Real Analysis II

Chapter 7 Calculus

7.1 Sets of Measure Zero

Definition 7.1A The subset E of $\mathbf{R}$ is said to be of **measure zero** if for each $\epsilon > 0$, there exists a finite or a countable number of open intervals $I_1,\ I_2, \cdots$ such that

$$E \subset \cup_n I_n \qquad \text{and} \qquad \sum_n |I_n| < \epsilon.$$

Theorem 7.1 B If each of the subsets $E_1,\ E_2,\ \cdots$ of $\mathbf{R}$ is of measure zero, then $\cup_n E_n$ is also of measure zero.

Corollary 7.1C Every countable set is of measure zero. In particular the set of rational numbers is of measure zero.

Definition 7.1D A statement is said to **hold at almost every point of** $[a, b]$ (or **almost everywhere in** $[a, b]$), if the set of points of $[a, b]$ at which the statement does not hold is of measure zero.

Notation It is common to write a.e for almost everywhere. Thus "f if continuous almost everywhere in $[a, b]$" can be written as "f is continuous a.e in $[a, b]$. "

7.2 Definition of the Riemann Integral

Definition 7.2A Let I be a bounded interval of real numbers and let f be a bounded function defined on I. We define $M[f; I]$ and $m[f; I]$by

$$M[f; I] = \sup_{x \in I} f(x)$$
$$m[f; I] = \inf_{x \in I} f(x)$$

Definition 7.2B A **subdivision** of the closed bounded interval $[a, b]$ we mean a finite subset $\{x_0,\ x_1,\ \cdots x_n$ of $[a, b]$ such that

$$a = x_0 < x_1 < x_2 < \cdots x_n = b.$$

If σ and τ are two subdivisions of $[a, b]$, we say that τ is a **refinement** of σ if $\sigma \subset \tau$.

If $\sigma = \{\{x_0,\ x_1,\ \cdots x_n\}$ is a subdivision of $[a, b]$, then the closed intervals

$$I_1 = [x_0,\ x_1],\ \ I_2 = [x_1,\ x_2],\ \ \cdots\ \ I_n = [x_{n-1},\ x_n]$$

are called the **components** of σ.

Definition 7.2C Let f be a bounded function on $[a, b]$ and let σ be a subdivision of $[a, b]$ with components $I_1,\ I_2,\ \cdots I_n$. We define $U[f; \sigma]$, called the **upper sum for f corresponding to** σ, by

$$U[f; \sigma] = \sum_{k=1}^{n} M[f; I_k]|I_k|.$$

Here $|I_k|$ is the length of I_k which is given by $I_k = x_k - x_{k-1}$.

The **lower sum for f corresponding to** σ, denoted by $L[f;\sigma]$ is defined by

$$L[f;\sigma] = \sum_{k=1}^{n} m[f;I_k]|I_k|.$$

Remarks. 1) For any subdivision σ we have $L[f;\sigma] \le U[f;\sigma]$

2) If f is continuous and positive on $[a,b]$, then $U[f;\sigma]$ is the area of the n rectangles each of which has one the I_k as base and whose height is equal to the maximum value of the function on the interval I_k. In other words, $U[f;\sigma]$ is the sum of the areas of the "circumscribed rectangles". Similarly, $L[f;\sigma]$ is the sum of the area of the "inscribed rectangles". (Draw a graph to show this)

Lemma 7.2D Let f be a bounded function on $[a,b]$. Then every upper sum for f is greater than or equal to every lower sum for f. That is, if τ and σ are any two subdivisions of $[a,b]$, then

$$U[f;\sigma] \ge L[f;\tau].$$

Remarks. 1) It follows that

$$g.l.b\, U[f;\tau] \ge l.u.b\, L[f;\sigma]$$

where the g.l.b and the l.u.b are taken over all subdivisions of $[a,b]$. (This assumption will be in effect throughout the chapter.)

Definition 7.2E Let f be a bounded function on $[a,b]$. We define the **upper integral of f over** $[a,b]$ by

$$\overline{\int_a^b} f(x)\,dx = g.l.b\, U[f;\tau]$$

We define the **lower integral of f over** $[a,b]$ by

$$\underline{\int_a^b} f(x)\,dx = l.u.b\, L[f;\tau]$$

It is common to denote upper integrals and lower integrals of f, respectively, by

$$\overline{\int_a^b} f \quad \text{and} \quad \underline{\int_a^b} f.$$

Remarks. 1) For any bounded function f on $[a,b]$ we have

$$\underline{\int_a^b} f \le \overline{\int_a^b} f.$$

For continuous functions, we will show that upper integral and lower integrals are equal.

