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Bit-Error-Rate Performance of Companding Transforms for OFDM

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***Abstract*—This paper provides a comprehensive analytical framework to assess the relative bit-error-rate (BER) performance of companding transforms (CTs) employed to reduce the peak-to-average-power ratio (PAPR) in orthogonal frequency-division multiplexing (OFDM) systems. This paper provides a quantitative basis for several claims, which are reported in the literature, based solely on simulation results. In particular, we consider three main classes of CTs and provide a set of necessary and sufficient conditions for the superiority of one CT relative to the others. The conditions are given in terms of the companding parameters, which are usually selected to achieve a target PAPR. Our analytical derivations are supported by simulation results.**

***Index Terms*—Companding transforms (CTs), orthogonal frequency-division multiplexing (OFDM), peak-to-average power ratio (PAPR).**

I. Introduction

Despite the significant advantages offered by orthogonal frequency-division multiplexing (OFDM), it has the major inherited drawback of fluctuating envelope with high peaks, which leads to a high peak-to-average-power ratio (PAPR) for the transmitted signal. High peaks drive the transmitter's *power amplifier* (PA) into the nonlinear or saturation regions of operation, hence causing distortions and out-of-band radiation. They also demand *analog-to-digital converters* (ADC) with wide dynamic ranges. Many PAPR reduction techniques have been proposed in the literature, such as clipping and filtering, companding transforms (CTs), selective mapping, partial transmit sequences, tone injection, tone reservation, and linear block coding [1]–[3]. PAPR reduction capability is usually measured by the empirical *complementary cumulative distribution function* (ccdf), which is defined as the probability that the signal's PAPR exceeds a specific threshold. In most methods, PAPR is reduced at the expense of increasing the bit error rate (BER), complexity, or data overhead.

CTs form an attractive and widely used PAPR reduction technique due to their flexibility and low complexity, regardless of the number of subcarriers in the OFDM signal. CTs attenuate the high peaks and amplify the low amplitudes, thus decreasing the PAPR of the signal prior to the PA. However, CTs increase the BER due to the distortion incurred by the modulating symbols at the transmitter and the expansion of channel's noise by the decompander at the receiver. Fig. 1 shows the block diagram of an OFDM transceiver with a compander inserted between the *parallel-to-serial* converter and the

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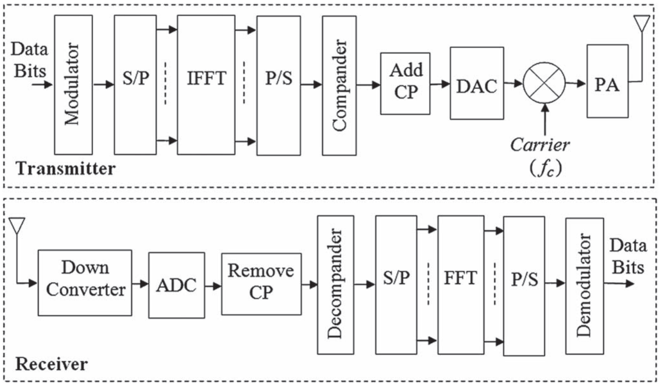

Fig. 1. OFDM transceiver system with CT.

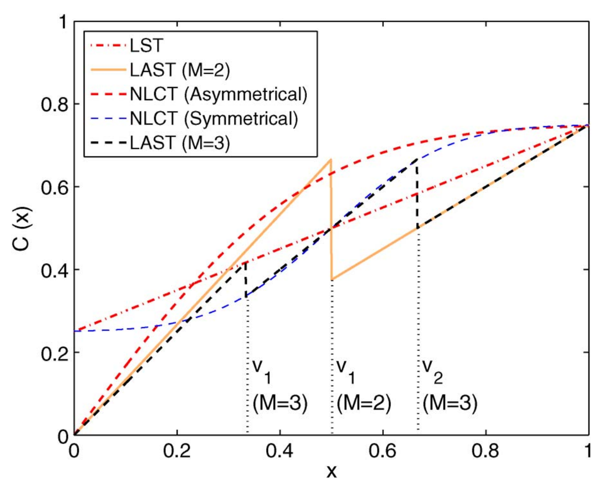

Fig. 2. Profiles of LST, LAST with $M = 2$ and $M = 3$, and NLCTs.

cyclic prefix (CP) adder. The decompander is inserted after the ADC and the CP remover and before the *serial-to-parallel* converter.

