Real Analysis II

Chapter Zero   Things Past

0.1   Real Number System

Assumption: We shall assume familiarity with the basic properties of the operations of addition and multiplication on the set of real numbers $\mathbf{R}$. In particular, we shall use freely the fact that $(\mathbf{R}, +, \cdot)$ is an **ordered field**. We also assume familiarity with operations (union, intersection, complement) on sets and the definition and operations of real-valued functions. In particular, you should read Section 1.1 through Section 1.4. The following theorems will be used freely.

Theorem 0. 1 a) A countable union of countable sets is countable. In particular, the set of rational numbers is countable.

b) For real numbers $a < b$, the interval (a, b) is uncountable. In particular, $\mathbf{R}$ is uncountable.

Note: For the proof of this theorem and further results on countable sets, please read Sections 1.5 and 1.6. We now review the important concepts of least upper bound and greatest lower bound of sets of real numbers. A detailed discussion can be found in Section 1.7.

Definition: 1) A subset $A \subset \mathbf{R}$ is said to be **bounded from above** if there is a number $N \in \mathbf{R}$ such that $x \le N$ for all $x \in A$. A subset $A \subset \mathbf{R}$ is said to be **bounded from below** if there is a number $M \in \mathbf{R}$ such that $M \le x$ for all $x \in A$. A subset $A \subset \mathbf{R}$ is said to be **bounded** if A is bounded from above and from below.

2) If $A \subset \mathbf{R}$, we say a real number N is an **upper bound** for A if $x \le N$ for all $x \in A$. If $A \subset \mathbf{R}$, we say a real number M is an **lower bound** for A if $M \le x$ for all $x \in A$.

3) Suppose $A \subset \mathbf{R}$ is bounded above. A number L is called the **least upper bound** of A if

1) L is an upper bound of A
2) no number smaller than L is an upper bound for A.

The least upper bound of A is denoted by $l.u.b$ or $l.u.b_{x\in A}\, x$ or $\sup A$.

4) Suppose $A \subset \mathbf{R}$ is bounded above. A number l is called the **greatest lower bound** of A if

1) l is an lower bound of A
2) no number greater than l is an lower bound for A.

The greatest lower bound of A is denoted by $g.l.b$ or $g.l.b_{x\in A}\, x$ or $\inf A$.

The Completeness Property of the Real Numbers If A is a nonempty subset of $\mathbf{R}$ that is bounded from above, then A has a least upper bound.

Theorem 0. 2 If A is a nonempty subset of $\mathbf{R}$ that is bounded from below, then A has the greatest lower

0.2 Sequences

Definition 1) In what follows, **I** will be the set of positive integers. A **sequence** $S = \{s_n\}_{n\in\mathbf{I}}$ is a function whose domain is **I**.

2) A **subsequence** of **I** is a strictly increasing function $N : \mathbf{I} \longrightarrow \mathbf{I}$. We write N_i instead of $N(i)$ and note that if $i < j$, then $N_i < N_j$.

3) Let $\{n_k\}$ be a subsequence of **I**. Then the sequence $\{s_{n_k}\}$ is called a **subsequence** of a sequence $\{s_n\}$.

4) We say the sequence $\{s_n\}$ **converges** to a real number s, if for every $\epsilon > 0$, there exists $N > 0$ such that

$$|s_n - s| < \epsilon \qquad \text{whenever} \quad n > N.$$

In this case we write

$$\lim_{n\to\infty} s_n = s \qquad \text{or} \qquad s_n \to s \quad \text{as} \ \ n \to \infty \qquad \text{or simply} \qquad s_n \to s$$

If $\{s_n\}$ does not converge, then we say it **diverges**.

5) A sequence $\{s_n\}$ is called **bounded** if there exists a number M such that $|s_n| \leq M$ for all $n \in I$.

6) We say a sequence $\{s_n\}$ is **increasing** if $s_n \leq s_{n+1}$ for all $n \in \mathbf{I}$. We say a sequence $\{s_n\}$ is **decreasing** if $s_n \geq s_{n+1}$ for all $n \in \mathbf{I}$. We say a sequence $\{s_n\}$ is **monotone** if it is either increasing or decreasing.

7) A sequence $\{s_n\}$ is called a **Cauchy** sequence if for every $\epsilon > 0$, there exists a real number N such that $|s_m - s_n| < \epsilon$ for every $n \geq m > N$.

Note: The following theorems will be used freely. For their proofs, please read Sections 2.1 through 2.7.

Theorem 0.4 If $s_n \to s$ and $t_n \to t$, then $s_n \pm t_n \to s \pm t$, $s_n t_n \to st$, and if $t_n \neq 0$ and $t \neq 0$, then $s_n/t_n \to s/t$.

Theorem 0.5 Every sequence has a monotone subsequence.

Theorem 0.6 If a sequence $\{s_n\}$ converges to s, then every subsequence of $\{s_n\}$ convergence to s.

*Theorem 0.7 (**Monotone Convergence Theorem**)* A bounded monotone sequence is convergent.

Theorem 0.8 A sequence $\{s_n\}$ converges if and only if $\{s_n\}$ is a Cauchy sequence.

