\Theta(x)](x) \ge 0$. A similar argument can be used for showing the extensivity of the SVFP dilation. □

Scaling with respect to the spatially-variant structuring function mapping:

Proposition 7. *If* $\Theta_1 \le \Theta_2$, *then*

$$\mathcal{E}_{\Theta_2}(f) \le \mathcal{E}_{\Theta_1}(f) \quad \text{and} \quad \mathcal{D}_{\Theta_1}(f) \le \mathcal{D}_{\Theta_2}(f).$$

Proof. Since $\Theta_1 \le \Theta_2$, we have for all $x \in \mathbf{E}$ and $v \in \mathcal{T}$, $\Theta_1(x) + v \le \Theta_2(x) + v$. Thus, for a given $f \in USC(\mathbf{E})$, we have $\{v : \Theta_2(x) + v \le f\} \subseteq \{v : \Theta_1(x) + v \le f\}$. Hence, $\vee\{v : \Theta_2(x) + v \le f\} \le \vee\{v : \Theta_2(x) + v \le f\}$ or, equivalently, $\mathcal{E}_{\Theta_2}(f) \le \mathcal{E}_{\Theta_1}(f)$.

The increasing property of the SVFP dilation with respect to the structuring function mapping can be derived by using similar arguments and the fact that $\Theta_1 \le \Theta_2 \Leftrightarrow \Theta_1' \le \Theta_2'$. □

Serial composition. Consider two structuring function mappings: Θ_1 and Θ_2. We use $\mathcal{E}_{\Theta_1}(\Theta_2)$ and $\mathcal{D}_{\Theta_1}(\Theta_2)$ to denote the structuring function mapping given by $\mathcal{E}_{\Theta_1}(\Theta_2)(x) = \mathcal{E}_{\Theta_1}(\Theta_2(x))$ and $\mathcal{D}_{\Theta_1}(\Theta_2)(x) = \mathcal{D}_{\Theta_1}(\Theta_2(x))$, $\forall x \in \mathbf{E}$. We have

$$\mathcal{E}_{\Theta_2}[\mathcal{E}_{\Theta_1}(f)] = \mathcal{E}_{\mathcal{D}_{\Theta_1}(\Theta_2)}(f), \tag{50}$$

and

$$\mathcal{D}_{\Theta_2}[\mathcal{D}_{\Theta_1}(f)] = \mathcal{D}_{\mathcal{D}_{\Theta_2}(\Theta_1)}(f). \tag{51}$$

Proof. We have

$$\begin{aligned} &\mathcal{E}_{\Theta_2}[\mathcal{E}_{\Theta_1}(f)](x) \\ =\ & \vee\{v : [\Theta_2(x)](u) + v \le [\mathcal{E}_{\Theta_1}(f)](u)],\ \forall u\} \\ =\ & \vee\{v : [\Theta_2(x)](u) + v \le \wedge_t\{f(t) - [\Theta_1(u)](t)\},\ \forall u\} \\ =\ & \vee\{v : [\Theta_2(x)](u) + v \le f(t) - [\Theta_1(u)](t),\ \forall u,\ \forall t\} \\ =\ & \vee\{v : [\Theta_2(x)](u) + [\Theta_1(u)](t) \le f(t) - v,\ \forall u,\ \forall t\} \\ =\ & \vee\{v : \vee_u\{[\Theta_2(x)](u) + [\Theta_1(u)](t)\} \le f(t) - v,\ \forall t\} \\ =\ & \vee\{v : \mathcal{D}_{\Theta_1}[\Theta_2(x)](t) \le f(t) - v,\ \forall t\} \\ =\ & \mathcal{E}_{\mathcal{D}_{\Theta_1}(\Theta_2)}(f). \end{aligned}$$

A similar argument can be used for obtaining the serial composition of the SVFP dilation. □

A.2 Properties of the SVFP Opening and Closing

From the properties of the SVFP erosion and dilation, it follows that the SVFP opening and closing are increasing dual operators. Moreover, the SVFP opening is anti-extensive, and the SVFP closing is extensive.