CTs can be classified into three classes: *linear symmetrical transform* (LST), *linear asymmetrical transform* (LAST), and *nonlinear CT* (NLCT). Fig. 2 shows the profiles of these classes. Both symmetrical and asymmetrical NLCT profiles and a LAST profile with two and three segments ($M = 2$ and $M = 3$) are shown. NLCTs such as the error function [4], [5], exponential [6], [7], logarithmic, hyperbolic tangent [8], and μ-law [9], are studied in the literature. In [10], the three aforementioned classes of CTs are considered, and it is claimed that LAST can achieve the lowest BER, whereas LST achieves the highest for a given PAPR reduction constraint. However, these claims are substantiated with mathematical explanations only for the special case of *quadrature phase-shift keying* modulation and *additive white Gaussian noise* (AWGN) channel. For other cases, only simulation results are provided. In [11], LAST with two discontinuity points is introduced. It has been claimed, based on simulation results, that the proposed transform has a better PAPR reduction and BER performance than the basic LAST with one discontinuity point.

Two immediate questions arise from the previous work on CTs: First, what companding parameters ensure that one CT outperforms another in terms of BER performance given that both achieve the same PAPR reduction? Second, does the BER of LAST depend only on the number of discontinuity points?

This paper tackles these questions by deriving a comprehensive analytical framework to study the BER performance of CTs in terms of companding parameters. In particular, this paper evaluates the relative BER performance of different CTs for a fixed target PAPR. The benefit of our approach is twofold. First, for a set of CTs, which achieve the same PAPR reduction, we can predict which one yields the least BER. Second, if a companding transformation can achieve the same desired PAPR reduction with more than one combination of companding parameters, we provide a means for choosing that combination, which yields the least BER among the given combinations.

Parts of this paper have been presented earlier [12], [13]. In [12], we compared the BER of LST versus LAST and derived a sufficient condition for the superiority of LAST over LST for the special case of the AWGN channel. In [13], we did the same for LST versus NLCT and derived a necessary condition for the superiority of one over the other. However, no comparison between LAST and NLCT was made before, and no conditions were derived under which one class of CTs outperforms another with respect to BER performance. In this paper, we extend our previous effort by deriving both necessary and sufficient sets of conditions, which ensure the BER performance superiority of one class of CTs over the other two. Unlike in [12], the derived necessary conditions here are more specific and given in terms of both the segment slopes (u_k's) and positions of discontinuity points (v_k's) for LAST, whereas in [12], the sufficient condition ignores the discontinuity points, hence leading to a looser bound than the one provided in this paper. In addition, the conditions derived here extend our previous derivations to the fading wireless environment. Therefore, this paper offers a substantial enhancement over our previous work and provides a comprehensive analytical framework and a quantitative basis for several claims, which are reported in the literature, based solely on simulation results without adequate analytical basis. For example, both the derived conditions and simulations show that the results claimed in [10] and [11] are not always true, and the results depend on companding parameters. In particular, our analysis shows that the BER of LAST depends on the slopes of the segments and the positions of the discontinuity points, and not on the number of discontinuities. We extend this result to a general class of LAST with $M - 1$ discontinuities (M segments). We also show that NLCT can sometimes outperform LAST if a proper function and parameters are used.

The remainder of this paper is organized as follows. Section II derives necessary and sufficient conditions, in terms of the companding parameters, to ensure that the expected absolute error for one CT is the minimum relative to the other CTs. Section III discusses simulation results, which validate the derived analytical findings. Finally, Section IV summarizes and concludes the work presented in this paper.

II. BER Performance Evaluation

Here, we derive necessary and sufficient conditions to ensure the BER performance superiority of one class of CTs relative to the others. Specifically, we derive the expected absolute error, at the OFDM receiver, when different companding classes are used. Since the error in the received time-domain signal is proportional to the error in the recovered frequency-domain symbols, which in turn is proportional to the BER, we compare the expected absolute error expressions to study the relative BER performance of the companding classes for a given PAPR constraint.

