Chapter Seven: Linear Algebra: Matrices, Vectors, and Linear Systems

September 9, 2014

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For $n = 3$, we write

$$ax + by + cz = d$$

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$(s_1, s_2, s_3, \cdots s_n)$ is a **solution** of $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ if

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Example $(2, 5, 4)$ is a solution of $3x - 4y + 5z = 6$.

Find a solution of $2x + 4y = 0$.

Definition

A **system of linear equations**:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
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$(s_1, s_2, s_3, \cdots s_n)$ is **solution** if

$$a_{1j}s_1 + a_{2j}s_2 + \cdots + a_{nj}s_n = b_j \quad \text{for all} \quad j.$$

When $m = n = 2$ we write

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

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Solve

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Augmented Matrices and Elementary Row Operations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \vdots & & \\ & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m. \end{bmatrix}$$

is called the **augmented matrix** of

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

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3) $$\begin{aligned} x+y+2z&=9 \\ 2x+4y-3z&=1 \\ 3x+6y-5z&=0 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

7.2 Matrices and Matrix Operations

DEFINITION An $m \times n$ (read as m by n) **matrix** is a rectangular array of mn numbers arranged in m horizontal **rows** and n vertical **columns**.

An $m \times n$ matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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For brevity, we write

$$A = [a_{ij}], \quad i = 1, 2, \cdots m; \ j = 1, 2, \cdots n.$$

More Terminology

If $m = n$, the matrix A is called a **square** matrix.

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The j-th row of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdots \\ a_{mj} \end{bmatrix}.$$

Examples

For each of the following matrices find the size, the rows, the columns, and entries.

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 21 & 3 \\ -3 & 15 & 6 \\ 9 & 2 & 0 \end{bmatrix}$$

$$F = [101] \quad G = \begin{bmatrix} 0 & 20 & 3 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 2 \\ 9 \\ 3 \\ 0 \end{bmatrix}$$

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Definition Two matrices are defined to be **equal** if they have the **same size** and their corresponding **entries are equal**.

Addition of Matrices

DEFINITION 3 If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$, then

$$A + B = [a_{ij} + b_{ij}] \qquad A - B = [a_{ij} - b_{ij}]$$

Scalar Multiplication

DEFINITION If $A = [a_{ij}]$ is $m \times n$, then

$$cA = [ca_{ij}]$$

Matrix Multiplication

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If $A = [a_{ij}]$ is $m \times r$ and $B = [b_{ij}]$ are $r \times n$, then

$$AB = [c_{ij}] \quad \text{where} \;\; c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}, \;\; \text{for each } i,\; j.$$

Examples

Given

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \\ 0 & 7 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 21 & 3 \\ -3 & 15 & 6 \\ 9 & 2 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 2 & -3 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad H = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$$

find the dimensions, the sum, and products of the matrices. If you cannot add or multiply any two, state the reason(s).

Transpose

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If $A = [a_{ij}]$ is $m \times n$, then $A^T = [a_{ji}]$ is $n \times m$

Algebraic Properties of Matrices

THEOREM

(a) $A + B = B + A$

(b) $A + (B + C) = (A + B) + C$

(c) $A(BC) = (AB)C$

(d) $A(B \pm C) = AB \pm AC$

(e) $(B \pm C)A = BA \pm CA$

Properties of Matrix Addition and Multiplication

(f) $\alpha(B \pm C) = \alpha B \pm \alpha C$

(g) $(\alpha \pm \beta)(B = \alpha B \pm \beta B$

(h) $\alpha(B \pm C) = \alpha B \pm \alpha C$

(1) $\alpha(\beta C) = (\alpha\beta)C$

(j) $\alpha(BC) = (\alpha B)C = B(\alpha C)$

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3) AB and BA may both be defined and have the same size, but the two matrices may be different.

Properties of the Transpose

THEOREM If the sizes of the matrices are such that the stated operations can be performed, then:

(a) $\left(A^T\right)^T = A$

(b) $(A \pm B)^T = A^T \pm B^T$

(c) $(kA)^T = kA^T$

(d) $(AB)^T = B^T A^T$

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4) A square matrix $A = [a_{ij}]$ is lower triangular if and only if the jth column starts with at least $j - 1$ zeros for every j.