Idempotence. For every structuring function mapping Θ, the spatially-variant function-processing morphological opening and closing are idempotent; that is,

$$\Gamma_\Theta^2 = \Gamma_\Theta \quad \text{and} \quad \Phi_\Theta^2 = \Phi_\Theta. \tag{52}$$

Proof.

$$\begin{aligned} \Gamma_\Theta[\Gamma_\Theta(f)] &= \underset{(u,v)}{\vee}\left\{\Theta(u) + v \le [\underset{(a,b)}{\vee}\{\Theta(a) + b \le f]\right\} \\ &= \underset{(u,v)}{\vee}\{\Theta(u) + v \le \Theta(a) + b \le f; (a,b) \in \mathbf{E} \times \mathcal{T}\} \\ &= \underset{(u,v)}{\vee}\{\Theta(u) + v \le f\} \\ &= \Gamma_\Theta(f). \end{aligned}$$

A similar argument can be used for obtaining the idempotence of the SVFP closing. □

It was shown in [1] and [32] that the translation-invariant set-processing opening and closing can be exactly specified from their fixed points. In the following, we provide a characterization of Θ-open and Θ-closed functions, which are the fixed points of the SVFP opening and closing, respectively.

Definition 5. *A function* f *is* Θ *open (respectively,* Θ *closed) if* $\Gamma_\Theta(f) = f$ *(respectively,* $\Phi_\Theta(f) = f$*).*

A useful characterization of Θ-open and Θ-closed functions is given by the following:

Proposition 8. *A function* f *is* Θ *open (respectively,* Θ *closed) if and only if there exists a function* g *such that* $f = \mathcal{D}_\Theta(g)$ *(respectively,* $f = \mathcal{E}_\Theta(g)$*).*

Proof. Assume first that $\Gamma_\Theta(f) = f$. Take $g = \mathcal{E}_\Theta(f)$. Then, we have $f = \mathcal{D}_\Theta(g)$. Assume now that $f = \mathcal{D}_\Theta(g)$ for some function g. By the antiextensivity of the SVFP opening and the increasing property of the SVFP dilation, we have

$$\mathcal{D}_\Theta(g) \ge \Gamma_\Theta(\mathcal{D}_\Theta(g)) = \mathcal{D}_\Theta(\mathcal{E}_\Theta(\mathcal{D}_\Theta(g))) = \mathcal{D}_\Theta(\Phi_\Theta(g)) \\ \ge \mathcal{D}_\Theta(g).$$

Hence, we have $\Gamma_\Theta(\mathcal{D}_\Theta(g)) = \mathcal{D}_\Theta(g)$, which is equivalent to $\Gamma_\Theta(f) = f$. Therefore, we obtain a characterization of Θ-open functions. A similar argument can be used for obtaining a characterization of Θ-closed functions. □

Appendix B
Proof of Propositions

Proof of Proposition 1. Consider $f \in USC(\mathbf{E})$ and $t \in \mathcal{T}$. We have

$$\begin{aligned} \mathcal{X}_t[\mathcal{E}_\Theta(f)] &= \left\{z : \underset{u\in\mathrm{Spt}[\Theta(z)]}{\wedge}\{f(u)\} \ge t\right\} \\ &= \{z : \forall\ u \in \mathrm{Spt}[\Theta(z)],\ f(u) \ge t\} \\ &= \{z : \forall\ u \in \theta(z),\ f(u) \ge t\} \\ &= \{z : \theta(z) \subseteq \mathcal{X}_t(f)\} \\ &= \mathcal{E}_\theta[\mathcal{X}_t(f)]. \end{aligned}$$