2) If f is denied on $[0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

then

$$\overline{\int_0^1} f = 1 \quad \text{and} \quad \underline{\int_0^1} f = 0.$$

Definition 7.2F If f is bounded on $[a, b]$, we say that f is **Riemann integrable on** $[a, b]$ if

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

In this case, we define the **Riemann integral of f over** $[a, b]$ as

$$\int_a^b f(x)\,dx = \int_a^b f = \underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

The class of all Riemann integrable functions is doted by $\mathcal{R}[a, b]$

Theorem 7.2G Let f be a bounded function on $[a, b]$. Then $f \in \mathcal{R}[a, b]$ if and only if, for each $\epsilon > 0$, there exists a subdivision σ of $[a, b]$ such that

$$U[f; \sigma \ < \ L[f; \sigma] + \epsilon.$$

7.3 Existence of the Riemann Integral

Theorem 7.3A Let f be a bounded function on $[a, b]$. Then $f \in \mathcal{R}[a, b]$ if and only if f is continuous at almost every point of $[a, b]$.

7.4 Properties of the Riemann Integral

Theorem 7.4 A If $f \in \mathcal{R}[a, b]$ and $a < c < b$, then $f \in \mathcal{R}[a, c]$, $f \in \mathcal{R}[c, b]$ and

$$\int_a^b f = \int_a^c f \ + \ \int_c^b f.$$

Theorem 7.4B If $f \in \mathcal{R}[a, b]$ and λ is any real number, then $\lambda f \in \mathcal{R}[a, b]$ and

$$\int_a^b \lambda f = \lambda \int_a^b f.$$

Theorem 7.4C If $f, \ g \in \mathcal{R}[a, b]$, then $f + g \in \mathcal{R}[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f \ + \ \int_a^b g.$$

Lemma 7.4D If $f \in \mathcal{R}[a, b]$ and if

$$f(x) \geq 0$$

almost everywhere in $[a, b]$, then

$$\int_a^b f \geq 0.$$

Corollary 7.4E If $f \in \mathcal{R}[a,b]$ and if

$$f(x) \le g(x)$$

almost everywhere in $[a,b]$, then

$$\int_a^b f \ge\le \int_a^b g.$$

Corollary 7.4F If $f \in \mathcal{R}[a,b]$, then If $|f| \in \mathcal{R}[a,b]$ and

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Remark If $b < a$, then we define

$$\int_a^b f \quad \text{to be} \quad -\int_b^a f.$$

7.5 Derivative

Definition 7.5A Let f be defined on an interval J. If $c \in J$, we say that **f has a derivative at** c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. If this limit exists, we denote it by $f'(c)$. We also say that f is **differentiable at** c**.** We may also say that $f'(c)$ **exists.** Note that the above limit could be restated as

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Theorem 7.5B If f has a derivative at c, then f is continuous at c.

Remark If f is not continuous at c then f is not differentiable at c. On the other hand, the function $f(x) = |x|$ is continuous at $c = 0$ but not differentiable at $c = 0$.

Theorem 7.5 C If f and g both have derivative at c, then do $f + g$, $f - g$, fg, and

$$\begin{aligned} (f+g)'(c) &= f'(c) + g'(c) \\ (f-g)'(c) &= f'(c) - g'(c) \\ (fg)'(c) &= f'(c)g(c) + f(c)g'(c) \end{aligned}$$

Furthermore, if $g(c) \neq 0$, then f/g has derivative at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Theorem 7.5 D(Chair Rule) If g has derivative at c and f has derivative at $g(c)$, then $f \circ g$ has derivative at c and

$$(f \circ g)'(c) = f'(g(c))g'(c).$$

Theorem 7.5 E Let f be one-to-one function and let ϕ be the inverse of f. If f is continuous at c and if ϕ has derivative at $d = f(c)$ with $\phi'(d) \neq 0$, then $f'(c)$ exists and

$$f'(c) = \frac{1}{\phi'(d)}.$$

7.6 Rolle's Theorem

Theorem 7.6 A Let f be a continuous function on $[a, b]$. If the maximum value of f is attained at $c \in (a, b)$ and if $f'(c)$ exists, then $f'(c) = 0$.

Theorem 7.6 B Let f be a continuous function on $[a, b]$. If the minimum value of f is attained at $c \in (a, b)$ and if $f'(c)$ exists, then $f'(c) = 0$.

Theorem 7.6 C (Rolle's Theorem) Let f be continuous on $[a, b]$ with $f(a) = f(b) = 0$. If $f'(x)$ exists for every $x \in (a, b)$, then there exists some point $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 7.6 E If f has derivative at every point of $[a, b]$, then f' takes on every value between $f'(a)$ and $f'(b)$.

7.7 The Mean Value Theorem

Theorem 7.7 A (The Mean Value Theorem If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 7.7 B If f is continuous on $[a, b]$ and differentiable on (a, b) and if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.

Theorem 7.7 C (Generalized Mean Value Theorem If f, g are continuous on $[a, b]$, $g(a) \neq g(b)$, if f, g are both differentiable on (a, b), and if $f'(x)$ and $g'(x)$ are not both zero for any $x \in (a, b)$, then there exists a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

7.8 The Fundamental Theorem of Calculus

Theorem 7.8 A (The First Fundamental Theorem of Calculus) If f is continuous on $[a, b]$, and if

$$F(x) = \int_a^x f(t)\, dt,$$

for each $x \in (a, b)$, then $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem 7.8 C If $f'(x) = 0$ for every $x \in (a, b)$, then $f(x) = f(a)$ for all $x \in [a, b]$.

