Short Papers

M-Idempotent and Self-Dual Morphological Filters

Nidhal Bouaynaya, *Member*, *IEEE*,
Mohammed Charif-Chefchaouni, and
Dan Schonfeld, *Fellow*, *IEEE*

Abstract—In this paper, we present a comprehensive analysis of self-dual and m-idempotent operators. We refer to an operator as m-idempotent if it converges after m iterations. We focus on an important special case of the general theory of lattice morphology: spatially variant morphology, which captures the geometrical interpretation of spatially variant structuring elements. We demonstrate that every increasing self-dual morphological operator can be viewed as a morphological center. Necessary and sufficient conditions for the idempotence of morphological operators are characterized in terms of their kernel representation. We further extend our results to the representation of the kernel of m-idempotent morphological operators. We then rely on the conditions on the kernel representation derived and establish methods for the construction of m-idempotent and self-dual morphological operators. Finally, we illustrate the importance of the self-duality and m-idempotence properties by an application to speckle noise removal in radar images.

Index Terms—Mathematical morphology, spatially-invariant mathematical morphology, duality, idempotence.

1 Introduction

MOST morphological operators occur in pairs of dual operators, such as erosion/dilation and opening/closing. In the binary case, duality refers to processing of the background instead of the foreground of the image. For example, erosion of the background of an image is equivalent to dilation of its foreground. An operator which acts on the foreground and background in the same fashion is called a *self-dual* operator. Examples of self-dual morphological operators are median filters, self-dual connected operators such as levelings [17], [18], and merging-based autodual connected operators [25]. However, self-dual operators are not necessarily morphological filters, i.e., idempotent. In fact, in the case of the median filter, not only is it not idempotent, but it may not converge and thus repeated application of median filtering could enter into a cycle [5]. The importance of idempotence in image analysis has been emphasized by Serra [28]. Classical examples of idempotent operators include ideal (low-pass, high-pass, and band-pass) filters and morphological filters (opening and closing, alternating filters, alternating sequential filters (ASF)).

The quest for self-dual and/or idempotent operators has been the focus of many investigators. We refer to some important work in the area in chronological order. A morphological operator that is idempotent and self-dual has been proposed in [28] by using the notion of the middle element. The middle element, however, can only be obtained through repeated (possibly infinite) iterations. Meyer and Serra [19] established conditions for the idempotence of the class of the contrast mappings. Heijmans [8] proposed a general method for the construction of morphological operators that are self-dual, but not necessarily idempotent. Heijmans and Ronse [10] derived conditions for the idempotence of the self-dual annular operator, in which case it will be called an annular filter. An alternative framework for morphological image processing that gives rise to image operators which are intrinsically self-dual is based upon the definition of a new self-dual partial ordering. Heijmans and Keshet [9] and Mehmet and Jackway [16] proposed alternative self-dual orderings on the gray-scale values of images. Self-dual morphological operators result in a natural way if the underlying partial ordering is self-dual. The price paid for this property is that the underlying algebraic structure of the image space is less rich. One ends up with semilattices rather than with lattices [9]. More discussion on morphology for images with alternative ordering, including geodesic reconstructions, can be found in [21]. Soille revisited the notion of self-duality using the more general concept of flat zones [30]. Recently, tree-based approaches to producing self-dual morphological operators were investigated in [12] and [36].

1.1 Motivation

Self-duality is a desirable property for at least two reasons: The first reason is that self-dual operators do not require a priori delineation of the image in terms of foreground and background. Such operators are well suited to applications where we desire to separate two components, one of which is sometimes lighter and sometimes darker than the other component. An example is the case in which we want to eliminate background noise [28]. Another scenario which requires self-dual operators arises when filtering images where the designation of foreground and background is unclear, e.g., textures, natural scenes, earth observation imagery [31], and radar images [16]. The second reason is that Yli-Harja et al. [37] have shown that self-dual binary filters are statistically unbiased in the sense of median, i.e., the median of the input is also the median of the output in the case of i.i.d. random variables. Similar results have also been presented experimentally by Stevenson and Arce [33], who demonstrated that median filters which are self-dual operators lead to unbiased estimates, whereas filtering using morphological operators that are not self-dual (e.g., opening and closing) results in biased estimates.

In many applications, it is also desirable to impose a stability condition: The operator ψ should be idempotent ($\psi^2 = \psi$) or at least rapidly converging ($\psi^{m+1} = \psi^m$, for some positive integer m) [24]. The idempotence or rapid convergence properties may be of great importance in situations where repeated filtering is undesirable due, for instance, to processing time [24]. In practice, idempotence implies that the transformation is complete and no further processing is required to achieve the aim of the filter [23]. Nonidempotent operators, on the other hand, must be repeated over several iterations and it is generally very difficult to predict how many iterations must be applied before the transformed signal (or image) reaches a desired state. Even worse, repeated iteration of nonidempotent operators often does not reach steady state and such a state may not exist [8].

In this paper, we formalize the notion of rapid convergence by introducing the concept of m-idempotence, namely, we refer to an

- *N. Bouaynaya is with the Department of Systems Engineering, George W. Donaghey College of Engineering and Information Technology, University of Arkansas at Little Rock, 2801 S. University Ave., Little Rock, AR 72204. E-mail: nxbouaynaya@ualr.edu.*
- *M. Charif-Chefchaouni is with the Institut National des Postes et Télécommunications, 2, av Allal El Fasse—Madinat AL Irfane, Rabat, Morocco. E-mail: charifm@inpt.ac.ma.*
- *D. Schonfeld is with the Department of Electrical and Computer Engineering, University of Illinois at Chicago, Room 1020 SEO (M/C 154), 851 South Morgan Street, Chicago, IL 60607-7053. E-mail: dans@uic.edu.*

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operator as m-idempotent provided that it converges after exactly m iterations, where m is a positive finite integer. We develop methods for construction of spatially variant morphological operators that are both self-dual and m-idempotent. SV morphology was first introduced by Serra in [28, Chapters 2 and 4], recently extended by Bouaynaya et al. in [2], [3], [4], and is currently playing a significant role in various signal and image processing applications [34], [35]. Our construction of morphological operators relies on a spatially variant version of a generalized morphological center. We demonstrate that self-dual operators form a subclass of the generalized morphological centers that can be used to establish an isomorphism between increasing self-dual operators and morphological centers. Self-duality and m-idempotence will therefore be characterized by constraints on the spatially variant structuring elements of the kernel of the generalized morphological center. In particular, we will show that Heijman's self-dual morphology based on switch operators [8] is a special case of the proposed self-duality theory. Moreover, we will establish methods that rely on the spatially-variant kernel representation of the generalized morphological center for construction of m-idempotent and self-dual morphological operators.

This paper is organized as follows: In Section 2, we briefly recall the basics of spatially variant morphology [2], [3], [4]. In Section 3, a method for the construction of self-dual morphological operators based on the notion of the morphological center and its kernel representation is presented. In Section 4, we provide necessary and sufficient conditions for the construction of overfilters and underfilters given by their kernel representation. In Section 5, sufficient conditions for the construction of m-idempotent self-dual morphological centers given by their kernel representation are provided. In Section 6, the results of the previous section are extended to obtain sufficient conditions for the construction of m-idempotent (not necessarily self-dual) morphological centers given by their kernel representation. Section 8 presents an application to speckle noise removal in radar images.