0.3 Series of Real Numbers

<u>*Definition*</u> 1) A sum of the form

$$a_1 + a_2 + a_3 + \cdots$$

is called an **infinite series** and is denoted by $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$. The sum

$$s_n = a_1 + a_2 + \cdots a_n$$

is called the **nth partial sum** of the infinite series. We say the infinite series $\sum a_n$ **convergence** to a real number s if the sequence of partial sums $\{s_n\}$ converges to s. In this case, we write $s = \sum a_n$.

2) We say $\sum a_n$ **convergence absolutely** if $\sum |a_n|$ convergence. If $\sum a_n$ convergence but $\sum |a_n|$ diverges, then we say $\sum a_n$ **converges conditionally**.

3) Suppose $a_n > 0$ and $b_n > 0$. We say $\sum a_n$ is **dominated** by $\sum b_n$, if $a_n \leq b_n$.

<u>*Note:*</u> The following theorems will be used freely. For their proofs, please read Sections 3.1 through 3.6.

<u>*Theorem 0.9*</u> If $\sum a_n$ convergence, then $a_n \to 0$.

<u>*Theorem 0.10*</u> If $\sum a_n$ and $\sum b_n$ are convergent, then so are $\sum(a_n \pm b_n)$ and $\sum(ca_n)$.

<u>*Theorem 0.11*</u> Assume $a_n \geq 0$. If the sequences of partial sums $\{s_n\}$ is bounded, then $\sum a_n$ is convergent.

<u>*Theorem 0.12* **(Alternating Series Test)**</u> If $a_n > 0$, $\{a_n\}$ is decreasing, and $a_n \to 0$, then the **alternating series** $\sum(-1)^n a_n$ is convergent. Furthermore, if $s = \sum(-1)^n a_n$, then

$$|s - s_n| < a_{n+1}$$

for all n.

<u>*Theorem 0.13*</u> If $|x| < 1$, then $\sum x^n = 1/(1-x)$.

<u>*Theorem 0.14*</u> If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

<u>*Theorem 0.15* **(Comparison Test)**</u> Suppose $\sum a_n$ is dominated by $\sum b_n$. If $\sum b_n$ is convergent, then $\sum a_n$ converges. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

<u>*Theorem 0.16* **(Limit Comparison Test)**</u> Suppose $b_n \neq 0$ and let

$$L = \lim_{n\to\infty} \left(\frac{a_n}{b_n}\right).$$

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

<u>*Theorem 0.17* **(Ratio Test)**</u> Let

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Theorem 0.18 **(Root Test)** Let

$$r = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then

1) If $r < 1$, then $\sum a_n$ convergence absolutely.
2) If $r > 1$, then $\sum a_n$ diverges.
3) If $r = 1$, then no conclusion can be drawn.

Theorem 0.19 If $\sum a_n$ is conditionally convergent, then for every real number x, there is a rearrangement of $\sum a_n$ that converges to x.

Theorem 0.20 Let

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then

1) If $|x| < 1/r$, then $\sum a_n x^n$ convergence absolutely.
2) If $|x| > 1/r$, then $\sum a_n x^n$ diverges.
3) If $|x| = 1/r$, then no conclusion can be drawn about the convergence of $\sum a_n x^n$.

0.4 Continuous Functions

Definition 1) We say a function f has a limit L at $x = a$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{whenver} \qquad 0 < |x - a| < \delta.$$

In this case we write $\lim_{x\to a} f(x) = L$.

2) We say f is continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$.

3) We say a function f has a limit L at ∞ if and only if for every $\epsilon > 0$, there exists a real number $M > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{whenver} \qquad x > M.$$

In this case we write $\lim_{x\to\infty} f(x) = L$. We define $\lim_{x\to-\infty} f(x) = L$ in a similar manner. Give the technical definition of $\lim_{x\to a} = \pm\infty$ and also that of the right-hand and left-hand limits.

Note: The following theorems will be used freely. For their proofs, please read Section 4.1.

Theorem 0.21 $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}$ that converges to a the sequence $\{f(x_n)\}$ converges to L.

Theorem 0.22 If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

1) $\lim_{x\to a} (f(x) + g(x)) = L + M$

Theorem 0.23 If f and g are continuous at $x = a$, then so are $f \pm g$, fg, and f/g (provided $g(a) \neq 0$). Also the composition of continuous functions is continuous.

Theorem 0.24 A function f is continuous at $x = a$ if and only if for every sequence $\{x_n\}$ that converges to a the sequence $\{f(x_n)\}$ converges to $f(a)$.

Theorem 0.25 **(Extreme Value Theorem or EVT)** Let f be a continuous function on a closed bounded interval $[a, b]$. Then f attains both the minimum and maximum values on $[a, b]$. In other words, there are points $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

Theorem 0.26 **(Intermediate Value Theorem or IVT)** Let f be a continuous function on a closed and bounded interval $[a, b]$. If K is any number between $f(a)$ and $f(b)$, then there exists a point $c \in [a, b]$ such that $f(c) = K$. In particular, if $f(a)f(b) < 0$, then f has a root in $[a, b]$.