The following proposition shows that the relative BER performance superiority of NLCT, LST, and LAST can be determined, respectively, by the derivative of the nonlinear companding function, the slope of LST, and the slopes and discontinuity points of different segments in LAST.

Proposition 1: Consider a sampled OFDM signal with envelope $x[n]$. It is known that $x[n]$ follows a Rayleigh distribution with parameter σ. Denote by C_{LST} the LST CT given by $C_{\mathrm{LST}}(x[n]) = a\, x[n] + b$, with $0 < a < 1$ and $b > 0$. Let C_{LAST} denote a general

LAST CT with $M-1$ discontinuity points (M segments), defined piecewise by

$$C_{\mathrm{LAST}}(x[n]) = \begin{cases} u_1\, x[n], & \text{if } 0 \le x[n] \le v_1 \\ u_2\, x[n], & \text{if } v_1 < x[n] \le v_2 \\ \vdots & \\ u_M\, x[n], & \text{if } x[n] > v_{M-1} \end{cases} \tag{1}$$

where $0 < v_1 < v_2 < \cdots < v_{M-1} < \max(x[n])$. In addition, let C_{NL} denote any nonlinear continuously differentiable companding function, for which the inverse exists. Assuming that the fading channel response is available at the receiver, the relative BER performance of LST, LAST, and NLCTs can be compared by comparing their expected absolute errors at the receiver E_{LST}, E_{LAST}, and E_{NL}, respectively, as follows.

1) $E_{\mathrm{LST}} \le \min\{E_{\mathrm{LAST}}, E_{\mathrm{NL}}\}$

$$\text{iff} \quad \frac{1}{a} \le \min\left\{\eta\left(\{u_k\}, \{v_k\}, \sigma\right), \frac{1}{C'_{\mathrm{NL}}\left(\sqrt{\frac{\pi}{2}}\sigma\right)}\right\}.$$

2) $E_{\mathrm{LAST}} \le \min\{E_{\mathrm{LST}}, E_{\mathrm{NL}}\}$

$$\text{iff} \quad \eta\left(\{u_k\}, \{v_k\}, \sigma\right) \le \min\left\{\frac{1}{a}, \frac{1}{C'_{\mathrm{NL}}\left(\sqrt{\frac{\pi}{2}}\sigma\right)}\right\}.$$

3) $E_{\mathrm{NL}} \le \min\{E_{\mathrm{LST}}, E_{\mathrm{LAST}}\}$

$$\text{iff} \quad \frac{1}{C'_{\mathrm{NL}}\left(\sqrt{\frac{\pi}{2}}\sigma\right)} \le \min\left\{\frac{1}{a}, \eta\left(\{u_k\}, \{v_k\}, \sigma\right)\right\}.$$

where

$$\eta\left(\{u_k\}, \{v_k\}, \sigma\right) = \sum_{k=1}^{M} \frac{\exp\left(-\frac{v_{k-1}^2}{2\sigma^2}\right) - \exp\left(-\frac{v_k^2}{2\sigma^2}\right)}{u_k} \tag{2}$$

is a function of the LAST slopes $\{u_k\}_{k=1}^{M}$; the LAST discontinuity points $\{v_k\}_{k=1}^{M-1}$, with the consent that $v_0 = 0$ and $v_M = \infty$; and the Rayleigh parameter σ of the OFDM envelope. C'_{NL} is the first derivative of the nonlinear companding function C_{NL}.

Proof of proposition 1: Let $h[n]$ denote the impulse response of the fading channel and $w[n]$ denote the AWGN. The OFDM transmitter transmits the companded signal $C(x[n])$. The received signal is a distorted and noisy version of the companded signal $(h * C(x))[n] + w[n]$. To compensate for the fading channel distortions, the received signal has to be equalized. This can be done through the preamble or embedded pilots within the OFDM signal. Similar to what has been done in [5], we simplify the analysis by assuming that perfect knowledge of the channel response is achieved at the OFDM receiver. Then, the recovered signal after decompanding becomes $y[n] = C^{-1}[(h^{-1} * (h * C(x) + w))[n]]$. Therefore, the error between the transmitted and recovered signals is given by

$$\begin{aligned} e[n] &= y[n] - x[n] = C^{-1}\left[\left(h^{-1} * (h * C(x) + w)\right)[n]\right] - x[n] \\ &= C^{-1}\left[C(x[n]) + (h^{-1} * w)[n]\right] - x[n] \end{aligned} \tag{3}$$

where C^{-1} is the inverse of the companding function C. In what follows, we will consider the three companding classes LST, LAST, and NLCT for (3) and find the expected value of the absolute received error, to obtain a BER-related measure.

