# HOMEWORK SOLUTIONS FALL 2015

HW 1

**1.1** We use the first three steps of Eq. 1.11

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$
$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$
$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

Adding the above, we get

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Adding and subtracting $\nu\frac{\sigma_x}{E}$ from the first equation,

$$\varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Similar expressions can be obtained for $\varepsilon_y$, and $\varepsilon_z$.

From the relationship for $\gamma_{yz}$ and Eq. 1.12,

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} \quad \text{etc.}$$

Above relations can be written in the form

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

where **D** is the material property matrix defined in Eq. 1.15. ■

**1.4** Displacement field

$$u = 10^{-4}\left(-x^2 + 2y^2 + 6xy\right)$$
$$v = 10^{-4}\left(3x + 6y - y^2\right)$$
$$\frac{\partial u}{\partial x} = 10^{-4}\left(-2x + 6y\right) \quad \frac{\partial u}{\partial y} = 10^{-4}\left(4y + 6x\right)$$
$$\frac{\partial v}{\partial x} = 3\times 10^{-4} \qquad \frac{\partial v}{\partial y} = 10^{-4}\left(6 + 2y\right)$$
$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

at $x = 1, y = 0$

$$\varepsilon = 10^{-4}\begin{Bmatrix} -2 \\ 6 \\ 9 \end{Bmatrix}$$

■

**1.8**

$$\sigma_x = 40 \text{ MPa} \quad \sigma_y = 20 \text{ MPa} \quad \sigma_z = 30 \text{ MPa}$$
$$\tau_{yz} = -30 \text{ MPa} \quad \tau_{xz} = 15 \text{ MPa} \quad \tau_{xy} = 10 \text{ MPa}$$
$$\mathbf{n} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} & \dfrac{1}{\sqrt{2}} \end{bmatrix}^{\mathrm{T}}$$

From Eq.1.8 we get

$$T_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$
$$= 35.607 \text{ MPa}$$
$$T_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$
$$= -6.213 \text{ MPa}$$
$$T_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$
$$= 13.713 \text{ MPa}$$
$$\sigma_n = T_x n_x + T_y n_y + T_z n_z$$
$$= 24.393 \text{ MPa}$$

■

HW 2

**1.12** Following the steps of Example 1.1, we have

$$\begin{bmatrix} (80+40+50) & -80 \\ -80 & 80 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 50 \end{Bmatrix}$$

Above matrix form is same as the set of equations:

$$170\, q_1 - 80\, q_2 = 60$$

$$-80\, q_1 + 80\, q_2 = 50$$

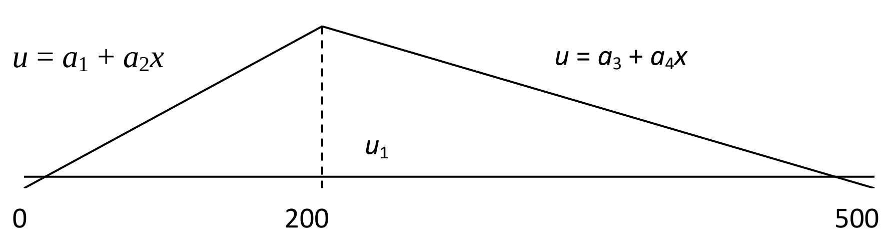
Solving for $q_1$ and $q_2$, we get

$$q_1 = 1.222 \text{ mm}$$

$$q_2 = 1.847 \text{ mm}$$

■

**1.17** Let $u_1$ be the displacement at $x$ = 200 mm. Piecewise linear displacement that is continuous in the interval $0 \le x \le 500$ is represented as shown in the figure.