Theorem 7.8 D If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists a constant C such that

$$f(x) = g(x) + C \qquad \text{for all } x \in [a, b].$$

Theorem 7.8 E (The Second Fundamental Theorem of Calculus) If f is a continuous function on $[a, b]$ and if

$$\Phi'(x) = f(x) \qquad \text{for all } x \in (a, b),$$

then

$$\int_a^b f(x)\, dx = \Phi(b) - \Phi(a).$$

Theorem 7.8 G Let ϕ be a function on $[a, b]$ such that ϕ' is continuous on $[a, b]$. Let $A = \phi(a)$ and $B = \phi(b)$. Then, if f is continuous on $\phi([a, b])$, we have

$$\int_A^B f(x)\, dx = \int_a^b f[\phi(u)]\phi'(u)\, du.$$

7.9 Improper Integral

Definition Suppose $f \in \mathcal{R}[a, s]$ for every $s > a$. We define

$$F(s) = \int_a^s f(x)\, dx.$$

We say

$$\int_a^\infty f(x)\, dx$$

is convergent to A if

$$\lim_{s \to \infty} F(s) = A$$

If $\int_a^\infty f(x)\, dx$ does not converge, we say it **diverges**.

Example

$$\int_1^\infty \frac{1}{x^2} dx \quad \text{convegres while} \quad \int_1^\infty \frac{1}{\sqrt{x}} dx \quad \text{diverges.}$$

Remark. If

$$\int_a^\infty f \qquad \text{and} \qquad \int_a^\infty g$$

converge, the so does

$$\int_a^\infty (f \pm g),$$

and

$$\int_a^\infty (f \pm g) = \int_a^\infty f \;\pm\; \int_a^\infty g$$

We also have

$$\int_a^\infty \lambda f = \lambda \int_a^\infty f.$$

Definition If $f \in \mathcal{R}[a, s]$ for every $s > a$ and if

$$\int_a^\infty |f(x)|\, dx$$

convegres, we say that

$$\int_a^\infty f$$

converges absolutely

If

$$\int_a^\infty f$$

convegres but

$$\int_a^\infty |f|$$

divegres, we sat that **converges conditionally**.

Example The improper integral

$$\int_\pi^\infty \frac{\sin x}{x}\, dx$$

converges conditionally. (For the proof see page 213-214 of the text.)

Theorem 7.9 A The improper integral

$$\int_1^\infty \frac{1}{x}\, dx$$

diveges.

Theorem 7.9 B Let f be a decreasing fucntion on $[1, \infty]$ such that $f(x) \geq 0$ for all x. Then

$$\int_1^\infty f$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

convegres.

Definition We define

$$\int_{-\infty}^a f(x)\, dx = \lim_{s \to \infty} \int_{-s}^a f(x)\, dx$$

provided the limit exists. If the limit exists we say

$$\int_{-\infty}^a f$$

converges and if the limit does not exist, we say the

$$\int_{-\infty}^a f$$

diverges.

7.10 Improper Integral (Continued)

Defintion Suppose $f \in\mathcal{R}[a+\epsilon, b]$ for every ϵ such that $0 < \epsilon < b-a$ but $f \not\in\mathcal{R}[a, b]$. Define

$$F(\epsilon) = \int_{a+\epsilon}^{b} f(x)\, dx.$$

We say the **improper integral**

$$\int_a^b f(x)\, dx$$

converges to A if

$$\lim_{\epsilon\to 0} F(\epsilon) = A.$$

If the limit does not exist, we say

$$\int_a^b f$$

divegres.

An improper integral of the form

$$\int_a^b f$$

is called an **improper intgeral of second kind**.

Example

$$\int_0^1 (1/\sqrt{x}) dx \quad \text{converges while} \quad \int_0^1 (1/x^2) dx \quad \text{diverges.}$$

Theorem 7.10 A The improper intgeral

$$\int_0^1 \frac{1}{x}$$

is divergent.

Example Show that the improper intgeral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

is convergenet.

Remrak There are other types of improper integrals that do not fall under either of the first or the second kind. For example, the improper integral

$$\int_0^\infty \frac{1}{x^2}$$

is neither of fisrt kind or secon kind. Hoewever, we can break it up in the form

$$\int_0^\infty \frac{1}{x^2} = \int_0^1 \frac{1}{x^2} \quad \int_1^\infty \frac{1}{x^2}.$$

In general if, for a fixed $b > a$, $f \in\mathcal{R}[a+\epsilon, b]$ for every ϵ such that $0 < \epsilon < b-a$ but $f \not\in\mathcal{R}[a, b]$, we define

$$\int_a^\infty f(x)\, dx \;=\; \int_a^b f(x)\, dx \;+\; \int_b^\infty f(x)\, dx$$