2 Preliminaries

In this paper, we consider the continuous or discrete euclidean space $\mathbf{E} = \mathbb{R}^n$ or $\mathbb{Z}^n$ for some $n > 0$. A binary image can be represented as a subset of $\mathbf{E}$. The set $\mathcal{P}(\mathbf{E})$ denotes the set of all subsets of $\mathbf{E}$. Elements of the set $\mathbf{E}$ will be denoted by lower case letters, e.g., a, b, c. Elements of the set $\mathcal{P}(\mathbf{E})$ will be denoted by upper case letters, e.g., A, B, C. An order on $\mathcal{P}(\mathbf{E})$ is imposed by the inclusion $\subseteq$. We use $\cup$ and $\cap$ to denote the union and intersection in $\mathcal{P}(\mathbf{E})$, respectively. "$\Rightarrow, \Leftrightarrow, \forall, \exists$" denote, respectively, "implies," "if and only if (iff)," "for all," and "there exist(s)." X^c denotes the complement of X, and $\check{X} = -X = \{-x : x \in X\}$ denotes the reflected set of X. The translate of the set X by the element $a \in \mathbf{E}$ is defined by $X + a = \{x + a : x \in X\}$. We use $\mathcal{O} = \mathcal{P}(\mathbf{E})^{\mathcal{P}(\mathbf{E})}$ to denote the set of all operators mapping $\mathcal{P}(\mathbf{E})$ into itself. Elements of the set $\mathcal{O}$ will be denoted by lower case Greek letters, e.g., α, β, γ. Id will denote the identity operator of $\mathcal{P}(\mathbf{E})$ into itself. An order on $\mathcal{O}$ is imposed by the inclusion $\subseteq$, i.e., $\alpha \subseteq \beta$ if and only if $\alpha(X) \subseteq \beta(X)$ for every $X \in \mathcal{P}(\mathbf{E})$. The symbol "$\backslash$" will denote, at the same time, the difference between sets and the induced difference between operators in $\mathcal{O}$. We shall restrict our attention to nondegenerate operators, i.e., $\psi(\mathbf{E}) = \mathbf{E}$ and $\psi(\emptyset) = \emptyset$ for every $\psi \in \mathcal{O}$.

The mapping ψ^* in $\mathcal{O}$ is the dual of the mapping ψ in $\mathcal{O}$ iff $\psi^*(X) = (\psi(X^c))^c$, for all $X \in \mathcal{P}(\mathbf{E})$. A *self-dual* operator is an operator such that $\psi^* = \psi$.

Throughout the paper, we provide references to known results and limit the presentation of proofs to new contributions. All proofs are presented in the supplemental material, which can be found in the Computer Society Digital Library at http://doi.ieeecomputersociety.org/10.1109/TPAMI.2011.244.

2.1 Spatially Variant Mathematical Morphology

In the SVMM framework, the structuring element is not fixed but varies (in size, shape, and other characteristics) in space. The spatially variant structuring element θ is given by a mapping from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ such that to every $z \in \mathbf{E}$ we can associate a "local" structuring element $\theta(z)$. The transposed spatially variant structuring element θ' is given by a mapping from $\mathbf{E}$ into $\mathcal{P}(\mathbf{E})$ such that

$$\theta'(y) = \{z \in \mathbf{E} : y \in \theta(z)\} \qquad (y \in \mathbf{E}). \tag{1}$$

The SV erosion and dilation are defined as follows [1], [3], [22], [28]:

Spatially variant erosion: The spatially-variant erosion $\mathcal{E}_\theta \in \mathcal{O}$ is defined as

$$\mathcal{E}_\theta(X) = \{z \in \mathbf{E} : \theta(z) \subseteq X\} = \bigcap_{x \in X^c} \theta'^c(x)(X \in \mathcal{P}(\mathbf{E})). \tag{2}$$

Spatially variant dilation: The spatially variant dilation $\mathcal{D}_\theta \in \mathcal{O}$ is defined as

$$\mathcal{D}_\theta(X) = \{z \in \mathbf{E} : \theta'(z) \cap X \neq \emptyset\} = \bigcup_{x \in X} \theta(x)(X \in \mathcal{P}(\mathbf{E})). \tag{3}$$

A kernel representation theorem [6] that extends Matheron's representation theorem to increasing (but not necessarily translation invariant) operators can also be proven [3].

Theorem 1 [3]. *A nondegenerate operator $\alpha \in \mathcal{O}$ is increasing if and only if α can be exactly represented as union of spatially variant erosions by mappings in its kernel or equivalently as intersection of spatially variant dilations by the transposed mappings in the kernel of its dual α^*, i.e.,*

$$\alpha(X) = \bigcup_{\theta \in \,Ker\,(\alpha)} \mathcal{E}_\theta(X) = \bigcap_{\theta \in \,Ker\,(\alpha^*)} \mathcal{D}_{\theta'}(X), \tag{4}$$

for every $X \in \mathcal{P}(\mathbf{E})$, where

$$Ker(\alpha) = \{\theta : z \in \alpha(\theta(z)),\ for\ every\ z \in \mathbf{E}\}. \tag{5}$$

The SV kernel representation, given in (5), is redundant in the sense that a smaller subset of the kernel is sufficient for the representation of increasing operators [3]. In this paper, we say that an increasing operator α is *generated by* a family of spatially variant structuring elements $\{\theta_i\}_{i \in I}$ if

$$\alpha = \bigcup_{i \in I} \mathcal{E}_{\theta_i}. \tag{6}$$

Observe that the family $\{\theta_i\}_{i \in I}$ is a subset of the kernel of α such that for every $\theta \in Ker(\alpha)$, there exists $i \in I$ such that $\theta_i \subseteq \theta$. Furthermore, we assume that $\theta_j \subseteq \theta_i \Rightarrow \theta_j = \theta_i$. The family $\{\theta_i\}_{i \in I}$ is also called *a basis* of the kernel of α [13], [3]. The requirement that the kernel of α has a basis, i.e., that every member of the kernel is bounded by a maximal element, requires some continuity condition on α [15], [3]. It is verified if α is locally finitary, i.e., $p \in \alpha(X) \Rightarrow p \in \alpha(Y)$, for a finite subset Y of X.

3 Self-Duality

We assume throughout this section that all operators in $\mathcal{O}$ are increasing. In particular, extension of increasing set operators into flat operators follows from [11], [14], [20], [27], [28], [32]. Following Serra's work on self-dual filtering [28], we shall enlarge the concept of the self-dual mapping to that of the central mapping, or center, which does not require the existence of complementation. The morphological center, ρ, of two operators $\alpha_1 \subseteq \alpha_2$ is defined by Serra [28]:

$$\rho = (Id \cap \alpha_2) \cup \alpha_1 = (Id \cup \alpha_1) \cap \alpha_2. \tag{7}$$

More generally, let $\{\alpha_i\}_{i\in I}$ be an arbitrary family of operators in $\mathcal{O}$. The morphological center ρ with respect to this family is defined as [28]

$$\rho = \left(Id \cap \bigcup_{i\in I}\alpha_i\right) \cup \left(\bigcap_{i\in I}\alpha_i\right). \tag{8}$$

We will show that every increasing self-dual morphological operator can be viewed as a morphological center. The following theorem provides a necessary and sufficient condition for the self-duality of the morphological center.

