
Chapter 7 Techniques of Integration

■ 7.1 Numerical Integration

Students should read Section 7.1 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

Numerical integration is the process of approximating a definite integral using appropriate sums of function values. We already saw in Chapter 5 of this text formulas for Right, Left, and Midpoint Rules and their subroutines **LRSUM**, **RRSUM**, and **MRSUM**, respectively. In this section, we will develop two additional rules: the Trapezoidal Rule and Simpson's Rule.

■ 7.1.1 Trapezoidal Rule

The Trapezoidal Rule approximates the definite integral $\int_a^b f(x) dx$ by using areas of trapezoids and is given by the formula:

$$T_n = .5 ((b - a)/n) (y_0 + 2 y_1 + \dots + 2 y_{n-1} + y_n)$$

where n is the number of trapezoids and $y_i = f(a + i(b - a)/n)$. This formula can be found in your calculus text. Here is a *Mathematica* subroutine, called **TRAP**, for implementing the Trapezoidal Rule:

```
In[1]:= Clear[f, a, b, n]
```

```
In[2]:= TRAP[a_, b_, n_] :=  
  (f[a] + 2 Sum[f[a + i * (b - a) / n], {i, 1, n - 1}] + f[b]) (.5 (b - a) / n)
```

Example 7.1. Calculate the area under the function $f(x) = x^2$ on $[0, 1]$ using the Trapezoidal Rule for various values of n .

Solution: The following output gives a table of approximations of $\int_0^1 x^2 dx$ based on the Trapezoidal Rule for $n = 10, 20, \dots, 100$.

```
In[3]:= f[x_] := x^2  
TableForm[Table[{n, N[TRAP[0, 1, n]]}, {n, 10, 100, 10}],  
  TableHeadings -> {{}, {"n", "T_n"}}]
```

Out[4]/TableForm=

n	T_n
10	0.335
20	0.33375
30	0.333519
40	0.333438
50	0.3334
60	0.33338
70	0.333367
80	0.333359
90	0.333354
100	0.33335

It is clear that these values are converging to $1/3$, which is the exact value of our definite integral:

```
In[5]:= ∫01 x2 dx
Out[5]:= 1/3
```

■ 7.1.2 Simpson's Rule

One difference between Simpson's Rule and all the other rules we have developed so far (**TRAP**, **LRSUM**, **RRSUM**, and **MRSUM**) is that the number of partition points, n , in this case, must be even. The other difference is that Simpson's Rule is a quadratic approximation based on parabolas, whereas the other rules are linear approximations based on lines. The formula for Simpson's Rule is given by (refer to your calculus text for details):

$$S_n = (1/3) [y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n] (b-a)/n$$

$$= (1/3) [(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \dots + (y_{n-2} + 4y_{n-1} + y_n)] (b-a)/n$$

where $y_i = f(a + i(b-a)/n)$. Here is a *Mathematica* subroutine, called **SIMP**, for implementing Simpson's Rule:

```
In[6]:= Clear[a, b, n]

In[7]:= SIMP[a_, b_, n_] :=
  (1/3) Sum[f[a + (2 i - 2) (b - a) / n] + 4 f[a + (2 i - 1) (b - a) / n] +
    f[a + 2 i (b - a) / n], {i, 1, n/2}] (b - a) / n
```

Example 7.2. Calculate the area under the function $f(x) = x^2$ on $[0, 1]$ using Simpson's Rule for various values of n .

Solution: We use the same set of values of n as in the previous example. This will allow us to compare Simpson's Rule with the Trapezoidal Rule.

```
In[8]:= f[x_] := x2
TableForm[Table[{n, N[SIMP[0, 1, n]]}, {n, 10, 100, 10}],
  TableHeadings -> {{}, {"n", "sn"}]]
```

Out[9]/TableForm=

n	s _n
10	0.333333
20	0.333333
30	0.333333
40	0.333333
50	0.333333
60	0.333333
70	0.333333
80	0.333333
90	0.333333
100	0.333333

Notice how fast **SIMP** converges to the actual value of the integral (1/3) compared to **TRAP**.

Example 7.3. Calculate the definite integral of $f(x) = \sin(25x^2)$ on $[0, 1]$ using Simpson's Rule and approximate it to five decimal places. What is the minimum number of partition points needed to obtain this level of accuracy?