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c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

d) The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Symmetric Matrices

DEFINITION A square matrix A is said to be symmetric if $A = A^T$. In other words, $A = [a_{ij}]$ us symmetric if and only

$$a_{ij} = a_{ji} \quad \text{for all} i,\ j = 1, 2, \cdots, n.$$

Properties of Symmetric Matrices

Find A^T, B^T, $5A$, and $B - A$ if $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

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c) kA is symmetric.

System of Equations as Matrix Equation

A system

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m. \end{array}$$

can be represented as a matrix equation

$$AX = b$$

where

System of Equations as Matrix Equation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Examples

Express the following systems in matrix form:

1) $$\begin{gathered} x - 4y = -5 \\ 3x - y = 2 \end{gathered}$$

2) $$\begin{gathered} x + 2y = 1 \\ 2x + 4y = 4 \\ 3x - 4y = 8 \end{gathered}$$

3) $$\begin{gathered} x + y + 2z = 9 \\ x + 4y - 3z = 13 \end{gathered}$$

4) $$\begin{gathered} x - 4y + z = -5 \\ 3x - y = 2 \\ 2y - 3z = 4 \end{gathered}$$

5) $$\begin{gathered} x + 2y + 3z - w = 1 \\ 2x + 4y - 4w = 0 \\ 3x - 4y + z + w = 8 \\ x - w = 4 \end{gathered}$$

7.3 Linear Systems of Equations: Gauss Elimination

Elementary row operations on a matrix are

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Elementary row operations on a matrix are

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

Example

$$
\begin{array}{ccccccc}
x & + & y & + & 2z & = & 9 \\
2x & + & 4y & - & 3z & = & 1 \\
3x & + & 6y & - & 5z & = & 0
\end{array}
\longrightarrow
\begin{bmatrix}
1 & 1 & 2 & 9 \\
2 & 4 & -3 & 1 \\
3 & 6 & -5 & 0
\end{bmatrix}
$$

Example

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Add -2 times the first equation (row) to the second (row):

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$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 9 \\ & & 2y & - & 7z & = & -17 \\ 3x & + & 6y & - & 5z & = & 0 \end{array}$$

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$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 9 \\ 2x & + & 4y & - & 3z & = & 1 \\ 3x & + & 6y & - & 5z & = & 0 \end{array} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

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$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 9 \\ 2x & + & 4y & - & 3z & = & 1 \\ 3x & + & 6y & - & 5z & = & 0 \end{array} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

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$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 9 \\ & & 2y & - & 7z & = & -17 \\ 3x & + & 6y & - & 5z & = & 0 \end{array} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

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Or we could continue.....

Example

Add -1 times the second equation(row) to the first:

$$\begin{array}{ccccccc} x & & & + & \frac{11}{2}z & = & \frac{35}{2} \\ & & y & - & \frac{7}{2}z & = & -\frac{17}{2} \\ & & & & z & = & 3 \end{array}$$

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Add $-11/2$ times the third to the first **and** $7/2$ time the third to the second:

$$\begin{array}{ccccc} x & & & = & 1 \\ & y & & = & 2 \\ & & z & = & 3 \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

A matrix is said to be in **reduced row echelon form** if it satisfies the following properties:

1) The first nonzero number in the row is a 1 (a **leading** 1).

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$A = \begin{bmatrix} 1 & 4 & 2 & 6 \\ 0 & 1 & 6 & 8 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 2 & 8 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ are in row echelon forms but not reduced row echelon forms. (Why?)

Row Equivalent

Two matrices A and B are called **row equivalent** if one is obtained from the other by performing a series of elementary row operation.

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Note: If the augmented matrices of two linear systems are row equivalent, then the systems have the same solutions.

Examples

Which of the following are in reduced row echelon form?

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -17 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 8 & -3 & 2 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

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Note: Every $m \times n$ nonzero matrix can be reduced to a **unique** matrix that is in reduced row echelon form (or **rref**.