Theorem 2. *Consider the morphological center ρ of two operators $\alpha_1 \subseteq \alpha_2$. Then, ρ is self-dual if and only if*

$$Id \cap \alpha_1^* \subseteq \rho \subseteq Id \cup \alpha_2^*, \tag{9}$$

$$\textit{and} \tag{10}$$

$$\alpha_2^* \subseteq \rho \subseteq \alpha_1^*. \tag{11}$$

From Theorem 2, we derive two corollaries which will be used for the construction of self-dual increasing operators.

Corollary 1. *The operator ρ is self-dual if and only if*

$$Id \cap \alpha_1^* \subseteq \alpha_2 \;\; \textit{and} \;\; \alpha_1 \subseteq Id \cup \alpha_2^* \tag{12}$$

$$\textit{and} \tag{13}$$

$$\alpha_2^* \subseteq \rho \subseteq \alpha_1^*. \tag{14}$$

Lemma 1. *If $\alpha_2 = \alpha_1^*$, then the morphological center ρ is self-dual.*

The following theorem provides a method for the construction of increasing self-dual operators based on the result of Lemma 1. We assume that α_1 is generated by the family $\{\theta_i\}_{i\in I}$, i.e.,

$$\alpha_1(X) = \bigcup_{i\in I}\mathcal{E}_{\theta_i}(X). \tag{15}$$

Before stating the main theorem, we prove a lemma giving a necessary and sufficient condition on the kernel of α_1 such that $\alpha_1 \subseteq \alpha_1^*$.

Lemma 2. *$\alpha_1 \subseteq \alpha_1^*$ if and only if $\theta_i(z) \cap \theta_j(z) \neq \emptyset$, for every $z \in \mathbf{E}$ and for every $i, j \in I$.*

Theorem 3 (Self-duality of the SV morphological center). *If the operator α_1, given by $\alpha_1 = \bigcup_{i\in I}\mathcal{E}_{\theta_i}$, is such that $\theta_i(z) \cap \theta_j(z) \neq \emptyset$ for all $i, j \in I$ and all $z \in \mathbf{E}$, then the morphological center $\rho = (Id \cap \alpha_1^*) \cup \alpha_1$ is self-dual.*

In the translation-invariant case, Theorem 3 reduces to the following corollary.

Corollary 2 (Self-duality of the TI morphological center). *If the operator α_1, given by $\alpha_1(X) = \bigcup_{i\in I}(X \ominus A_i)$, for every $X \in \mathcal{P}(\mathbf{E})$, is such that $A_i \cap A_j \neq \emptyset$ for every $i, j \in I$, then the morphological center $\rho = (Id \cap \alpha_1^*) \cup \alpha_1$ is self-dual.*

The following example illustrates the self-duality of the morphological center in the translation-invariant case.

Example 1. Let A, B be two structuring elements given, respectively, by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{16}$$

where the origin is taken at the center of the matrices. From Corollary 2, it can be seen that the morphological center ρ given by

$$\rho(X) = X \cap \big((X \oplus \check{A}) \cap (X \oplus B)\big) \cup \big((X \ominus A) \cup (X \ominus B)\big), \tag{17}$$

for every $X \in \mathcal{P}(\mathbf{E})$, is a self-dual operator.

We end this section with a theorem showing that the class of all morphological centers of α_1 and α_1^* satisfying $\alpha_1 \subseteq \alpha_1^*$ contains all self-dual operators. The following lemma will be useful for the proof of the theorem.

Lemma 3. *Let $\alpha \in \mathcal{O}$. For any $\alpha_1 \subseteq \alpha_2$, we have $\alpha = (Id \cap \alpha_2) \cup \alpha_1$ if and only if $\alpha \cap Id^c \subseteq \alpha_1 \subseteq \alpha$ and $\alpha \subseteq \alpha_2 \subseteq \alpha \cup Id^c$.*

Theorem 4 (Self-duality of SV operators). *The operator $\alpha \in \mathcal{O}$ is self-dual if and only if there exists $\alpha_1 \in \mathcal{O}$ such that $\alpha_1 \subseteq \alpha_1^*$ and $\alpha = (Id \cap \alpha_1^*) \cup \alpha_1 = (Id \cup \alpha_1) \cap \alpha_1^*$.*

Theorem 4 states that every self-dual operator α can be viewed as the morphological center of the operator $\alpha_1 = \alpha \cap Id^c$ and its dual $\alpha_1^* = \alpha \cup Id^c$. Note that for a given α, there can be several α_1, with $\alpha = (Id \cup \alpha_1) \cap \alpha_1^*$. Clearly, among such α_1, α is the greatest one and $\alpha \cap Id^c$ is the least one. One should also notice that as long as α is not identically equal to Id or Id^c, the operator α_1 is nontrivial. In particular, for $\alpha = Id$ (respectively, Id^c), one gets $\alpha_1 = \emptyset$ (respectively, Id^c). This provides a procedure for the construction of arbitrary self-dual operators. By considering all the operators α_1 such that $\alpha_1 \subseteq \alpha_1^*$, we are sure to cover all the self-dual operators by considering all the centers of the form $(Id \cap \alpha_1^*) \cup \alpha_1$.

Now, the algorithm for the construction of self-dual operators is straightforward.

Algorithm for the construction of self-dual operators:

1) Consider a family of mappings $\{\theta_i\}_{i\in I}$ such that $\theta_i(z) \cap \theta_j(z) \neq \emptyset, \forall i, j \in I, \forall z \in \mathbf{E}$.
2) Construct the operator $\alpha_1 = \cup_{i\in I}\mathcal{E}_{\theta_i}$
3) The operator $\alpha = (Id \cap \alpha_1^*) \cup \alpha_1$ is self-dual.

Moreover, every increasing self-dual operator can be constructed using the above algorithm.

We now show that Heijmans' method of the construction of self-dual operators in [8] is a special case of the proposed self-duality theory, corresponding to the choice of $\alpha_1 = \alpha \cap Id^c$.

Example 2 (The switching operator [8]). A switch operator σ associated with a self-dual operator α is defined by

$$\sigma(X) = X \cap \alpha(X^c), (X \in \mathcal{P}(\mathbf{E})). \tag{18}$$

The term "switch operator" has been used by Heijmans to denote two properties of σ, that are satisfied if and only if the self-dual operator α is increasing. The self-dual operator α can be reconstructed from σ by means of the following formula [8]:

$$\alpha(X) = (X \cap \sigma(X)^c) \cup \sigma(X^c), (X \in \mathcal{P}(\mathbf{E})). \tag{19}$$

By letting $\alpha_1(X) = (\alpha \cap Id^c)(X) = \sigma(X^c)$, it is easy to see that we have

$$\begin{aligned} \alpha(X) &= (X \cap \sigma(X)^c) \cup \sigma(X^c) \\ &= (X \cap \alpha_1^*(X)) \cup \alpha_1(X). \end{aligned} \tag{20}$$

Therefore, Heijman's construction of self-dual operators is a special case of the proposed self-duality theory, corresponding to the choice of $\alpha_1 = \alpha \cap Id^c$ in Theorem 4.

