Main Points from Linear Algebra

I System of Equations.

a) Express an equation of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

in matrix form and find its augmented matrix.

b) What are elementary row operations?

c) What do we mean by two matrices are (row) equivalent?

d) To solve $AX = b$, apply **rref** to the augmented matrix $[A|b]$. Suppose the **rref** of the augmented matrix is $[C|d]$. Then the solution of $AX = b$ is the same as that of $CX = d$.

e) If A is an invertible matrix, then the solution of $AX = b$ is given by $X = A^{-1}b$.

II The Determinant. (In what follows all matrices are $n \times n$).

a) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

b) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei - afh - bdi + dfg + cdh - ceg$

c) Give a procedure for the determinant of a 4×4 matrix and then generalize to any $n \times n$ matrix.

d) If B is obtained from A by interchanging two rows of A, then $\det(B) = -\det(A)$

e) If B is obtained from A by multiplying a row of A by k, then $\det(B) = k\det(A)$.

f) If B is obtained from A by adding a multiple of a row of A to another row of A, then $\det(B) = \det(A)$.

g) $det(kA) = k^n \det(A)$. h) $\det(A^T) = \det(A)$. i) $\det(AB) = \det(A)\det(B)$.

j) If $\det(A) \neq 0$, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

k) If A has a zero row, then $\det(A) = 0$.

l) If a row of A is a multiple of another row of A , then $\det(A) = 0$.

m) If $\det(A) \neq 0$, then the system $AX = b$ has a unique solution and the solution can be found by the formula

$$x_i = \frac{\det(A_i)}{\det(A)},$$

where A_i is the matrix obtained by replacing the ith column of A by b.

III Linear Independence and Basis

a) A vector v in a vector space V is called a linear combination of vector $v_1, v_2, v_3, \cdots, v_m$, iff we can find numbers $x_1, x_2, x_3, \cdots, x_m$ such that

$$v = x_1 v_1 + x_2 v_2 + \cdots + x_m v_m.$$

b) If $S = \{v_1, v_2, v_3, \cdots, v_m\}$ is a set of m vectors, we say S is a linearly independent set iff

$$x_1 v_1 + x_2 v_2 + \cdots + x_m v_m = 0$$

holds only if

$$x_1 = x_2 = x_3 = \cdots = x_n = 0.$$

If S is not linearly independent, we call it a linearly dependent set.

c) Any set of vectors that contains the zero vector is a linearly dependent set. Also if $S = \{v_1, v_2\}$ has two vectors in it, then S is linearly independent iff $v_1 = cv_2$ for some number c.

d) A set $S\{v_1, v_2, v_3, \cdots, v_m\}$ is said to span V if any vector v in V is a linear combination of the vectors in S.

e) A set $S\{v_1, v_2, v_3, \cdots, v_m\}$ is said to be a basis for V if S is linearly independent and also spans V.

f) Any two bases of a vector space V have the same number of elements. This number is called the dimension of V and is denoted by $\dim \mathbf{V}$.

g) Suppose $\dim V = n$ and let $S = \{v_1, v_2, v_3, \cdots, v_m\}$ be a set of m vectors. If $m > n$, the S is linearly dependent. On the other hand, if $m < n$, then S does not span V. **Thus, a basis of V contains the maximum number of linearly independent vectors and the minimum number of vectors that can span V.**

h) If $v_1, v_2, v_3, \cdots, v_n$, are n vectors in R^n, then they are linearly independent iff $\det(A) \neq 0$, where A is the matrix whose columns are $v_1, v_2, v_3, \cdots, v_m$.

i) Let $V = R^m$ and let $S = \{v_1, v_2, v_3, \cdots, v_n\}$ Suppose $W = \text{Span}S$. Here is the procedure to find a basis of W consisting of elements of S.

Step 1 Construct a matrix A whose columns are the vectors $v_1,\ v_2,\ \cdots, v_n$.

Step 2 Apply **rref** to A.

Step 3 The vectors corresponding to the columns of containing the leading $1's$ form a basis for W.

j) Suppose $S = \{v_1, v_2, v_3, \cdots, v_n\}$ is a linearly independent set of vectors in V, where $\dim V = m$, and $n < m$. Here is a procedure to find a basis for V containing the elements of S.

Step 1 Let $T = \{v_1, v_2, v_3, \cdots, v_n, e_1, e_2, , \cdots, e_m\}$, where $e_1 = (1, 0, 0, \cdots, 0), e_2 = (0, 1, 0, \cdot 0), \cdots e_m = (0, 0, 0, \cdots 1)$.

Step 2 Apply the steps of (i) above to T.

IV Eigenvalues and Eigenvectors

a) Let A be an $n \times n$ matrix. We say that a number λ is an eigenvalue of A iff there exits a **nonzero** vector X such that $AX = \lambda X$. The vector X is called an eigenvector corresponding to the eigenvalue λ.

b) To find eigenvalues of a given matrix A, we solve its **characteristic** equation:

$$\det(A - \lambda I_n) = 0.$$

c) To find an eigenvector corresponding to an eigenvalue $lambda_1$, we solve the matrix equation

$$(A - \lambda_1 I_n)X = 0.$$

Apply rref to the coefficient matrix $A - \lambda_1 I_n$. At least one row at the bottom of the rref must be a zero row. Then you pick a nonzero solution from the infinitely many possible solutions!.

d) A matrix A is diagonalizable(that is , there exists a nonsingular matrix P and a diagonal matrix D such that $PAP^{-1} = D$) if all the roots of its characteristic polynomial are real and distinct. In fact, if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct real eigenvalues of A and $v_1, v_2, \cdot, v_n$ are eigenvectors corresponding to these eigenvalues, the we can take P to be the matrix whose columns are $v_1, v_2, \cdot, v_n$ and D is the diagonal matrix whose elements on the diagonal are $\lambda_1, \lambda_2, \cdots, \lambda_n$.

V Sqaure Matrices

1) The **row rank** of a matirx is defined to be the number of linearly independent row vectors of A, while the **column rank** of A is the number of linearly independent column vectors of A. It is known that $\mathbf{rowrank(A)} = \mathbf{columnrank(A)}$. The rank of a matix A is then defined to be the row rank of A.

2) The nullity of a matrix A is defined to be the dimension of the solution space of $AX = 0$. The rank and nullity of A are related by $rank(A) + Nullity(A) = n$.

3) If A is an $n \times n$ matrix, the following are equivalent.

a) A is invertible.
b) $AX = 0$ has only trivial solution.
c) The reduced row-echelon form of A is I_n.
d) $AX = b$ has a unique solution for any b.
e) $det(A) \neq 0$
f) The column vectors of A are linearly independent.
g) The row vectors of A are linearly independent.
h) The row vector of A span R^n.
i) The column vectors of A span R^n.
j) The column vectors of A form a basis for R^n.
k) The row vectors of A form a basis for R^n.
l) The rank of A is n.
m) The nullity of A is 0.
n) $\lambda = 0$ is **not** an eigenvalue of A.

NOTE: The web site http://www.rowan.edu/math/HASSEN/Mathematica/index.html contains a helpful manual for a TI 89(6 pages) for linear algebra.