If $\mathrm{Spt}(\Theta'(x))$ is compact for all $x \in \mathbf{E}$, then the supremum on the set $\mathrm{Spt}(\Theta'(x))$ is achieved. Then, we have

$$\begin{aligned}\mathcal{X}_t[\mathcal{D}_\Theta(f)] &= \left\{z : \bigvee_{u\in \mathrm{Spt}(f)\cap \mathrm{Spt}[\Theta'(z)]} \{f(u)\} \geq t\right\}\\ &= \{z : \exists\, u \in \mathrm{Spt}(f)\cap \mathrm{Spt}[\Theta'(z)], f(u)\geq t\}\\ &= \{z : \exists\, u \in \mathrm{Spt}(f)\cap \theta'(z), f(u)\geq t\}\\ &= \{z : \theta'(z)\cap \mathcal{X}_t(f)\neq \emptyset\}\\ &= \mathcal{D}_\theta[\mathcal{X}_t(f)].\end{aligned}$$

□

Proof of Proposition 2. Let Ψ be a V-system and consider its umbra processing system ψ_u. Define the umbra SE mapping Θ^U from $\mathbf{E}$ to the set of all umbras on $\mathbf{E}\times\mathcal{T}$ by $\Theta^U(x,y) = U[\Theta(x)+y] = U[\Theta(x)]+y$. ψ_u has an SV kernel $Ker(\psi_u)$, as defined in [11]. We have

$$\begin{aligned}&\Theta^U \in \mathrm{Ker}\ (\psi_u)\\ \iff &(x,y)\in \psi_u(\Theta^U(x,y)),\ \forall (x,y)\in \mathbf{E}\times\mathcal{T}\\ \iff &(x,y)\in \psi_u(U[\Theta(x)+y]),\ \forall (x,y)\in \mathbf{E}\times\mathcal{T}\\ \iff &(x,y)\in U[\Psi(\Theta(x)+y)],\ \forall (x,y)\in \mathbf{E}\times\mathcal{T}\\ \iff &[\Psi(\Theta(x)+y)](x)\geq y,\ \forall (x,y)\in \mathbf{E}\times\mathcal{T}\\ \iff &[\Psi(\Theta(x))](x)+y\geq y,\ \forall (x,y)\in \mathbf{E}\times\mathcal{T}\\ \iff &\Psi[\Theta(x)](x)\geq 0,\ \forall x\in\mathbf{E}\\ \iff &\Theta\in\mathcal{K}(\Psi).\end{aligned}$$

□

Proof of Proposition 3.

1. *Increasing*. Let $f\leq g$. In particular, $f(y)\leq g(y)\ \forall y\in B(x)$ and $\forall x\in\mathbf{E}$. This implies that $\forall x\in\mathbf{E}$, $[\Phi_r(f,B)](x)\leq[\Phi_r(g,B)](x)$. Thus, Φ_r is an increasing system.
2. *Duality*. Let $f\in USC(\mathbf{E})$. For every $x\in\mathbf{E}$, we have

$$\begin{aligned}[\Phi^*(f,B)](x) &= [-\Phi(-f,B)](x)\\ &= -r\text{th largest value of}\\ &\quad\ \{-f(y), y\in B(x)\}\\ &= (n-r+1)\text{th largest value of}\\ &\quad\ \{f(y), y\in B(x)\}\\ &= [\Phi_{n-r+1}(f,B)](x).\end{aligned}$$

   Hence, $\Phi^*(\bullet,B)=\Phi_{n-r+1}(\bullet,B)$.
3. *Commuting with thresholding*. Consider a function $f\in USC(\mathbf{E})$. We have

$$\begin{aligned}&z\in\mathcal{X}_t[\Phi_r(f,B)]\\ \iff &[\Phi_r(f,B)](z)\geq t\\ \iff &r\text{th largest value of}\{f(y): y\in B(z)\}\geq t\\ \iff &|\mathcal{X}_t(f)\cap B(z)|\geq r \iff z\in\phi_r(\mathcal{X}_t(f),B).\end{aligned}$$

   Thus, $\mathcal{X}_t[\Phi_r(f,B)]=\phi_r(\mathcal{X}_t(f),B)$. From (11), we conclude that Φ_r commutes with thresholding. □