$0 \le x \le 200$

$u = 0$ at $x = 0 \Rightarrow a_1 = 0$

$u = u_1$ at $x = 200 \Rightarrow a_2 = u_1/200$

$\Rightarrow u = (u_1/200)x \qquad du/dx = u_1/200$

$200 \le x \le 500$

$u = 0$ at $x = 500 \Rightarrow a_3 + 500\, a_4 = 0$

$u = u_1$ at $x = 200 \Rightarrow a_3 + 200\, a_4 = u_1$

$\Rightarrow a_4 = -u_1/300 \qquad a_3 = (5/3)u_1$
$\Rightarrow u = (5/3)u_1 - (u_1/300)x \qquad du/dx = - u_1/200$

$$\Pi = \frac{1}{2}\int_0^{200} E_{al} A_1 \left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_{200}^{500} E_{st} A_2 \left(\frac{du}{dx}\right)^2 dx - 10000u_1$$

$$\Pi = \frac{1}{2} E_{al} A_1 \left(\frac{u_1}{200}\right)^2 200 + \frac{1}{2} E_{st} A_2 \left(-\frac{u_1}{300}\right)^2 300 - 10000u_1$$

$$= \frac{1}{2}\left(\frac{E_{al} A_1}{200} + \frac{E_{st} A_2}{300}\right) u_1^{\,2} - 10000u_1$$

$$\frac{\partial \Pi}{\partial u_1} = 0 \;\Rightarrow\; \left(\frac{E_{al} A_1}{200} + \frac{E_{st} A_2}{300}\right) u_1 - 10000 = 0$$

Note that using the units MPa (N/mm$^2$) for modulus of elasticity and mm$^2$ for area and mm for length will result in displacement in mm, and stress in MPa.

Thus, $E_{al}$ = 70000 MPa, $E_{st}$ = 200000, and $A_1$ = 900 mm$^2$, $A_2$ = 1200 mm$^2$. On substituting these values into the above equation, we get

$u_1$ = 0.009 mm

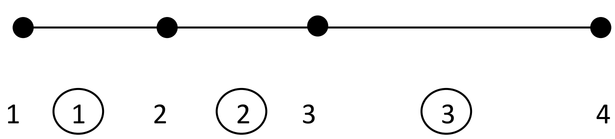
This is precisely the solution obtained from strength of materials approach ■

HW 3

**3.2**

NBW = max [(3-1), (4-3), (5-4), **(5-2)**] + 1 = 4 ■

**3.9** We introduce a node at the point of load application, and a node at the point where there is a change of cross section. The finite element configuration is shown below.




$E$ = 200 × 10$^3$ N/mm$^2$ (MPa)

$A_1$ = 250 mm$^2$ $A_2$ = 250 mm$^2$ $A_3$ = 400 mm$^2$

$L_1$ = 150 mm $L_2$ = 150 mm $L_3$ = 300 mm

Load $P$ = 300000 N is applied at node 2. $Q_1 = 0$, $Q_4 = 0$.

$$\begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} & 0 \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2}+\frac{E_3A_3}{L_3} & -\frac{E_3A_3}{L_3} \\ 0 & 0 & -\frac{E_3A_3}{L_3} & \frac{E_3A_3}{L_3} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ 0 \\ 0 \end{Bmatrix}$$

Since $Q_1$ and $Q_4$ are zero, using the elimination approach, as indicated above,

$$10^5\begin{bmatrix} 6.667 & -3.333 \\ -3.333 & 6 \end{bmatrix}\begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 30000 \\ 0 \end{Bmatrix}$$

On solvind, we get $Q_2$ = 0.623 mm and $Q_3$ = 0.346 mm.