Solution: We first evaluate **SIMP** using values for n in increments of 20.

```
In[10]:= f[x_] := Sin[25 x^2]
TableForm[Table[{n, N[SIMP[0, 1, n]]}, {n, 20, 200, 20}],
  TableHeadings -> {{}, {"n", "s_n"}}]
```

Out[11]/TableForm=

n	s_n
20	0.0958943
40	0.10526
60	0.105526
80	0.105566
100	0.105576
120	0.10558
140	0.105582
160	0.105582
180	0.105583
200	0.105583

Based on the output our approximation, accurate to five decimal places, is 0.10558. This first occurs between $n = 100$ to $n = 120$. We evaluate SIMP inside this range to zoom in on the minimum number of partition points needed.

```
In[12]:= f[x_] := Sin[25 x^2]
TableForm[Table[{n, N[SIMP[0, 1, n]]}, {n, 100, 120, 2}],
  TableHeadings -> {{}, {"n", "s_n"}}]
```

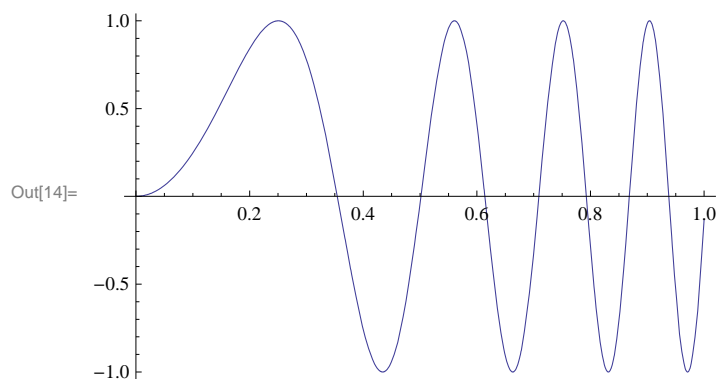
Out[13]/TableForm=

n	s_n
100	0.105576
102	0.105577
104	0.105577
106	0.105578
108	0.105578
110	0.105579
112	0.105579
114	0.105579
116	0.10558
118	0.10558
120	0.10558

Thus, we see that the minimum number of points needed is $n = 116$. How does this compare with the minimum number of points needed by TRAP to obtain the same level of accuracy?

NOTE: Observe that SIMP does not converge as fast in this example as in the previous example. This is because the function $f(x) = \sin(25x^2)$ is oscillatory as the following graph demonstrates:

```
In[14]:= Plot[f[x], {x, 0, 1}]
```



Try increasing the frequency of this function, say to $\sin(100x^2)$, to see how well SIMP performs.

■ 7.1.3 Midpoint Rule

Since most calculus texts include again the Midpoint Rule in the section on numerical integration, for completeness, we will too. The Riemann sum using the midpoints of each subinterval is given by the following formula:

```
In[15]:= Clear[f]
MRSUM[a_, b_, n_] := Sum[f[a + (i - 1 / 2) * (b - a) / n] * (b - a) / n, {i, 1, n}]
```

Example 7.4. Calculate the area under the function $f(x) = x^2$ on $[0, 1]$ using the Midpoint Rule for various values of n .

Solution:

```
In[17]:= f[x_] := x^2
TableForm[Table[{n, N[MRSUM[0, 1, n]]}, {n, 10, 100, 10}],
  TableHeadings -> {{}, {"n", "Midpoint rule"}}]
```

Out[18]//TableForm=

n	Midpoint rule
10	0.3325
20	0.333125
30	0.333241
40	0.333281
50	0.3333
60	0.33331
70	0.333316
80	0.33332
90	0.333323
100	0.333325

■ Exercises

1. Consider the definite integral $\int_1^2 \ln(x) dx$.

a) Using the Trapezoidal Rule, Simpson's Rule, and Midpoint Rule, approximate this integral for $n = 10, 20, \dots, 100$.

b) Compare how fast each subroutine (**TRAP**, **SIMP**, **MRSUM**) converges to $\int_a^b \ln(x) dx$ and decide which of these rules is "best."

2. Repeat Exercise 1 for the following definite integrals:

a) $\int_0^2 \frac{e^x}{x+1} dx$ b) $\int_0^1 \cos(x^2) dx$ c) $\int_0^1 e^{x^2} dx$

Can you make any general conclusions about which rule (Trapezoidal, Simpson's, Midpoint) is best?