For the LST case, the expected absolute error is given by

$$\begin{aligned} E_{\mathrm{LST}} &= E\left|e_{\mathrm{LST}}[n]\right| \\ &= E\left|\frac{a\, x[n] + b + (h^{-1} * w)[n] - b}{a} - x[n]\right| \\ &= E\left|\frac{(h^{-1} * w)[n]}{a}\right| = \frac{E\left|(h^{-1} * w)[n]\right|}{a}. \end{aligned} \tag{4}$$

The received error for LAST is given by

$$e_{\mathrm{LAST}}[n] = \begin{cases} \frac{(h^{-1} * w)[n]}{u_1}, & \text{if } n \in \phi_1 \\ \vdots & \\ \frac{(h^{-1} * w)[n]}{u_M}, & \text{if } n \in \phi_M \end{cases} \tag{5}$$

where ϕ_k is the index set of the OFDM samples with amplitudes that fall in the region $v_{k-1} < x[n] \le v_k$.

Since ϕ_k's are disjoint sets (i.e., $\phi_i \bigcap \phi_j = \emptyset$ for $i \ne j$), (5) can be concisely rewritten as

$$e_{\mathrm{LAST}}[n] = \sum_{k=1}^{M} \frac{(h^{-1} * w)[n] I_{\phi_k}[n]}{u_k} = (h^{-1} * w)[n] \sum_{k=1}^{M} \frac{I_{\phi_k}[n]}{u_k} \tag{6}$$

where

$$I_{\phi_k}[n] = \begin{cases} 1, & \text{if } n \in \phi_k \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } v_{k-1} < x[n] \le v_k \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Thus, the expected absolute error is given by

$$E_{\mathrm{LAST}} = E\left|e_{\mathrm{LAST}}[n]\right| = E\left|(h^{-1} * w)[n] \sum_{k=1}^{M} \frac{I_{\phi_k}[n]}{u_k}\right|. \tag{8}$$

Since $w[n]$ and $x[n]$ are uncorrelated and $(I_{\phi_k}[n]/u_k) \ge 0$, $\forall\, k$ and $\forall\, n$, (8) becomes

$$E_{\mathrm{LAST}} = E\left|(h^{-1} * w)[n]\right| \sum_{k=1}^{M} \frac{E\left[I_{\phi_k}[n]\right]}{u_k}. \tag{9}$$

From (7), $E[I_{\phi_k}[n]]$ is given by

$$\begin{aligned} E\left[I_{\phi_k}[n]\right] &= \Pr\left(v_{k-1} < x[n] \le v_k\right) \\ &= \exp\left(\frac{-v_{k-1}^2}{2\,\sigma^2}\right) - \exp\left(\frac{-v_k^2}{2\,\sigma^2}\right) \end{aligned} \tag{10}$$

and σ is the positive Rayleigh parameter of the probability density function characterizing the envelope of the OFDM signal, with the consent that $v_0 = 0$ and $v_M = \infty$.

To obtain the expected absolute error of the NLCT class, we use the first-order Taylor series expansion of function $C_{\mathrm{NL}}^{-1}[C_{\mathrm{NL}}(x[n]) + (h^{-1} * w)[n]]$ around point $C_{\mathrm{NL}}(x[n])$. By noting that

$$\left[C_{\mathrm{NL}}^{-1}\left(C_{\mathrm{NL}}\left(x[n]\right)\right)\right]' = \frac{1}{C'_{\mathrm{NL}}\left(x[n]\right)} \tag{11}$$

the expected absolute error becomes

$$\begin{aligned} E_{\mathrm{NL}} = E\Big| & C_{\mathrm{NL}}^{-1}\left[C_{\mathrm{NL}}\left(x[n]\right)\right] + (h^{-1} * w)[n] \\ & \times \left[C_{\mathrm{NL}}^{-1}\left[C_{\mathrm{NL}}\left(x[n]\right)\right]\right]' - x[n]\Big| \\ = E & \left|\frac{(h^{-1} * w)[n]}{C'_{\mathrm{NL}}\left(x[n]\right)}\right|. \end{aligned} \tag{12}$$