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Example Find the rref of the matrix

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 3 & 1 -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

Elimination Method

Example Solve the following using the method of elimination.

$$
\begin{aligned}
x + 2y + 3z &= 6 \\
2x - 3y + 2z &= 14 \\
3x + y - z &= -2
\end{aligned}
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We now apply elementary row operations to reduce it to a reduced row echelon form.

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Add -2 times row 1 to row 2 **and** -3 times row 1 to row 3:

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Add 7 times row 2 to row 3: $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{bmatrix}$

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The solution is $x =$

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The solution is $x = 1$, $y = -2$ and $z = 3$

Gauss-Jordan Elimination

Definition The method used in the above example is called **Gauss-Jordan Elimination** method.

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Example Use the Gauss-Jordan elimination method to solve the system whose augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 0 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

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Use Gauss-Jordan elimination method to the solve the homogeneous system:

$$\begin{array}{ccccccccccccc}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & + & 5x_3 & + & 2x_4 & + & 4x_5 & + & 3x_6 & = & 0 \\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 0 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 0
\end{array}$$

Theorem Every homogeneous system of linear equations is **consistent**. The solution

$$x_1 = 0, x_2 = 0, \cdots, x_n = O$$

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Note Any homogeneous system has either the trivial solution or infinitely many solutions.

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Example Solve

$$\begin{array}{ccccccccc} x & + & 2y & + & 3z & + & w & = & 0 \\ x & - & y & + & z & - & w & = & 0 \\ x & + & 5y & + & 5z & + & 3w & = & 0 \\ 5x & + & 4y & + & 11z & + & w & = & 0 \end{array}$$

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Theorem A homogeneous linear system with more unknowns than equations has infinitely many solutions.

Example Solve

$$\begin{array}{ccccccccc} x & + & y & + & z & + & w & = & 0 \\ x & & & & & + & w & = & 0 \\ x & + & 2y & + & z & & & = & 0 \end{array}$$

7.4 Linear Independence and Rank of a Matrix

DEFINITION A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ of vectors in $\mathbb{R}^n$ is called a **linearly independent set** if the only solution of the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

is the trivial solution

$$k_1 = 0,\ k_2 = 0,\ k_3 = 0,\ + \cdots +\ k_r = 0.$$

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If there are solutions in addition to the trivial solution, then S is said to be a **linearly dependent set**.

Example The most basic linearly independent set in $\mathbf{R}^n$ is the set of standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, 0, \cdots, 1)$$

EXAMPLES

Determine whether the vectors are linearly independent or linearly dependent

1. $\mathbf{v}_1 = (1, -2, 3)$, $\mathbf{v}_2 = (5, 6, -1)$, $\mathbf{v}_3 = (3, 2, 1)$ in $\mathbf{R}^3$.

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2. $\mathbf{v}_1 = (1, 2, -2, -1)$, $\mathbf{v}_2 = (4, 9, 9, -4)$, $\mathbf{v}_3 = (5, 8, 9, -5)$ in $\mathbf{R}^4$.

Some Properties of Linear Independence

THEOREM (a) A set S with two or more vectors is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

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THEOREM Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ be a set of vectors in $\mathbf{R}^n$. If $r > n$, then S is linearly dependent.

Row Space, Column Space, and Null Space

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The solution space of the homogeneous system of equations $AX = 0$, which is a subspace of $\mathbf{R}^n$, is called the **null space of** A.

Let $\mathbf{b} = (1, -9, -3)$ and $A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}$.

Show that $\mathbf{b}$ is in the column space of A.

THEOREM

If X_0 is any solution of a consistent linear system $AX = b$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a basis for the null space of A,then every solution of $AX = b$ can be expressed in the form

$$X = X_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

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$$X = X_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Conversely, for all choices of scalars $c_1, c_2, \cdots, c_k$, the vector X in this formula is a solution of $AX = b$.

Example Find the general solution of $AX = b$, where

$$\mathbf{b} = (0, -1, 5, 6) \text{ and } A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}.$$

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Example Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

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Apply REF to get $R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

Rank and Nullity

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THEOREM If A is a matrix with n columns, then

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n.$$

7.7 Determinants and Cramer's Rule

DEFINITION 1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The we define the **determinant** of A by

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

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3) The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the **cofactor** of entry a_{ij} .