3. For each of the functions given below, set up a definite integral for the volume of the solid of revolution obtained by revolving the region under $f(x)$ along the given interval and about the given axis. Then use the subroutines **TRAP**, **SIMP**, and **MRSUM** to approximate the volume of each solid accurate to two decimal places (use various values of n to obtain the desired accuracy).

a) $f(x) = \cos x$; $[0, \pi/2]$; x -axis b) $f(x) = e^{-x^2}$; $[0, 1]$; y -axis c) $f(x) = \sin x$, $[0, \pi]$, x -axis

■ 7.2 Techniques of Integration

Students should read Sections 7.2 through 7.4 and 7.6 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

All calculus texts have at least a chapter devoted to "Techniques of Integration." When using *Mathematica*, these techniques are usually not necessary since *Mathematica* automatically gives you the answer.

■ 7.2.1 Substitution

On occasion, we do need to use techniques of integration, even when using *Mathematica*.

Example 7.5. Evaluate the following integral: $\int 2^x \sqrt{(2^x)^2 - 1} dx$.

Solution: We evaluate this integral in *Mathematica*:

$$\begin{aligned} \text{In[19]:= } & \int 2^x \sqrt{(2^x)^2 - 1} dx \\ \text{Out[19]= } & \frac{2^x \sqrt{-1 + 4^x} - \text{Log}[2^x + \sqrt{-1 + 4^x}]}{\text{Log}[4]} \end{aligned}$$

To students in a first-year calculus course, this answer makes no sense. There are many integrals that *Mathematica* cannot evaluate at all, or cannot evaluate in terms of elementary functions (such as the integral above). Some of these integrals are doable in terms we should understand, once we first use an appropriate technique of integration. In the above example, all we need to do is first make the following substitution: $u = 2^x$ and $du = (\ln 2) 2^x dx$, which transforms the integral to:

$$\begin{aligned} \text{In[20]:= } & \frac{1}{\text{Log}[2]} \int \sqrt{u^2 - 1} du \\ \text{Out[20]= } & \frac{\frac{1}{2} u \sqrt{-1 + u^2} - \frac{1}{2} \text{Log}[u + \sqrt{-1 + u^2}]}{\text{Log}[2]} \end{aligned}$$

This is the correct answer. All we need to do is substitute 2^x for u , and add the arbitrary constant of integration, getting:

$$\frac{1}{2 \text{Log}[2]} (2^x \sqrt{-1 + (2^x)^2} - \text{Log}[2^x + \sqrt{-1 + (2^x)^2}]) + C$$

Note that the *Mathematica* function $\text{Log}[x]$ is equivalent to the standard form $\ln x$.

■ 7.2.2 Trigonometric Substitution

Example 7.6. Evaluate $\int \frac{1}{x^2 \sqrt{x^2-9}} dx$.

Solution: By hand, the integral $\int \frac{1}{x^2 \sqrt{x^2-9}} dx$ would normally be evaluated with a trigonometric substitution of the form $x = 3 \sec \theta$. But with *Mathematica*, we can do this directly:

$$\text{In}[21]:= \int \frac{1}{x^2 \sqrt{x^2-9}} dx$$

$$\text{Out}[21]= \frac{\sqrt{-9+x^2}}{9x}$$

This, of course, is the correct answer, when we remember that *Mathematica* does not add an arbitrary constant to indefinite integrals.

■ 7.2.3 Method of Partial Fractions

Integrals of rational expressions often require the Method of Partial Fraction Decomposition to evaluate them (by hand). For example:

$$\int \frac{3x-3}{x^2+5x+4} dx = \int \left(\frac{5}{x+4} - \frac{2}{x+1} \right) dx = 5 \ln |x+4| - 2 \ln |x+1| = \ln \left| \frac{(x+4)^5}{(x+1)^2} \right|$$

On the other hand, *Mathematica* will give us essentially the same answer for this integral, but does its work behind the scenes without revealing its technique:

$$\text{In}[22]:= \text{Simplify} \left[\int \frac{3x-3}{x^2+5x+4} dx \right]$$

$$\text{Out}[22]= -2 \text{Log}[1+x] + 5 \text{Log}[4+x]$$

If we would like to see the partial fraction decomposition of the integrand, $\frac{3x-3}{x^2+5x+4}$, *Mathematica* will also do that for us without strain by using the **Apart** command:

$$\text{In}[23]:= \text{Apart} \left[\frac{3x-3}{x^2+5x+4} \right]$$

$$\text{Out}[23]= -\frac{2}{1+x} + \frac{5}{4+x}$$

Example 7.7. Evaluate $\int \frac{2x^3+x^2-2x+2}{(x^2+1)^2} dx$.