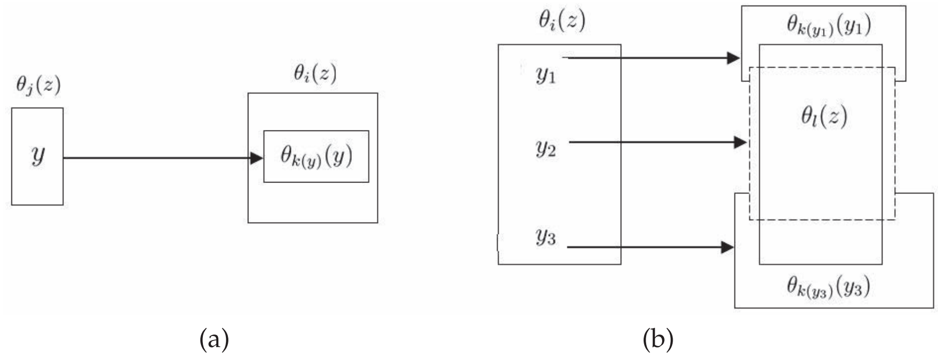


Fig. 1. Kernel structure for (a) an overfilter and (b) an underfilter.

4 Overfilters and Underfilters

Definition 1 [28]. *An increasing operator α is called an* overfilter *(respectively,* underfilter*) if $\alpha \subseteq \alpha^2$ (respextively, $\alpha^2 \subseteq \alpha$). An increasing operator α is an* inf-overfilter *(respestively,* sup-underfilter*) if $\alpha(Id \cap \alpha) = \alpha$ (respectively, $\alpha(Id \cup \alpha) = \alpha$).*

Observe that an operator that is simultaneously an overfilter and an underfilter is an idempotent operator.

4.1 Overfilters

In the following, we derive a theorem which provides a necessary and sufficient condition on the elements of the kernel of $\alpha = \bigcup_{i \in I} \mathcal{E}_{\theta_i}$ in order for α to be an overfilter.

Lemma 4. *Consider $\alpha \in \mathcal{O}$. We have $\alpha \subseteq \alpha^2$ if and only if for every $z \in \mathbf{E}$ and for every $i \in I$ there exists $j \in I$ such that $\theta_j(z) \subseteq \alpha(\theta_i(z))$.*

Theorem 5 (SV overfilters). *The operator $\alpha \in \mathcal{O}$ is an overfilter if and only if for every $z \in \mathbf{E}$ and for every $i \in I$ there exists $j \in I$ such that for every $y \in \theta_j(z)$ there exists $k(y) \in I$ such that $\theta_{k(y)}(y) \subseteq \theta_i(z)$, where k is a mapping from $\mathbf{E}$ to I.*

Fig. 1a provides a simple summary of Theorem 5. For every SVSE i of z, we can always find a SVSE j of z such that for every element y in the jth SVSE, it is possible to come back to the ith SVSE of z by using some other SVSE k of y. This condition generalizes the concept of triple adjacency introduced by Heijmans [7].

The following corollary is a direct consequence of Theorem 5 for the translation-invariant case.

Corollary 3 (TI overfilters). *The translation-invariant operator α is an overfilter if and only if for every $i \in I$ there exists $j \in I$ such that for every $y \in A_j$ there exists $k(y) \in I$ such that $(A_{k(y)})_y \subseteq A_i$.*

In the following, we provide some examples showing how one can build overfilters in the increasing and translation-invariant case by using Corollary 3.

Example 3 (Translation-invariant morphological opening). Let γ be the translation-invariant opening by a structuring element B. It is known that the basis of the kernel of γ is given by $\{B_{-z}\}_{z \in B}$ [28]. For every $A_i = B_{-z}$, it is possible to choose $A_j = B_{-z}$ satisfying Corollary 3. This proves the known fact that the translation-invariant opening is idempotent and in particular an overfilter.

Example 4. Let $A, \{A_{-z}\}_{z \in B}, B, \{B_{-z}\}_{z \in B}$ be the elements of the kernel of some operator α. It is straightforward to see that A satisfies the condition of Corollary 3. The structuring element A_{-z} also satisfies this condition because, for every $t \in B_{-z}$, we have $A_{-z-t} = A_{-(z+t)} = A_b$ with $b \in B$. Also, from Example 1, $B, \{B_{-z}\}_{z \in B}$ satisfy the condition. Therefore, from Corollary 3, the operator α given by

$$\alpha(X) = (X \ominus A) \cup (X \ominus B) \cup \gamma_B(X) \cup (X \ominus A \oplus B),$$

for every $X \in \mathcal{P}(\mathbf{E})$, is an overfilter.

The following proposition gives the structure of the kernel of an inf-overfilter.

Proposition 1 (SV inf-overfilters). *The operator α is an inf-overfilter if and only if for every $z \in \mathbf{E}$, for every $i \in I$ there exists $j \in I$ such that $\theta_j(z) \subseteq \theta_i(z) \cap \alpha(\theta_i(z))$.*

Moreover, if we assume that there is no inclusion between the elements of the kernel generating α, i.e., $\theta_j \subseteq \theta_i \Rightarrow \theta_j = \theta_i$, then α is an inf-overfilter if and only if for every $i \in I$ for every $z \in \mathbf{E}$ we have $\theta_i(z) \subseteq \alpha(\theta_i(z))$.

In the translation invariant case, Proposition 1 reduces to the following corollary:

Corollary 4. *The operator α is an inf-overfilter if and only if for every $i \in I$ we have $A_i \subseteq \alpha(A_i)$.*

From Proposition 1 and Corollary 4, we verify the known fact that an inf-overfilter is an overfilter.

4.2 Underfilters

We now provide a necessary and sufficient condition on the elements of the kernel of α in order for α to be an underfilter.

Theorem 6. *The operator α is an underfilter if and only if for every $z \in \mathbf{E}$ for every $i \in I$ and for every mapping $k : \theta_i(z) \mapsto I$ there exists $l \in I$ such that*

$$\theta_l(z) \subseteq \bigcup_{y \in \theta_i(z)} \theta_{k(y)}(y). \tag{21}$$

Fig. 1b shows the mechanism involved in Theorem 6. To every element y of every SVSE i of z we can associate any SVSE k of y such that the union of all the associated SVSEs covers some SVSE l of z.

The following corollary provides a necessary and sufficient condition on the elements of the kernel of α in order for α to be an underfilter in the increasing and translation-invariant case.

Corollary 5. *The operator $\alpha \in \mathcal{O}$ is an underfilter if and only if for every $i \in I$ and for every mapping $k : A_i \mapsto I$ there exists $l \in I$ such that*

$$A_l \subseteq \bigcup_{y \in A_i} (A_{k(y)})_y. \tag{22}$$

In general, the construction of underfilters based on (21) (respectively, (22)) is not easy. A simple case is when α is antiextensive. In this case, we have for every $i \in I$ and for every

$z \in \mathbf{E}$, $z \in \theta_i(z)$ (respectively, $0 \in A_i$, for every $i \in I$) and therefore Theorem 6 (respectively, Corollary 5) is satisfied.

The following proposition gives the structure of the kernel of a sup-underfilter in the translation-invariant case.