Proof of Proposition 4. Assume first that the function $h(\tau)\geq 0$ for all $\tau\in\mathbf{E}$. Consider two functions f and g such that $f\geq g$. Then, $[\Psi(f)-\Psi(g)](t)=\int(f-g)(\tau)[h(\tau)](t)d\tau\geq 0$, since the integrand is a nonnegative function. Hence, the LTV system Ψ is increasing. Assume now that Ψ is increasing, that is, $f\leq g\Rightarrow\Psi(f)\leq\Psi(g)$. Thus, $\int R(\tau)[h(\tau)](t)d\tau\geq 0$ for every nonnegative function R. Let $p_k(x)$ be a sequence of triangular functions such that their width goes to zero, and their height goes to $+\infty$, satisfying $\int p_k(x)dx=1$, for all $k\in\mathbb{N}$. For a given τ, we have $\forall t$

$$\begin{aligned}[h(\tau)](t) &= \int[h(\tau)](z)\delta(z-t)dz=\\ &= \int[h(\tau)](z)\lim_{k\longrightarrow+\infty}p_k(z-t)dz\\ &= \lim_{k\longrightarrow+\infty}\int[h(\tau)](z)p_k(z-t)dz\geq 0.\end{aligned}$$

Thus, the function $h(\tau)$ is nonnegative. □

Proof of Proposition 6. Consider a mapping Θ satisfying $\sum_k[h'(n)](k)[\Theta(n)](k)=0, \forall n\in\mathbb{N}$. We want to show that Θ is a minimal element. Assume that there exists $\Lambda\in\mathcal{K}(\Psi)$ such that $\Lambda<\Theta$. Then, from the fact that $\Lambda\in\mathcal{K}(\Psi)$ and the fact that $[h'(n)](k)\geq 0$ for all n, $k\in\mathbb{N}$, we have $0\leq\sum_k[h'(n)](k)[\Lambda(n)](k)\leq\sum_k[h'(n)](k)[\Theta(n)](k)=0\Longrightarrow\sum_k[h'(n)](k)[\Lambda(n)](k)=0$. From the fact that $\Lambda<\Theta$, there exists n and j such that $[\Lambda(n)](j)<[\Theta(n)](j)$ and, hence, $[h'(n)](j)[\Lambda(n)](j)<[h'(n)](j)[\Theta(n)](j)$. Thus, $\sum_k[h'(n)](k)[\Lambda(n)](k)<\sum_k[h'(n)](k)[\Theta(n)](k)=0$. This is a contradiction. Therefore, Θ is a minimal element.

Consider now a minimal element Θ. We want to show that Θ satisfies $\sum_k[h'(n)](k)[\Theta(n)](k)=0, \forall n\in\mathbb{N}$. Assume that there exists $n_0\in\mathbb{N}$ such that $\sum_k[h'(n_0)](k)[\Theta(n_0)](k)=q>0$. Consider the mapping $\Lambda:\mathbb{Z}^m\to\mathbb{Z}$ defined by

$$\Lambda(n)=\begin{cases}\Theta(n), & \text{if } n\neq n_0;\\ \Theta(n_0)-q, & \text{if } n=n_0.\end{cases}$$

Then, $\Lambda<\Theta$, and $\Lambda\in\mathcal{K}(\Psi)$, since $\sum_k[h'(n_0)](k)[\Lambda(n_0)](k)=0$, and $\sum_k[h'(n)](k)[\Lambda(n)](k)=\sum_k[h'(n)](k)[\Theta(n)](k)\geq 0$ for $n\neq n_0$. This contradicts the minimality of Θ. Hence, Θ satisfies $\sum_k[h'(n)](k)[\Theta(n)](k)=0, \forall n\in\mathbb{N}$. We conclude that the class of minimal elements of the kernel are exactly $\{\Theta:\sum_k[h'(n)](k)[\Theta(n)](k)=0, \forall n\in\mathbb{N}\}$. □

Appendix C
Proof of Theorems

Proof of Theorem 1. First, assume that $\Psi=\vee_{\Theta\in\mathcal{K}(\Psi)}\mathcal{E}_\Theta$. Then, Ψ is an increasing V-system as the supremum of increasing V-systems. Assume now that Ψ is an increasing V-system. Consider $f\in USC(\mathbf{E})$ and $t\in\mathcal{T}$. Let $f'=\vee_{\Theta\in\mathcal{K}(\Psi)}\mathcal{E}_\Theta(f)$. We will show that $[\Psi(f)](x)\geq t\iff f'(x)\geq t$.