Example

Find the minors and cofactors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

Definition of a General Determinant

THEOREM If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

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The **expansion along the** i**th row** is given by

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik}, \quad \text{for any } i.$$

Determinant of a Triangular Matrix

THEOREM If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Evaluating Determinants by Row Reduction

THEOREM Let A be a square matrix. If A has a row of zeros or a column of zeros, then

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Elementary Row Operations

THEOREM (a) Let A be an $n \times n$ matrix and if B is the matrix that results when a single row or single column of A is multiplied by a scalar k , then $\det(B) = k\det(A)$. That is

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

Elementary Row Operations

THEOREM (b) Let A be an $n \times n$ matrix and if B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$. That is

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & \vdots & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

Elementary Row Operations

THEOREM Let A be an $n \times n$ matrix.

(c) If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then

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Find $\begin{vmatrix} 3 & 6 & -9 & 15 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 1 & 1 & 4 \end{vmatrix}$

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Find $\begin{vmatrix} 3 & 6 & -9 & 15 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 1 & 1 & 4 \end{vmatrix}$ $\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$

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Example If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, then find

$$\begin{vmatrix} 2a & b & c \\ 2g & h & i \\ 2d & e & f \end{vmatrix}$$

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$$\begin{vmatrix} 2a & b & c \\ 2g & h & i \\ 2d & e & f \end{vmatrix} \qquad \begin{vmatrix} 3a & 3b & 3c \\ d+a & e+b & f+c \\ 4g & 4h & 4i \end{vmatrix}$$

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Properties of Determinants

THEOREM If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B).$$

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Example If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $E_1 = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$,

then find E_1A, E_2A, E_3A and their determinants.

Properties of Determinants

THEOREM A square matrix A is invertible if and only if

$$\det(A) \neq 0.$$

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Example Is $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$ invertible?

Properties of Determinants

THEOREM If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\,\det(B).$$

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Example Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
compute AB and the determinants of A, B, and AB.

Properties of Determinants

THEOREM 2.3.5 If A is invertible, then

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}.$$

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Cramer's Rule

If $AX = b$ is a system of n linear equations inn unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is given by

$$x_1 = \frac{\det(A_1)}{\det(A)} \quad x_2 = \frac{\det(A_2)}{\det(A)} \quad \cdots \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix b.

7.8. Inverse of a Matrix

Given

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -6 & 2 \\ 0 & 0 \end{bmatrix},$$

find AB.

Identity Matrices

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An $n \times n$ identity matrix is denoted by I_n.

THEOREM If R is the *reduced row echelon form* of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n.

Inverse of a Matrix

DEFINITION If A is a square matrix, and if a matrix B of the same size can be found such that

$$AB = I_n = BA$$

then A is said to be **invertible (or nonsingular)** and B is called an **inverse** of A.

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The inverse of A is denoted by A^{-1}.

If no such matrix B can be found, then A is said to be **singular (or non-invertible)**.

Which of these matrices have inverses?

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -6 & 2 & 0 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

Properties of Inverses

THEOREM If B and C are both inverses of the matrix A, then $B = C$.

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THEOREM 1 The matrix $A = \begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}$ is nonsingular if and only if

$$ad - bc \neq 0$$

and if this is the case, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} \text{d} & \text{-b} \\ \text{-c} & \text{a} \end{bmatrix}$$

Properties of Inverses

THEOREM If A and B are invertible matrices with the same size, then AB is invertible and

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Inversion Algorithm

To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

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Example Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

More on Linear Systems and Invertible Matrices

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EXAMPLE Find the solution of

$$\begin{array}{rcrcrcr} x & + & 3y & + & z & = & 4 \\ 2x & + & 2y & + & z & = & -1 \\ 2x & + & 3y & + & z & = & 3 \end{array}$$

EXAMPLE Find the solution of the systems

$$\begin{cases} -x + 4y + z = 0 \\ x + 9y - 2z = 1 \\ 6x + 4y - 8z = 0 \end{cases} \quad \text{and} \quad \begin{cases} -x + 4y + z = -3 \\ x + 9y - 2z = 4 \\ 6x + 4y - 8z = -5 \end{cases}$$

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