The data set for using program FEM1D is given below:

```
<< 1D STRESS ANALYSIS USING BAR ELEMENT >>
PROBLEM 3.7
NN NE NM NDIM NEN NDN
 4  3  1   1   2   1
ND NL  NMPC
 2  1   0
Node#   X-Coordinate
 1      0
 2    150
 3    300
 4    600
Elem#  N1  N2  Mat#  Area TempRise
 1     1   2    1    250   0
 2     2   3    1    250   0
 3     3   4    1    400   0
DOF#  Displacement
 1     0
 4     0
DOF#  Load
 2    300000
MAT#  E       Alpha
 1   200000     0
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

The output from the program is given below.

Results from Program FEM1D

PROBLEM 3.7

| Node# | Displacement |
|---|---|
| 1 | 3.11536E-05 |

| | |
|---|---|
| 2 | 0.623102751 |
| 3 | 0.346174349 |
| 4 | 1.38464E-05 |
| 5 | |

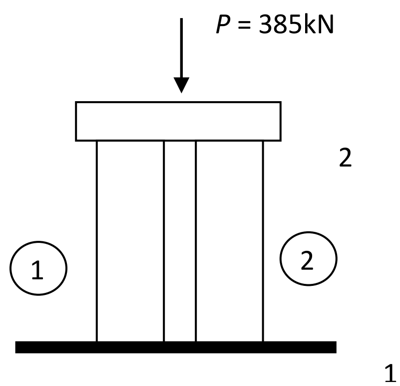
| Element# | Stress |
|---|---|
| 1 | 830.7621304 |
| 2 | -369.2378696 |
| 3 | -230.7736685 |

| Node# | Reaction |
|---|---|
| 1 | -207690.5326 |
| 4 | -92309.4674 |

The programs gives the reactions at the fixed nodes. ■

# HW 4

**3.11**




$E_1$ = 70000 MPa $A_1 = 30\times60 = 1800$ mm$^2$
$E_2$ = 105000 MPa $A_2$ = 1800 mm$^2$
$L_1 = L_2 = 200$ mm

This problem is easily formulated by defining same node numbers for each element.

| Elem# | Node1 | Node2 | Material# |
|---|---|---|---|
| 1 | 1 | 2 | 1 |
| 2 | 1 | 2 | 2 |

The unmodified system is

$$\begin{bmatrix} \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_1A_1}{L_1}-\frac{E_2A_2}{L_2} \\ -\frac{E_1A_1}{L_1}-\frac{E_2A_2}{L_2} & \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

On eliminating the first row and the first column,

$$\frac{1800}{200}\left(7\times10^4+10.5\times10^4\right)Q_2 = 385\times10^3$$

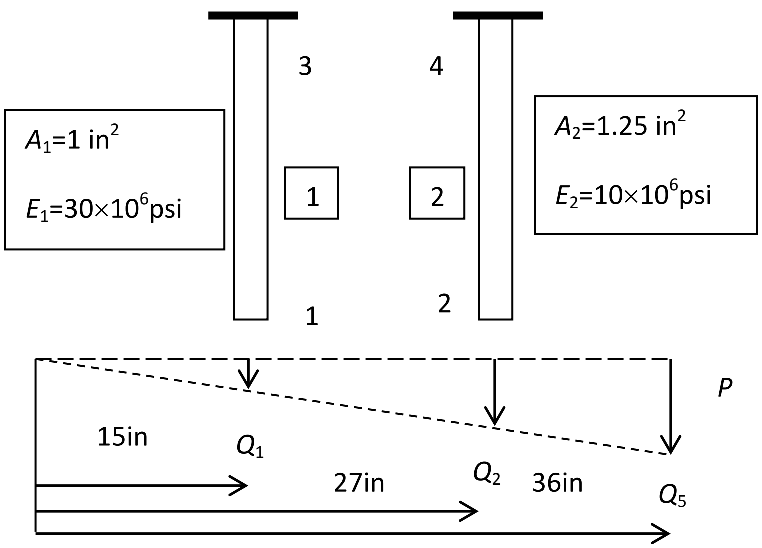
$$Q_2 = 0.244\ \text{mm}$$

$$\sigma_1 = \frac{E_1Q_2}{L_1} = 85.4\ \text{MPa}$$

$$\sigma_2 = \frac{E_2Q_2}{L_2} = 128.1\ \text{MPa}$$

■

**3.16** This problem follows the steps used in Example 3.6.




We denote $s_1 = \frac{E_1A_1}{L_1}$ and $s_2 = \frac{E_2A_2}{L_2}$. The unmodified stiffness is

$$\mathbf{K} = \begin{bmatrix} s_1 & 0 & -s_1 & 0 & 0 \\ 0 & s_2 & 0 & -s_2 & 0 \\ -s_1 & 0 & s_1 & 0 & 0 \\ 0 & -s_2 & 0 & s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The multipoint constraints are

$$Q_1 - \frac{15}{36} Q_5 = 0$$
$$Q_2 - \frac{27}{36} Q_5 = 0$$

The displacement constraints are

$Q_3 = 0$;   $Q_4 = 0$

These are now used in preparing the input file for the solution of the problem.

Input Data file