Assume first that $[\Psi(f)](x)\geq t$ for some $x\in\mathbf{E}$. Consider the mapping $\Theta_{f,t}$ given by

$$\Theta_{f,t}(x)=\begin{cases}f-t, & \text{if } [\Psi(f)](x)\geq t;\\ \mathcal{I}, & \text{otherwise.}\end{cases}\tag{53}$$

We have $\Theta_{f,t}(x)=f-t$. Moreover,

$$\Psi[\Theta_{f,t}(x)]=\begin{cases}\Psi(f)-t, & \text{if } [\Psi(f)](x)\geq t;\\ \mathcal{I}, & \text{otherwise.}\end{cases}\tag{54}$$

In particular, $\Psi[\Theta_{f,t}(x)](x)\geq 0$. Thus, $\Theta_{f,t}\in\mathcal{K}(\Psi)$. We have $[\mathcal{E}_{\Theta_{f,t}}(f)](x)=\vee\{v\in\mathcal{T}:\Theta_{f,t}(x)+v\leq f\}\geq t$, since $t\in\{v\in\mathcal{T}:\Theta_{f,t}(x)+v\leq f\}$. Hence,

$$f'(x)=[\vee_{\Theta\in\mathcal{K}(\Psi)}\mathcal{E}_\Theta(f)](x)\geq[\mathcal{E}_{\Theta_{f,t}}(f)](x)\geq t.$$

Assume now that $f'(x) \geq t$. We have

$$\begin{aligned} f'(x) \geq t &\iff \bigvee_{\Theta \in \mathcal{K}(\Psi)} \mathcal{E}_\Theta(f)(x) \geq t \\ &\implies \exists \Theta \in \mathcal{K}(\Psi) : \mathcal{E}_\Theta(f)(x) \geq t \\ &\iff \exists \Theta \in \mathcal{K}(\Psi) : \wedge_u \{f(u) - [\Theta(x)](u)\} \geq t \\ &\implies \exists \Theta \in \mathcal{K}(\Psi) : f - \Theta(x) \geq t \\ &\implies \exists \Theta \in \mathcal{K}(\Psi) : \Psi(f) \geq t + \Psi(\Theta(x)). \end{aligned}$$

Since $\Theta \in \mathcal{K}(\Psi)$, we have $\Psi[\Theta(x)](x) \geq 0$. Thus, $[\Psi(f)](x) \geq t + [\Psi(\Theta(x))](x) \geq t$.

Finally, we showed that

$$[\Psi(f)](x) \geq t \iff f'(x) \geq t, \ \forall t \in \mathcal{T} \tag{55}$$

$$\iff \mathcal{X}_t[\Psi(f)] = \mathcal{X}_t[f'], \ \forall t \in \mathcal{T} \tag{56}$$

$$\iff \Psi(f) = f', \tag{57}$$

where the last equivalence follows from the bijection of the threshold sets operators [2], [7]. This establishes the proof that a function-processing system is an increasing V-system if and only if it is the supremum of erosions by mappings in its kernel. The dual representation of Ψ in terms of SVFP dilations is easily obtained by duality. □