Since $w[n]$ and $x[n]$ are uncorrelated, $1/C'_{\mathrm{NL}}(x[n]) \ge 0 \;\; \forall\, x[n]$ and $E[1/C'_{\mathrm{NL}}(x[n])]$ can be approximated by $1/C'_{\mathrm{NL}}(E[x])$ using the first-order Taylor's series approximation [14, Ch. 3, pp. 64], [15, Ch. 5], (12) becomes

$$E_{\mathrm{NL}} = \frac{E\left|(h^{-1} * w)[n]\right|}{C'_{\mathrm{NL}}\left(E[x]\right)} = \frac{E\left|(h^{-1} * w)[n]\right|}{C'_{\mathrm{NL}}\left(\sqrt{\frac{\pi}{2}}\sigma\right)}. \tag{13}$$

Finally, by noting that $E|(h^{-1} * w)[n]|$ is a common term in (4), (9), and (13), we can easily compare the expected absolute errors of different CTs by comparing the factors $1/a$, $1/C'_{\mathrm{NL}}(\sqrt{(\pi/2)}\sigma)$, and $\eta(\{u_k\},\{v_k\},\sigma)$, as stated in Proposition 1. ■

A special case of Proposition 1 results in the following sufficient conditions using which one may evaluate the relative BER performance of the three companding classes.

Corollary 1: Let the conditions of Proposition 1 hold. Denote by $u_{\min} = \min_{k=1,\ldots,M} u_k$ and $u_{\max} = \max_{k=1,\ldots,M} u_k$. Then, the relative performance of LST, LAST, and NLCTs, as measured by the expected absolute error at the receiver, can be assessed using the following sufficient conditions.

1) If $a \geq \max\{u_{\max}, C'_{\mathrm{NL}}(\sqrt{(\pi/2)}\sigma)\}$, then $E_{\mathrm{LST}} \leq \min\{E_{\mathrm{LAST}}, E_{\mathrm{NL}}\}$.
2) If $u_{\min} \geq \max\{a, C'_{\mathrm{NL}}(\sqrt{(\pi/2)}\sigma)\}$, then $E_{\mathrm{LAST}} \leq \min\{E_{\mathrm{LST}}, E_{\mathrm{NL}}\}$.
3) If $C'_{\mathrm{NL}}(\sqrt{(\pi/2)}\sigma) \geq \max\{a, u_{\max}\}$, then $E_{\mathrm{NL}} \leq \min\{E_{\mathrm{LST}}, E_{\mathrm{LAST}}\}$.

Proof of Corollary 1: By noting that $(1/u_{\max}) \leq \eta(\{u_k\}, \{v_k\},\sigma) \leq (1/u_{\min})$, the proof can be easily derived from the results of Proposition 1. ■

Observe that the sufficient conditions in Corollary 1 depend solely on the slopes a, $\{u_k\}_{k=1}^{M}$, and the derivative of the NLCT. In particular, they do not depend on the discontinuity points $\{v_k\}_{k=1}^{M-1}$.

III. Simulation Results

We conduct computer simulations to validate the theoretical results in Proposition 1 and Corollary 1. We consider the Worldwide Interoperability for Microwave Access standard in the *downlink partial use subcarrier* mode with 1024 subcarriers. The nonlinearity of the transmitter's PA is modeled by the widely accepted *solid-state PA* model [16] given by

$$x_{\mathrm{out}} = \frac{x_{\mathrm{in}}}{\left[1+\left(\frac{x_{\mathrm{in}}}{A_{\mathrm{sat}}}\right)^{2p}\right]^{1/2p}} \tag{14}$$

where x_{in} and x_{out} are the input and output signals of the amplifier, respectively; p is a positive parameter controlling the nonlinearity level of the PA; and A_{sat} is a normalization factor specifying the saturation level of the amplifier. In all simulations, we set $p = 2$ and $A_{\mathrm{sat}} = 0.14$. We conducted two sets of simulations using the *Stanford University Interim*-1 (SUI-1) [17] and the International Telecommunication Union Radiocommunication Sector (ITU-R) pedestrian A channel models with AWGN to represent two wireless channels. The SUI-1 models fixed wireless applications with a mostly flat terrain with light tree density and has a low path loss and delay spread. The ITU-R pedestrian A models an outdoor-to-indoor pedestrian environment and has a low delay spread. These two models apply mild fading and, hence, suit our simulations in which no channel coding or diversity techniques were used to preserve the abstract effect of companding.