```
PROGRAM FEM1D  << BAR ANALYSIS
EXAMPLE 3.16
NN   NE      NM      NDIM    NEN     NDN
5    2       2       1       2       1
ND   NL      NMPC
2    1       2
NODE#        X-COORD
1    0
2    0
3    -36
4    -36
5    0
EL#  N1      N2      MAT#    AREA    TEMP RISE
1    1       3       1       1       0
2    2       4       2       1.25    0
DOF#DISP
3    0
4    0
DOF#LOAD
5    15000
MAT#         E       Alpha
```

| | | | | |
|---|---|---|---|---|
| 1 | 3.00E+07 | | 1.20E-05 | |
| 2 | 1.00E+07 | | 2.30E-05 | |
| B1 | I | B2 | J | B3 |
| 1 | 1 | -0.416665 | | 0 |
| 1 | 2 | -0.75 | 5 | 0 |

Results from Program FEM1D

EXAMPLE 3.14

| Node# | Displacement |
|---|---|
| 1 | 0.018383541 |
| 2 | 0.033092834 |
| 3 | 1.83817E-06 |
| 4 | 1.37881E-06 |
| 5 | 0.044125617 |

| Element# | Stress |
|---|---|
| 1 | 15318.08586 |
| 2 | 9192.070767 |

| Node# | Reaction |
|---|---|
| 3 | -15318.08586 |
| 4 | -11490.08846 |

■

**3.31**

The data file for running the program FEM1D is as follows:

PROGRAM FEM1D << BAR ANALYSIS

PROBLEM 3.23

| NN | NE | NM | NDIM | NEN | NDN |
|---|---|---|---|---|---|
| 4 | 3 | 3 | 1 | 2 | 1 |
| ND | NL | NMPC | | | |
| 2 | 2 | 0 | | | |
| NODE# | | X-COORD | | | |
| 1 | 0 | | | | |
| 2 | 800 | | | | |
| 3 | 1400 | | | | |
| 4 | 1800 | | | | |
| EL# | N1 | N2 | MAT# | AREA | TEMP RISE |
| 1 | 1 | 2 | 1 | 2400 | 80 |
| 2 | 2 | 3 | 2 | 1200 | 80 |
| 3 | 3 | 4 | 3 | 600 | 80 |
| DOF# | | DISP | | | |
| 1 | 0 | | | | |
| 4 | 0 | | | | |
| DOF# | | LOAD | | | |
| 2 | -60000 | | | | |
| 3 | -75000 | | | | |
| MAT# | | E | Alpha | | |
| 1 | 83000 | 1.89E-05 | | | |
| 2 | 70000 | 2.30E-05 | | | |

| | | | | |
|---|---|---|---|---|
| 3 | 200000 | 1.17E-05 | | |
| B1 | I | B2 | J | B3 |

Results from Program FEM1D
PROBLEM 3.23

| Node# | Displacement |
|---|---|
| 1 | -5.5931E-05 |
| 2 | 0.22120552 |
| 3 | -0.004053755 |
| 4 | 2.52492E-05 |

| Element# | Stress |
|---|---|
| 1 | -102.5401244 |
| 2 | -155.0802488 |
| 3 | -185.1604977 |

| Node# | Reaction |
|---|---|
| 1 | 246096.2986 |
| 4 | -111096.2986 |

■

## Homework 5

**4.6**

$$E = 70{,}000 \quad Mpa$$
$$A = 200 mm^2$$
$$L_1 = L_2 = 500 mm$$

Note: 1 N/mm$^2$ = 1 Mpa

Element connectivity Direction Cosines:

| | | $l$ | $m$ |
|---|---|---|---|
| 1 | $2-1$ | 1 | 0 |
| 2 | $1-3$ | 0.8 | $-0.6$ |

Element stiffness (Eq. 4.13)

$$\begin{array}{cc} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \mathbf{k}^1 = C & \left[\begin{array}{cc|cc} & & & \\ \hline & & 1 & 0 \\ & & 0 & 0 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \end{array}$$

$$
\begin{array}{c c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\