Proof of Theorem 2. Let Ψ be an increasing and upper-semicontinuous V-system and let ψ_u be its umbra processing system. From [9, Theorem 3], ψ_u is increasing and upper-semicontinuous. We showed in [11, Theorem 3] that an increasing upper-semicontinuous SP system has a minimal element. Therefore, ψ_u has a minimal element. Let Θ_M^U be a minimal element of ψ_u. Due to the one-to-one correspondence between $ker(\psi_u)$ and $\mathcal{K}(\Psi)$ (see the proof of Proposition 2), there exists a unique $\Theta_M \in \mathcal{K}(\Psi)$ such that $\Theta_M^U(x,y) = U[\Theta_M(x) + y]$, $\forall (x,y) \in \mathbf{E} \times \mathcal{T}$. We claim that Θ_M is a minimal element of $\mathcal{K}(\Psi)$, for otherwise, there exists $\Lambda \in \mathcal{K}(\Psi)$ such that $\Lambda \leq \Theta_M$. Let then $\Lambda^U(x,y) = U[\Lambda(x) + y]$, $\forall (x,y) \in \mathbf{E} \times \mathcal{T}$. From the one-to-one correspondence between $ker(\psi_u)$ and $\mathcal{K}(\Psi)$, we deduce that $\Lambda^U \in Ker(\psi_u)$, and $\Lambda^U \leq \Theta_M^U$. This contradicts the fact that Θ_M^U is a minimal element of $Ker(\psi_u)$. Therefore, we conclude that Θ_M is a minimal element of $\mathcal{K}(\Psi)$. □

Proof of Theorem 3. Let Ψ be an upper-semicontinuous V-system and consider $\Theta_A \in \mathcal{K}(\Psi)$. Then, there exists $\Theta_B \in \mathcal{K}(\Psi)$ such that $\Theta_B \leq \Theta_A$, for otherwise, Θ_A is a minimal element. Therefore, for every $\Theta_A \in \mathcal{K}(\Psi)$, we can construct a decreasing family $\mathcal{L}$ of $\mathcal{K}(\Psi)$ containing Θ_A. From the fact that $\mathcal{L}$ is a totally ordered subset of $\mathcal{K}(\Psi)$ and from Hausdorff's maximality principle [4], there exists a maximal totally ordered subset $\mathcal{M}$ of $\mathcal{K}(\Psi)$ containing $\mathcal{L}$. Let $\Theta_M(x) = (\wedge\mathcal{M})(x) = \wedge_{\Theta \in \mathcal{M}} \Theta(x)$ for every $x \in \mathbf{E}$. From [33, Lemma 4.1], there exists a sequence $\{\Theta_n(x) : n \in \mathbb{N}, \Theta_n \in \mathcal{M}\}$ such that $\Theta_n(x) \downarrow \Theta_M(x)$ for every $x \in \mathbf{E}$. From the fact that Ψ is an upper-semicontinuous system and that $\Theta_n(x)$ is an upper-semicontinuous function for every $x \in \mathbf{E}$, we have

$$\Psi(\Theta_n(x)) \downarrow \Psi(\Theta_M(x)), \ \forall x \in \mathbf{E}.$$

We also have $\Psi(\Theta_n(x)) \downarrow \wedge_n \Psi(\Theta_n(x))$, $\forall x$. By the uniqueness of the limit, we have $[\Psi(\Theta_M(x))](x) = \wedge_n [\Psi(\Theta_n(x))](x)$. Since $\Theta_n \in \mathcal{K}(\Psi)$, we have $[\Psi(\Theta_n(x))](x) \geq 0$, $\forall n \in \mathbb{N}$, $\forall x \in \mathbf{E}$. Hence, $[\Psi(\Theta_M(x))](x) \geq 0$, $\forall x \in \mathbf{E}$. Thus, $\Theta_M \in \mathcal{K}(\Psi)$. We have $\Theta_M = \wedge\mathcal{M} \leq \wedge\mathcal{L} \leq \Theta_A$. We claim that Θ_M is a minimal element of $\mathcal{K}(\Psi)$. Otherwise, there exists $\Lambda \in \mathcal{K}(\Psi)$ such that $\Lambda \leq \Theta_M$. The set $\mathcal{M} \cup \{\Lambda\}$ is then a totally ordered subset of $\mathcal{K}(\Psi)$ containing $\mathcal{M}$. This contradicts the maximality of $\mathcal{M}$. Finally, we have shown that Θ_M is a minimal element of $\mathcal{K}(\Psi)$ and $\Theta_M \leq \Theta_A$. □