In the simulations, we assume perfect channel estimation and use a single-tap frequency-domain equalizer after the decompanding and fast Fourier transform blocks. We evaluate the BER performance of the OFDM system with companding functions: LST, LAST with $M = 2$, LAST with $M = 3$, the hyperbolic tangent transform $C_{\mathrm{NL}}(x[n]) = k_1 \tanh(k_2\, x[n])$, and the error function transform $C_{\mathrm{NL}}(x[n]) = k_1 \mathrm{erf}(k_2\, x[n])$, The mean of the Rayleigh distributed OFDM envelope is given by $\sqrt{\pi/2}\,\sigma$, and we estimated that $\sigma \approx 0.03$. Simulations are subject to the constraint that CTs must achieve a target ccdf of 10^{-6} at a PAPR threshold of 8 dB compared with 12 dB for the original OFDM signal. Table I shows some choices of companding parameters, which satisfy the imposed constraint, and the corresponding factors used to assess the expected absolute errors. From Proposition 1, we verify that the nonlinear (erf) CT has the lowest BER, which is followed by LAST (case 2), LAST (case 1), LAST (case 3), the nonlinear (tanh) CT, and LST. The sufficient conditions of Corollary 1 are also confirmed. We observe that, depending on the companding parameters (not the number of discontinuity points), LAST with $M = 2$ may have a higher or lower BER compared with LAST with $M = 3$. This implies that increasing the number of discontinuity points does not necessarily yield any BER advantage, unless proper sets of parameters are used. Fig. 3 shows the BER versus *signal-to-noise ratio* of the six CT examples listed in Table I when the SUI-1 channel model is used. Fig. 4 shows the same when the ITU-R pedestrian A channel model is used. The expected relative BER performance between different CTs obtained by the numerical calculations in Table I agrees with the simulation results in Figs. 3 and 4, which shows that the conditions derived in Proposition 1 are indeed useful in predicting the relative BER performance of OFDM when CTs with specific parameters are to be used.

TABLE I
Companding Parameters and Comparison Factors

Transform	Parameters	Comparison
LST	$a = 0.535$ $b = 2\sigma/3 = 0.02$	$1/a = 1.87$
NL (tanh)	$k_1 = 0.08$, $k_2 = 10$	$1/C'_{NL}(\sqrt{\pi/2}\,\sigma) = 1.435$
NL (erf)	$k_1 = 0.15$, $k_2 = 8$	$1/C'_{NL}(\sqrt{\pi/2}\,\sigma) = 0.8$
LAST ($M=2$) (case 1)	$u_2 = \frac{1}{u_1} = 0.8$ $v_1 = 2\sigma = 0.06$	$\eta\left(\{u_k\},\{v_k\},\sigma\right) = 0.86$
LAST ($M=3$) (case 2)	$v_1 = 2\sigma$, $v_2 = 3\sigma$ $(u_1, u_2, u_3) = (1.25, 1, 0.8)$	$\eta\left(\{u_k\},\{v_k\},\sigma\right) = 0.83$
LAST ($M=3$) (case 3)	$v_1 = \sigma$, $v_2 = 2\sigma$ $(u_1, u_2, u_3) = (1.2, 0.85, 0.55)$	$\eta\left(\{u_k\},\{v_k\},\sigma\right) = 1.128$