Solution: We simply evaluate this integral using *Mathematica*:

$$\text{In}[24]:= \int \frac{2x^3+x^2-2x+2}{(x^2+1)^2} dx$$

$$\text{Out}[24]= \frac{4+x}{2(1+x^2)} + \frac{3 \text{ArcTan}[x]}{2} + \text{Log}[1+x^2]$$

But again, if we would like to see the partial fraction decomposition of the integrand, $\frac{2x^3+x^2-2x+2}{(x^2+1)^2}$, then this is straightforward with *Mathematica*:

$$\begin{aligned} \text{In[25]:= } & \text{Apart}\left[\frac{2x^3 + x^2 - 2x + 2}{(x^2 + 1)^2}\right] \\ \text{Out[25]= } & \frac{1 - 4x}{(1 + x^2)^2} + \frac{1 + 2x}{1 + x^2} \end{aligned}$$

■ Exercises

1. Evaluate $\int (1 + \ln(x)) \sqrt{1 + (x \ln(x))^2} \, dx$ with *Mathematica*. If it doesn't give an understandable answer, use a technique of integration that changes the integral into one that *Mathematica* will evaluate.

In Exercises 2 through 5, use *Mathematica* to find the partial fraction decomposition of the given functions and then integrate them:

2. $\frac{x^2+3x-44}{(x-3)(x+5)(3x-2)}$

3. $\frac{3x^2-4x+5}{(x-1)(x^2+1)}$

4. $\frac{25}{x(x^2+2x+5)}$

5. $\frac{10}{x(x^2+2x+5)^2}$

In Exercises 6 through 10, use *Mathematica* to evaluate the given integrals.

6. $\int \frac{x^2}{(x^2-4)^{3/2}} \, dx$

7. $\int x^3 \sqrt{9-x^2} \, dx$

8. $\int \frac{1}{\sqrt{25+x^2}} \, dx$

9. $\int \sin^5 x \, dx$

9. $\int \frac{\tan^{-1} t}{1+t^2} \, dt$

10. $\int \sinh^3 x \cosh x \, dx$

■ 7.3 Improper Integrals

Students should read Section 7.7 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

Recall that there are two types of improper integrals.

Type I: If we assume that $f(x)$ is integrable over $[a, b]$ for all $b \geq a$, then the improper integral of $f(x)$ over $[a, \infty)$ is defined as

$$\int_a^\infty f(x) \, dx = \lim_{t \rightarrow \infty} \int_a^t f(x) \, dx,$$

provided this limit exists. Similarly, we define

$$\int_{-\infty}^b f(x) \, dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) \, dx,$$

provided this limit exists.

Type II: If $f(x)$ is continuous on $[a, b)$ but discontinuous at $x = b$, we define

$$\int_a^b f(x) \, dx = \lim_{t \rightarrow b^-} \int_a^t f(x) \, dx,$$

provided this limit exists. Similarly, if $f(x)$ is continuous on $(a, b]$ but discontinuous at $x = a$,

$$\int_a^b f(x) \, dx = \lim_{t \rightarrow a^+} \int_t^b f(x) \, dx,$$

provided this limit exists. Finally, if $f(x)$ is continuous for all x on $[a, b]$ except at $x = c$, where $a < c < b$, we define

$$\int_a^b f(x) \, dx = \lim_{t \rightarrow c^-} \int_a^t f(x) \, dx + \lim_{t \rightarrow c^+} \int_t^b f(x) \, dx,$$

provided both of these limits exist.

By using the **Limit** command in *Mathematica* along with **Integrate**, *Mathematica* eliminates the drudgery of having to evaluate these integrals by hand.

Example 7.8. Evaluate the following improper integrals:

a) $\int_{20}^{\infty} \frac{1}{y} dy$

b) $\int_2^{\infty} e^{-2x} dx$

c) $\int_0^1 x \ln x dx$

d) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Solution:

a) We evaluate

$$\text{In[26]:= } \int_{20}^{\infty} \frac{1}{y} dy$$

Integrate::idiv : Integral of $\frac{1}{y}$ does not converge on $\{20, \infty\}$. >>

$$\text{Out[26]= } \int_{20}^{\infty} \frac{1}{y} dy$$

Thus, evaluating this integral directly using *Mathematica* tells us it does not exist. Alternatively, we could have used the limit definition:

$$\text{In[27]:= } \text{Limit} \left[\int_{20}^t \frac{1}{y} dy, t \rightarrow \infty \right]$$

$$\text{Out[27]= } \infty$$

Observe the difference in the two outputs above. Both correctly express the answer as divergent; however, the second answer is better since it reveals the nature of the divergence (infinity), which is the answer we would expect if solving this problem by hand.