Proof of Theorem 4. Let Ψ be a upper-semicontinuous V-system. From Theorem 1, Ψ has a kernel representation as the supremum of SVFP erosions. From the fact that $\boldsymbol{B}_\Psi \subseteq \mathcal{K}(\Psi)$, we have $\vee_{\Theta \in \boldsymbol{B}_\Psi} \mathcal{E}_\Theta \leq \vee_{\Theta \in \mathcal{K}(\Psi)} \mathcal{E}_\Theta$. From Theorem 3, for every $\Theta \in \mathcal{K}(\Psi)$, there exists $\Theta_M \in \boldsymbol{B}_\Psi$ such that $\Theta_M \leq \Theta$. Thus, $\mathcal{E}_\Theta \leq \mathcal{E}_{\Theta_M}$. Therefore, $\vee_{\Theta \in \mathcal{K}(\Psi)} \mathcal{E}_\Theta \leq \vee_{\Theta \in \boldsymbol{B}_\Psi} \mathcal{E}_\Theta$. The result follows by antisymmetry of the partial order $\leq$. □

Appendix D

Proof of Corollaries

Proof of Corollary 1. Let Φ be an FSP system, which commutes with thresholding. From [9, Theorem 3], both Φ and its SP system ϕ are increasing and upper-semicontinuous. From the kernel representation of increasing SVSP systems given in [11], we have $\phi[\mathcal{X}_t(f)] = \mathcal{X}_t[\Phi(f)] = \bigcup_{\theta \in Ker(\phi)} \mathcal{E}_\theta\ (\mathcal{X}_t(f)) = \bigcap_{\theta \in Ker(\phi^*)} \mathcal{D}_{\theta'}(\mathcal{X}_t(f))$. Using Proposition 1 and the fact that the intersection (respectively, union) of threshold sets corresponds to $\wedge$ (respectively, $\vee$) of functions [5, (8) and (9)], we obtain

$$\Phi(f) = \bigvee_{\theta \in \mathrm{Ker}(\phi)} \mathcal{E}_\theta(f) = \bigwedge_{\theta \in \mathrm{Ker}(\phi^*)} \mathcal{D}_{\theta'}(f). \tag{58}$$

□

Proof of Corollary 2. Let $\Phi : USC(\mathbb{Z}^m) \to USC(\mathbb{Z}^m)$ be an increasing and upper-semicontinuous V-system. From Theorem 4, Φ has a basis representation in terms of SVFP erosions. Assume further that its dual system Φ^* is upper-semicontinuous. Φ^* is also increasing. Therefore, applying Theorem 4 to Ψ^* and since $-f$ is also upper-semicontinuous, we obtain $\Phi^*(f) = -\Phi(-f) = \vee_{\Theta \in \boldsymbol{B}_{\Phi^*}} \mathcal{E}_\Theta(f) \Longrightarrow \Phi(f) = \wedge_{\Theta \in \boldsymbol{B}_{\Phi^*}} \mathcal{D}_{\Theta'}(f)$. Thus, Φ also has a basis representation in terms of SVFP dilations. □

Proof of Corollary 3.

a. Let $\Phi : USC(\mathbf{E}) \to USC(\mathbf{E})$ be an SVFSP system that commutes with thresholding. Consider its SVSP system ψ. From [9, Theorem 3], both Φ and ϕ are increasing and upper-semicontinuous. Consider a function f. From the kernel representation of increasing SVSP systems in [11], we have

$$\begin{aligned} \phi[\mathcal{X}_t(f)] = \mathcal{X}_t[\Phi(f)] &= \bigcup_{\theta \in \mathcal{B}_\phi} \mathcal{E}_\theta(\mathcal{X}_t(f)) \\ &= \bigcup_{\theta \in \mathcal{B}(\phi)} \mathcal{X}_t[\mathcal{E}_\theta(f)] \Rightarrow \Phi(f) = \vee_{\theta \in \mathcal{B}_\phi} \mathcal{E}_\theta(f). \end{aligned}$$

b. Φ^* is also an increasing V-system. By assumption, it is also upper-semicontinuous. Applying Corollary 3a to Φ^*, we obtain the desired result. □

ACKNOWLEDGMENTS

N. Bouaynaya was with the Department of Electrical and Computer Engineering, University of Illinois at Chicago when this work was done.

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