Proposition 2. *The operator $\alpha \in \mathcal{O}$ is a sup-underfilter if and only if for every $i \in I$ for every $B_i \subseteq A_i$ and for every mapping k: $A_i \setminus B_i \mapsto I$ there exists $j \in I$ such that $A_j \subseteq B_i \cup \bigcup_{y \in A_i \setminus B_i} (A_{k(y)})_y$.*

From Proposition 2 and Corollary 5, we verify the known fact that a sup-underfilter is an underfilter.

5 SELF-DUALITY AND IDEMPOTENCE

5.1 Idempotence

In this section, we provide sufficient conditions on the elements of the kernel of α for the idempotence of the self-dual morphological center ρ of α and α^*. This result is subsequently extended to provide elements of the kernel of α in order to have $\rho^{m+1} = \rho^m$, for any positive integer m.

We assume throughout this section that α is given by $\alpha(X) = \bigcup_{i \in I} \mathcal{E}_{\theta_i}(X)$, for every $X \in \mathcal{P}(\mathbf{E})$, if α is increasing and by $\alpha(X) = \bigcup_{i \in I} (X \ominus A_i)$, for every $X \in \mathcal{P}(\mathbf{E})$, if α is increasing and translation invariant.

The mappings $\{\theta_i\}_{i \in I}$ (respectively, $\{A_i\}_{i \in I}$) are assumed to be *symmetric*, i.e., $\theta_i = \theta_i'$ (respectively, $A_i = \check{A}_i$) for all $i \in I$. The symmetry condition of the SV structuring element θ can be expressed as follows: For every $y, z \in \mathbf{E}$ and for every $i \in I$, we have $y \in \theta_i(z) \Longrightarrow z \in \theta_i(y)$. We also assume that for every $i, j \in I$ and for every $z \in \mathbf{E}$, $\theta_i(z) \cap \theta_j(z) \neq \emptyset$ (respectively, $A_i \cap A_j \neq \emptyset$). From Theorem 3 (respectively, Corollary 2), it can be concluded that the morphological center ρ of α and α^* is self-dual. In order to determine conditions on the mappings $\{\theta_i\}_{i \in I}$ (respectively, $\{A_i\}_{i \in I}$) for the morphological center ρ to be idempotent, we first start by studying the kernel of ρ. Its structure is given by the following proposition.

Proposition 3. *The kernel of ρ is generated by the following mapping $\theta : \mathbf{E} \mapsto \mathcal{P}(\mathbf{E})$ (respectively, structuring elements B) given by*

$$\theta(z) = \{z\} \cup \bigcup_{\substack{y_i \in \theta_i(z) \\ i \in I}} \{y_i\} \quad or \quad \theta(z) = \theta_i(z), \quad for \quad i \in I \tag{23}$$

for every $z \in \mathbf{E}$ (respectively, $B = \{0\} \cup \bigcup_{i \in I} \{a_i, a_i \in A_i\}$ or $B = A_i$ for $i \in I$).

The following theorem provides sufficient conditions on the elements of the kernel of α in order for the self-dual morphological center ρ to be idempotent.

Theorem 7 (Idempotence of self-dual SV centers). *The self-dual morphological center ρ of α and α^* is idempotent if for every $z \in \mathbf{E}$, we have*

1. *For every $i, j \in I$ and for every $x_i \in \theta_i(z) \setminus \theta_j(z)$ and $y_j \in \theta_j(z) \setminus \theta_i(z)$, $x_i \in \theta_i(y_j)$ and $y_j \in \theta_j(x_i)$; and*
2. *For every $i \in I$ there exists $x \in \bigcap_{k \in I} \theta_k(z)$ such that for every $j \in I$ there exists $y_j \in \theta_j(x)$ such that $z \in \theta_i(y_j)$.*

For clarity, we will translate the conditions provided in the previous Theorem into the concept of graph morphology. Conditions 1 and 2 are represented by Fig. 2.

The next corollary provides sufficient conditions for the self-dual morphological center to be idempotent in the increasing and translation-invariant case.

Corollary 6 (Idempotence of self-dual TI centers). *The self-dual morphological center ρ of α and α^* is idempotent if we have*

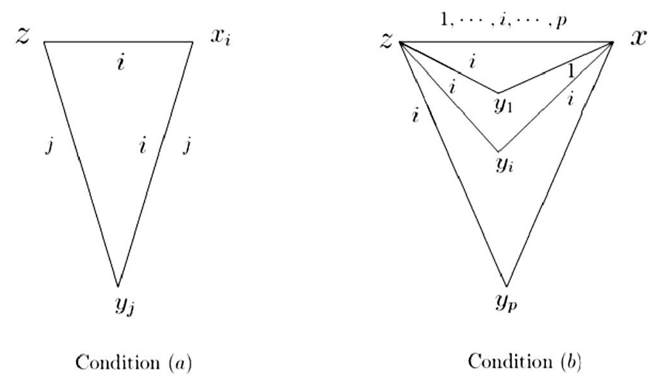


Fig. 2. Sufficient conditions for the idempotence of self-dual morphological centers.

1. *For every $i, j \in I$ and for every $x_i \in A_i \setminus A_j$ and $y_j \in A_j \setminus A_i$, $x_i - y_j \in A_i \cap A_j$; and*
2. *For every $i \in I$ there exists $x \in \bigcap_{k \in I} A_k$ such that for every $j \in I$ we have $0 \in A_i \oplus (A_j)_x$.*

The next example presents a special case of Corollary 6 with one structuring element, that is, $\alpha(X) = X \ominus A$, $X \in \mathcal{P}(\mathbf{E})$, is the erosion by the symmetric structuring element A.

Example 5 (The self-dual center of the TI erosion [10]). Let A be a symmetric structuring element. Then, the morphological center ρ given by

$$\rho(X) = (X \cap (X \oplus A)) \cup (X \ominus A), \tag{24}$$

for every $X \in \mathcal{P}(\mathbf{E})$, is idempotent if $0 \in A \oplus A \oplus A$.

For instance, let A be given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{25}$$

It is clear that the morphological center ρ is self-dual and idempotent.

The next example illustrates Corollary 6 in the case of two structuring elements.

Example 6. The morphological center ρ given by

$$\begin{aligned} \rho(X) = (X \cap ((X \oplus A) \cap (X \oplus B))) \\ \cup ((X \ominus A) \cup (X \ominus B)), \end{aligned} \tag{26}$$

for every $X \in \mathcal{P}(\mathbf{E})$, with A and B symmetric and $A \cap B \neq \emptyset$, is idempotent if

1. For every $x \in A \setminus B$ and $y \in B \setminus A$, $x - y \in A \cap B$; and
2. There exists $x, y \in A \cap B$ such that $0 \in A_x \oplus A$, $0 \in B_x \oplus A$, $0 \in A_y \oplus B$, and $0 \in B_y \oplus B$.

For instance, let A and B be given by

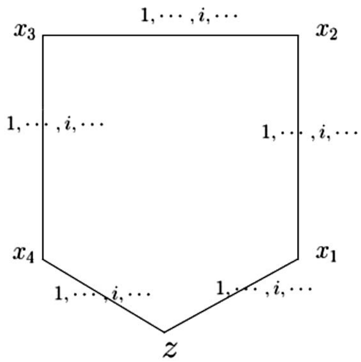
$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \tag{27}$$

The structuring elements A and B satisfy the above conditions and the morphological center ρ is self-dual and idempotent.