\mathbf{k}^2 = C \begin{bmatrix} .64 & -.48 & & \\ -.48 & -.36 & & \\ & & & \\ & & & \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

$$
C = \frac{EA}{L} = \frac{70000 \times 200}{500}
$$

Assembly of **K** :

$$
\begin{array}{c c}
\begin{array}{cc} 1 & 2 \end{array} & \\
\mathbf{K} = C \begin{bmatrix} 1.64 & -.48 \\ -.48 & .36 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array}
\end{array}
$$

Elimination Approach :

$$
C \begin{bmatrix} 1.64 & -.48 \\ -.48 & .36 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -12000 \end{bmatrix}
$$

Solving,

$Q_1 = -0.5714 in$

$Q_2 = -1.9524 in$

Stress in Element 2 :

Eq. (4.16):

$$
\sigma_2 = \frac{70000}{500} \begin{bmatrix} -.8 & .6 & .8 & -.6 \end{bmatrix} \begin{bmatrix} -.5714 \\ -.9524 \\ 0 \\ 0 \end{bmatrix} = -100 MPa \quad \text{(Compression)}
$$

Verify the above using program TRUSS2. ■

**4.7** Solution Using Truss Program -- Input File

Next line is problem title
Problem 4.7 – Truss

| | | | | | |
|---|---|---|---|---|---|
| NN | NE | NM | NDIM | NEN | NDN |
| 4 | 3 | 1 | 2 | 2 | 2 |
| ND | NL | NCH | NPR | NMPC | |
| 6 | 1 | 2 | 2 | 0 | |
| Node # | X | Y | | | |
| 1 | 0 | 0 | | | |
| 2 | -450 | 600 | | | |
| 3 | 800 | 600 | | | |
| 4 | 450 | 600 | | | |
| Elem# | N1 | N2 | Mat# | Area | TempRise |
| 1 | 2 | 1 | 1 | 250 | 0 |
| 2 | 1 | 3 | 1 | 250 | 0 |
| 3 | 1 | 4 | 1 | 250 | 0 |
| DOF# | Displacement | | | | |
| 3 | 0 | | | | |
| 4 | 0 | | | | |
| 5 | 0 | | | | |
| 6 | 0 | | | | |
| 7 | 0 | | | | |
| 8 | 0 | | | | |
| DOF# | Load | | | | |
| 2 | -18000 | | | | |
| MAT# | PROP1 | PROP2 | | | |
| 1 | 200e3 | | 12E-6 | | |
| Bl | i | B2j | B3 (Multi-point constr.) | | |

**Output**

Output for Input Data from File temp. inp
Problem 4.7

| Node# | X-Displ | Y-Displ |
|---|---|---|
| 1 | 5.6176E-02 | -1.8725E-01 |
| 2 | 7.1030E-06 | -9.4707E-06 |
| 3 | -2.6093E-06 | -1.9570E-06 |
| 4 | -4.4937E-06 | -5.9917E-06 |

| Elem# | Stress |
|---|---|
| 1 | 4.893E+01 |
| 2 | 1.348E+01 |

| | |
|---|---|
| 3 | 3.096E+01 |

| DOF# | Reaction |
|---|---|
| 3 | -7.3398E+03 |
| 4 | 9.7864E+03 |
| 5 | 2.6963E+03 |
| 6 | 2.0222E+03 |
| 7 | 4.6435E+03 |
| 8 | 6.1914E+03 |

■

**4.15**

The load at node 5 in Figure P4.15 is resolved into its X- and Y - components.

OUTPUT

| Node# | (in.)X – Displ | (in.)Y – Displ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0.0111 | 0 |
| 3 | 0.0138 | – 0.0126 |
| ⋮ | ⋮ | ⋮ |
| 8 | – 0.0088 | – 0.1288 |
| ⋮ | ⋮ | ⋮ |
| 10 | 0.0342 | – 0.0367 |

| Elem# | Stress(psi) |
|---|---|
| 1 | 695 |
| 4 | – 1870 |
| 9 | – 2012 |
| 14 | 2083 |
| 17 | – 1339 |