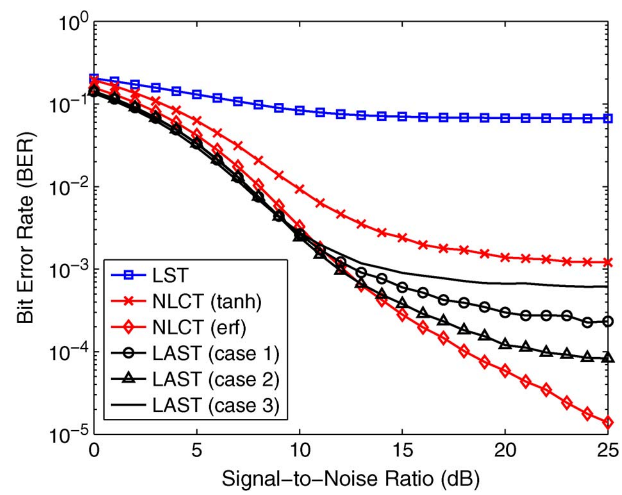

Fig. 3. BER performance for LST, LAST, and NLCTs with optimized companding parameters over the SUI-1 channel.

IV. Conclusion

This paper has provided a comprehensive analytical framework to investigate the relative BER performance of different CTs reported in the literature. Specifically, we derive necessary and sufficient sets of conditions to ensure the BER superiority of one CT over the others.

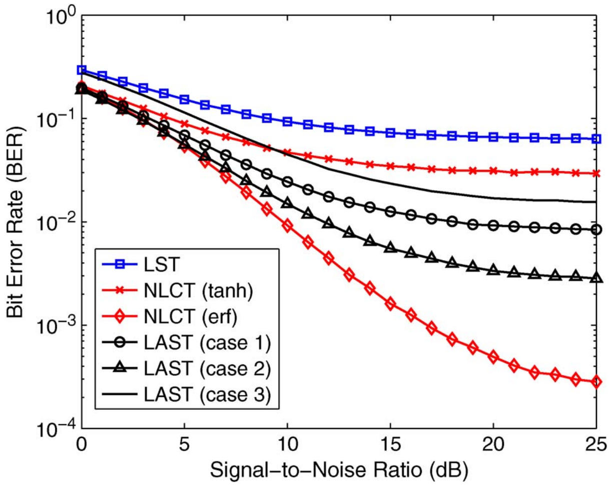

Fig. 4. BER performance for LST, LAST, and NLCTs with optimized companding parameters over the ITU-R pedestrian A channel.

Our derivations explain the different simulation results and claims reported in the literature, which were unsubstantiated with mathematical analysis. Furthermore, the proposed framework helps in selecting proper companding parameters to achieve specific design constraints. In particular, we show that the BER performance of LST depends only on the slope parameter, whereas that of a general LAST with $M - 1$ discontinuity points depends on the slopes of the segments and the positions of the discontinuity points, rather than on the number of discontinuities. Moreover, for any nonlinear continuously differentiable CT, for which the inverse exists, the BER performance depends on the derivative of the companding function. Our theoretical derivations are supported by the simulation results shown in Figs. 3 and 4. The results show perfect agreement with the conditions derived in Proposition 1. It is shown that the BER of LAST is related to the slopes of different segments and the position of discontinuities, as has been stated in Proposition 1. In particular, LAST with one discontinuity point can outperform LAST with two discontinuity points, given proper choices of the parameters, as shown in Figs. 3 and 4.

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Dynamic Power Allocation for Downlink Interference Management in a Two-Tier OFDMA Network

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***Abstract*—We study the downlink interference problem in a two-tier orthogonal frequency-division multiple-access (OFDMA) system. Assume that macro- and femtotiers share the same spectrum and that the femtotier uses a closed-access scheme. Cross-tier and intercell interference in the two tiers is investigated. Resorting to game theory and variational inequality (VI) theory, we formulate the problem mathematically and design algorithms for the solution. With the algorithms, the femto base stations (FBSs) can dynamically allocate their power according to the feedback from the macrotier to avoid cross-tier interference and to adapt to intercell interference. The overall power allocation of the femtotier reaches an equilibrium, provided that every FBS aims to maximize its own Shannon capacity. The algorithms can be distributively implemented and can mitigate the interference quickly without unnecessary performance loss. Simulation results are provided to demonstrate the performance of our algorithms and to compare with other related algorithms.**

***Index Terms*—Cross-tier interference, femto-cell, heterogeneous network, interference management, power control, variational inequality theory.**

I. INTRODUCTION

Traditionally, new base stations are carefully deployed and form the macrotier. Femtocells [1] with small low-power base stations are deployed in a plug-and-play manner without network planning and

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