5.2 M-Idempotence

The following theorem provides sufficient conditions under which the self-dual morphological center ρ is m-idempotent, i.e., the operator ρ is such that $\rho^{m+1} = \rho^m$, for some integer $m \geq 1$.

Theorem 8. *Consider an integer $m \geq 1$. The self-dual morphological center ρ of α and α^* is m-idempotent if for every $z \in \mathbf{E}$ we have*



Condition (2)

Fig. 3. Self-dual center: 5-idempotence.

1. *For every* $i, j \in I$ *and for every* $x_i \in \theta_i(z) \setminus \theta_j(z)$ *and* $y_j \in \theta_j(z) \setminus \theta_i(z)$, $x_i \in \theta_i(y_j)$ *and* $y_j \in \theta_j(x_i)$; *and*
2. *There exists* $(x_1, x_2, \ldots, x_{2m})$ *such that* $x_1 \in \bigcap_{j\in I}\theta_j(z), x_{i+1} \in \bigcap_{j\in I}\theta_j(x_i)$, *for* $i = 2, \ldots, 2m-1$ *and* $z \in \bigcap_{j\in I}\theta_j(x_{2m})$.

Condition 2 can be simply represented by Fig. 3. This condition generalizes the notion of triple adjacency introduced by Heijmans [7].

We shall now state the corresponding corollary for the increasing and translation-invariant case.

Corollary 7. *The self-dual morphological center* ρ *of the translation-invariant operators* α *and* α^* *is* m*-idempotent for some integer* $m \geq 1$ *if we have*

1. *For every* $(i, j) \in I \times I$ *and for every* $x_i \in A_i \setminus A_j$ *and* $y_j \in A_j \setminus A_i$, $x_i - y_j \in A_i \cap A_j$; *and*
2. $0 \in \oplus_{j=1,\ldots,2m+1}(\cap_{i\in I} A_i)$.

We end this section by providing a special case of Corollary 7 corresponding to one structuring element.

Example 7 (The self-dual center of the TI erosion). Consider a symmetric structuring element A. The morphological center ρ given by

$$\rho(X) = X \cap (X \oplus A) \cup (X \ominus A), \tag{28}$$

for every $X \in \mathcal{P}(\mathbf{E})$, is m-idempotent for some integer $m \geq 1$ if $0 \in \oplus_{i=1,\ldots,2m+1} A$.

For instance, let A be given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

It is easy to verify that $0 \notin A \oplus A \oplus A$ and $0 \in A \oplus A \oplus A \oplus A \oplus A$. Therefore, A satisfies the above condition and the self-dual morphological center ρ is 2-idempotent, i.e., $\rho^3 = \rho$.

6 Generalized Morphological Center

In this section, we extend the results obtained in the previous section to the case of the morphological center ρ given by

$$\rho(X) = \left(X \cup \bigcup_{i\in I} \mathcal{E}_{\alpha_i}(X) \right) \cap \bigcap_{j\in J} \mathcal{D}_{\beta_j}(X), \tag{30}$$

for every $X \in \mathcal{P}(\mathbf{E})$, for the increasing case, and by

$$\rho(X) = \left(X \cup \bigcup_{i\in I} (X \ominus A_i) \right) \cap \bigcap_{j\in J} (X \oplus B_j), \tag{31}$$

for every $X \in \mathcal{P}(\mathbf{E})$, for the increasing and translation-invariant case.

In order to ensure that ρ is a morphological center, we assume that for every $i \in I$, for every $j \in J$, and for every $z \in \mathbf{E}$, we have $\alpha_i(z) \cap \beta_j(z) \neq \emptyset$ (respectively, for every $i \in I$ and for every $j \in J$, we have $A_i \cap B_j \neq \emptyset$). Finally, we assume that the relations (respectively, structuring elements) corresponding to α_i (respectively, A_i) and β_j (respectively, B_j) are symmetric, for every $i \in I$ and $j \in J$. In this case, the dual ρ^* of ρ is given by

$$\rho^*(X) = \left(X \cap \bigcap_{i\in I} \mathcal{D}_{\alpha_i}(X) \right) \cup \bigcup_{j\in J} \mathcal{E}_{\beta_j}(X), \tag{32}$$

for every $X \in \mathcal{P}(\mathbf{E})$, for the increasing case, and by

$$\rho^*(X) = \left(X \cap \bigcap_{i\in I} (X \oplus A_i) \right) \cup \bigcup_{j\in J} (X \ominus B_j), \tag{33}$$

for every $X \in \mathcal{P}(\mathbf{E})$, for the increasing and translation-invariant case. Notice the symmetry in the structure of ρ and its dual. This symmetry implies that any result that is valid for ρ will be valid for its dual by simply exchanging the roles of α_i (respectively, A_i) and β_j (respectively, B_j).

In this section, we shall provide sufficient conditions on α_i (respectively, A_i) and β_j (respectively, B_j) in order for the morphological center ρ to be idempotent and, more generally, m-idempotent, for some positive integer m.

6.1 Idempotence

As in the previous section, we first begin by studying the structure of the kernel of ρ. This structure is given by the following proposition:

Proposition 4. *The kernel of* ρ *is generated by the mappings* θ *from* $\mathbf{E}$ *into* $\mathcal{P}(\mathbf{E})$ *given by* $\theta(z) = \{z\} \cup \bigcup_{j\in J}\{y_j : y_j \in \beta_j(z)\}$ *or* $\theta(z) = \alpha_i(z)$ *for* $i \in I$, *for every* $z \in \mathbf{E}$.

A similar result can be obtained for the dual ρ^* of ρ by exchanging the roles of α_i and β_j.

The following theorem provides sufficient conditions for the morphological center ρ to be idempotent.

Theorem 9. *The operator* ρ *is idempotent if, for every* $z \in \mathbf{E}$, *we have*

1. *For every* $k, l \in J$ *and for every* $x_k \in \beta_k(z) \setminus \beta_l(z)$ *and* $y_l \in \beta_l(z) \setminus \beta_k(z)$, $x_k \in \beta_k(y_l)$ *and* $y_l \in \beta_l(x_k)$;
2. *For every* $i \in I$, *there exists* $x \in \alpha_i(z) \cap \bigcap_{j\in J} \beta_j(z)$ *such that for every* $j \in J$ *there exists* $y_j \in \beta_j(x)$ *such that* $z \in \alpha_i(y_j)$;
3. *For every* $k, l \in I$ *and for every* $u_k \in \alpha_k(z) \setminus \alpha_l(z)$ *and* $v_l \in \alpha_l(z) \setminus \alpha_k(z)$, $u_k \in \alpha_k(v_l)$ *and* $v_l \in \alpha_l(u_k)$;
4. *For every* $j \in J$ *there exists* $y \in \beta_j(z) \cap \bigcap_{i\in I} \alpha_i(z)$ *such that for every* $i \in I$ *there exists* $y_i \in \alpha_i(y)$ *such that* $z \in \alpha_j(y_i)$.

We now state the corresponding corollary for the case of an increasing and translation-invariant morphological center.

Corollary 8. *The operator* ρ *is idempotent if, for every* $z \in \mathbf{E}$, *we have*

1. *For every* $k, l \in J$ *and for every* $x_k \in B_k \setminus B_l$ *and* $y_l \in B_l \setminus B_k$, $x_k - y_l \in B_k \cap B_l$;
2. *For every* $i \in I$, *there exists* $x \in A_i \cap \bigcap_{j\in J} B_j$ *such that for every* $j \in J$ *we have* $(B_j)_x \cap A_i \neq \emptyset$;

3. *For every $k, l \in I$ and for every $u_k \in A_k \setminus A_l$ and $v_l \in A_l \setminus A_k$, $u_k - v_l \in A_k \cap A_l$;*
4. *For every $j \in J$ there exists $y \in B_j \cap \bigcap_{i \in I} A_i$ such that for every $i \in I$ we have $(A_i)_y \cap B_j \neq \emptyset$.*

A special case of Corollary 8 is obtained when $A_i = A$ for all $i \in I$ and $B_j = B$ for all $j \in I$. The resulting morphological center is then called the *annular filter*. The following example provides sufficient conditions under which the annular filter is idempotent.

Example 8 (Idempotence of the annular filter [10]). The annular filter ρ given by

$$\rho(X) = X \cap (X \oplus B) \cup (X \ominus A), \tag{34}$$

with A and B symmetric, and $A \cap B \neq \emptyset$, is idempotent if $(A \cap B) \cap (A \oplus B) \neq \emptyset$.

6.2 M-Idempotence

In the next theorem, we provide sufficient conditions under which the morphological center ρ is m-idempotent for some positive integer m.

Theorem 10. *The operator ρ is m-idempotent for some positive integer m if, for every $z \in \mathbf{E}$, we have*

1. *For every $k, l \in J$ and for every $x_k \in \beta_k(z) \setminus \beta_l(z)$ and $y_l \in \beta_l(z) \setminus \beta_k(z)$, $x_k \in \beta_k(y_l)$ and $y_l \in \beta_l(x_k)$;*
2. *There exists $(x_1, x_2, \ldots, x_{2m})$ such that*

$$x_1 \in \bigcap_{j \in J} \beta_j(z) \cap \bigcap_{i \in I} \alpha_i(z), x_{l+1} \in \bigcap_{j \in J} \beta_j(x_l) \cap \bigcap_{i \in I} \alpha_i(x_l)$$

for $l = 2, \ldots, 2m$ and

$$z \in \bigcap_{j \in J} \beta_j(x_{2m}) \cap \bigcap_{i \in I} \alpha_i(x_{2m}); \textit{and}$$

3. *For every $k, l \in I$ and for every $u_k \in \alpha_k(z) \setminus \alpha_l(z)$ and $v_l \in \alpha_l(z) \setminus \alpha_k(z)$, $u_k \in \alpha_k(v_l)$ and $v_l \in \alpha_l(u_k)$.*

Finally, we provide the corresponding corollary for the case where the morphological center ρ is increasing and translation invariant

Corollary 9. *The operator ρ is m-idempotent for some positive integer m if we have*

1. *For every $k, l \in J$ and for every $x_k \in B_k \setminus B_l$ and $y_l \in B_l \setminus B_k$, $x_k - y_l \in B_k \cap B_l$;*
2. *$0 \in \oplus_{l=1,\ldots,2m+1} (\bigcap_{i \in I} A_i \cap \bigcap_{j \in J} B_j)$; and*
3. *For every $k, l \in I$ and for every $u_k \in A_k \setminus A_l$ and $v_l \in A_l \setminus A_k$, $u_k - v_l \in A_k \cap A_l$.*

We end this section by providing sufficient conditions for the annular filter to be m-idempotent, for some positive integer m.

Example 9 (m-idempotence of the annular filter). The annular filter [10] ρ is m-idempotent for some positive integer m if we have $0 \in \oplus_{l=1,\ldots,2m+1} (A \cap B)$.

7 Application: Speckle Noise Removal

Speckle noise is a granular noise that is inherent in radar images and causes a degradation of their quality. This noise results from sporadic variations caused by small phase shifts in the radar signal which may be related to ground irregularities, such as plowed soil in a field that results in point sources which backscatter in various directions. Fig. 4a shows a speckled radar image of a field. Note that the image consists of several dark and bright spots next to each other and hence there is no clear distinction between the foreground and background. In this case, self-dual filters are necessary to avoid dependence on the varying local contrast in the image. A method for reducing speckle noise based on directional alternating sequential filters has been proposed in [24]. Considering the large variation of the speckle, in order to preserve the local average value of the amplitude, it is necessary to alternatively suppress the local maxima and minima with neighborhoods of increasing size. Linear structuring elements are particularly suitable for this task as two-dimensional structuring elements may contain one or several very sharp maxima or minima so that an opening or a closing with respect to this element modifies the local value noticeably [24]. Let us denote by (θ, p) the linear structuring element of length p and angle, with respect to the x-axis, θ. The algebraic opening is then defined as [24]

$$\gamma_p = \text{Max}\,(\gamma_{(\theta_1,p)}, \gamma_{(\theta_2,p)}, \ldots, \gamma_{(\theta_n,p)}), \tag{35}$$

where $\theta_1, \theta_2, \ldots, \theta_n$ are the different directions chosen. Similarly, the algebraic closing is defined as [24]

$$\phi_p = \text{Min}\,(\phi_{(\theta_1,p)}, \phi_{(\theta_2,p)}, \ldots, \phi_{(\theta_n,p)}). \tag{36}$$

The alternating sequential filter corresponding to $m_p = \gamma_p \phi_p$ is

$$M_n = m_{p_n} \ldots m_{p_2} m_{p_1}, \tag{37}$$

where $p_1 < p_2 < \ldots < p_n$. The ASF defined in (37) is called a *multidirectional filter* [24].

Alternating sequential filters are generally not self-dual [26] and the final result depends on whether one starts the filtering with an opening or a closing. Figs. 4b and 4c show the outputs of the directional open-close and close-open ASF, respectively, with $\theta_i = 10^\circ, \ldots, 90^\circ$ and $p = 2, \ldots, 5$. In fact, the more directions in which θ_i will be chosen, the more details in the image will be preserved but the less efficient the operator will be. The maximum length p_n of the structuring element is given by the mean speckle grain width, which is 5 pixels for the radar image in Fig. 4a. Observe that the results of the multidirectional filter differ depending on whether one begins the sequence by an opening or a closing. Due to the nature of the speckle noise, the morphological center of the directional alternating sequential filters starting with an opening and a closing processes dark and bright structures equally, as shown in Fig. 4d (and not only the directional alternating sequential filter starting with a closing as was adopted in [24]). In comparison, the first and fifth iterations of the median filter are shown in Figs. 4e and 4f, respectively. Observe that repeated application of the median filter results in smoother image outputs. The 1-idempotent annular filter, given in Example 5, is both self-dual and idempotent. Its corresponding output is displayed in Fig. 4g. The 2-idempotent annular filter, given in Example 7, results in the filtered image in Fig. 4h at the first iteration and Fig. 4i at the second iteration. All further iterations of the 2-idempotent annular filter will result in outputs identical to the second iteration of the filter.

To assess the capability of the filters to remove speckle noise and preserve the edges of the image, we use two performance measures suggested by Sheng and Xia [29]: the speckle suppression index (SSI) and the edge enhancing index (EEI).

Speckle suppression index [29]: The speckle suppression index is defined as the coefficient of variance of the filtered image normalized by that of the original image:

$$SSI = \frac{\sqrt{Var(I_f)}}{Mean(I_f)} \frac{Mean(I_0)}{\sqrt{Var(I_0)}}, \tag{38}$$

where I_0 and I_f denote the original and filtered image, respectively. For most cases, $SSI < 1$, which means speckle is suppressed. The lower the SSI, the stronger is the suppression ability of the filter.

Edge enhancing index [29]: The EEI is defined as

$$EEI = \frac{\sum |I_{01} - I_{02}|}{\sum |I_{f1} - I_{f2}|}, \tag{39}$$

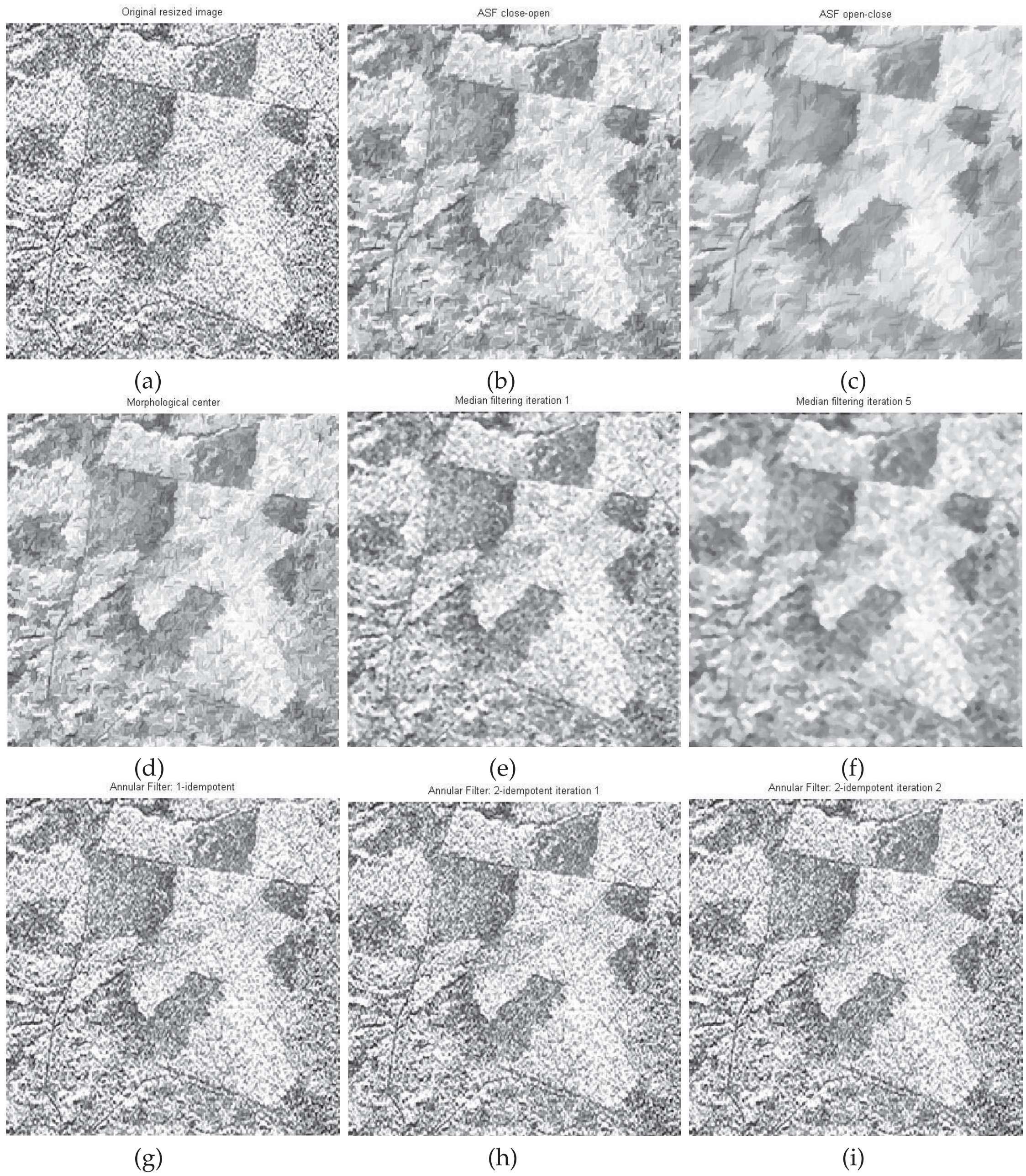


Fig. 4. Speckle noise removal: (a) original speckled radar image, (b) directional ASF close-open, (c) directional ASF open-close, (d) morphological center, (e) median filtering: iteration 1, (f) median filtering: iteration 5, (g) 1-idempotent annular filter, (h) 2-idempotent annular filter: iteration 1, (i) 2-idempotent annular filter: iteration 2.

where I_{01} and I_{02} are the original values of the pixels on either side of the edge, whereas I_{f1} and I_{f2} are the corresponding filtered values. The numerator is the absolute difference in intensity of the pixels on the two sides of the edge in the original image and the denominator is the same difference in the filtered image. Therefore, the EEI is usually greater than 1, and lower EEI values correspond to a better edge preserving capability.

Table 1 shows the SSI and EEI values of the filters in Fig. 4. The fifth iteration of the median filter removes most of the speckle noise, followed by the directional close-open ASF and the morphological center. The annular filters result in the poorest denoising. On the other hand, consecutive iterations of the median filter smooth the output image, thus resulting in indistinct edges. Because of their poor denoising capability, the annular filters lead to high edge enhancing index. The morphological center achieves an optimum tradeoff between speckle noise removal and edge preservation.

8 Conclusion

In this paper, we have presented a comprehensive analysis of m-idempotent and self-dual morphological operators. Our investigation was based on the morphological center in the general

TABLE 1
SSI and EEI of the Different Filters in Fig. 4

Filter	SSI	EEI
directional close-open ASF	0.7016	2.2168
directional open-close ASF	0.7144	2.2109
Morphological center	0.7115	2.1035
Median filter iteration 1	0.7585	1.8337
Median filter iteration 5	0.6466	3.7161
1-idempotent annular filter	0.9565	1.0503
2-idempotent annular filter	0.9500	1.0592

framework of spatially variant mathematical morphology. We have shown that the class of self-dual operators is contained in the class of morphological centers. We have relied on this framework to provide sufficient conditions for the representation of self-dual increasing operators characterized in terms of their kernel representation.

We have also derived necessary and sufficient conditions for the representation of idempotent operators by characterization of the kernel of underfilters and overfilters. Furthermore, we introduced m-idempotent operators by extending the notion of idempotence (i.e., operators that converge after m iterations) and characterized the kernel of m-idempotent operators. We finally used the conditions on the kernel representation derived to establish methods for the construction of m-idempotent and self-dual morphological operators.

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