b) We evaluate

$$\text{In[28]:= } \int_2^{\infty} e^{-2x} dx$$

$$\text{Out[28]= } \frac{1}{2 e^4}$$

Again, we obtain the same answer using the limit definition (as it should):

$$\text{In[29]:= } \text{Limit} \left[\int_2^t e^{-2x} dx, t \rightarrow \infty \right]$$

$$\text{Out[29]= } \frac{1}{2 e^4}$$

Mathematica will similarly handle discontinuities. In the following example, the function has a discontinuity at $x = 0$.

c) We evaluate

$$\text{In}[30]:= \int_0^1 \mathbf{x} \text{Log}[\mathbf{x}] \, d\mathbf{x}$$

$$\text{Out}[30]= -\frac{1}{4}$$

$$\text{In}[31]:= \text{Limit}\left[\int_t^1 \mathbf{x} \text{Log}[\mathbf{x}] \, d\mathbf{x}, t \rightarrow 0, \text{Direction} \rightarrow -1\right]$$

$$\text{Out}[31]= \text{ConditionalExpression}\left[-\frac{1}{4}, (t \notin \text{Reals} \mid 0 < \text{Re}[t] < 1 \mid \text{Re}[t] > 1) \&\& \left(\frac{t}{1-t} \notin \text{Reals} \mid \text{Re}\left[\frac{t}{1-t}\right] \geq 0 \mid \text{Re}\left[\frac{t}{1-t}\right] \leq -1\right)\right]$$

d) We evaluate

$$\text{In}[32]:= \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$$

$$\text{Out}[32]= \pi$$

Note that *Mathematica* does not require us to break the integral up into two integrals, which would be required according to its definition, if evaluated by hand. On the other hand, there is nothing wrong with dividing this integral into two in *Mathematica*:

$$\text{In}[33]:= \int_{-\infty}^0 \frac{1}{1+x^2} \, d\mathbf{x} + \int_0^{\infty} \frac{1}{1+x^2} \, d\mathbf{x}$$

$$\text{Out}[33]= \pi$$

NOTE: Observe that it does not matter where we divide the integral. It is valid to express $\int_{-\infty}^a \frac{1}{1+x^2} \, dx + \int_a^{\infty} \frac{1}{1+x^2} \, dx$ for the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$ for any real value a as long as they are convergent. However, evaluating this sum in *Mathematica* yields different expressions for the answer, which depend on the sign of a and whether it is real or complex. This is shown in the following output:

$$\text{In}[34]:= \text{Clear}[\mathbf{a}]$$

$$\int_{-\infty}^a \frac{1}{1+x^2} \, d\mathbf{x} + \int_a^{\infty} \frac{1}{1+x^2} \, d\mathbf{x}$$

$$\text{Out}[35]= \text{ConditionalExpression}\left[\frac{1}{2} \left(\pi + \mathbf{i} \text{Log}[1 + \mathbf{i} \mathbf{a}] - \mathbf{i} \left(\begin{cases} \text{Conjugate}[\text{Log}[1 - \mathbf{i} \mathbf{a}]] & \text{Re}[\mathbf{a}] == 0 \&\& \text{Im}[\mathbf{a}] < 0 \\ \text{Log}[1 - \mathbf{i} \mathbf{a}] & \text{True} \end{cases}\right)\right) + \frac{1}{2} \left(\pi + \mathbf{i} \text{Log}[1 - \mathbf{i} \mathbf{a}] - \mathbf{i} \left(\begin{cases} \text{Conjugate}[\text{Log}[1 + \mathbf{i} \mathbf{a}]] & \text{Re}[\mathbf{a}] == 0 \&\& \text{Im}[\mathbf{a}] > 0 \\ \text{Log}[1 + \mathbf{i} \mathbf{a}] & \text{True} \end{cases}\right)\right), -1 \leq \text{Im}[\mathbf{a}] \leq 1\right]$$

If instead, a is given a fixed value, then *Mathematica* will give us our answer of π :

In[36]:= **a = 1**

$$\int_{-\infty}^a \frac{1}{1+x^2} dx + \int_a^{\infty} \frac{1}{1+x^2} dx$$

Out[36]= 1

Out[37]= π

■ Exercises

In Exercises 1 through 8, evaluate the given improper integrals:

1. $\int_{-\infty}^4 e^{0.1t} dt$
2. $\int_{-3}^{\infty} \frac{1}{(x+4)^{3/2}} dx$
3. $\int_{-2}^4 \frac{1}{(x+2)^{1/3}} dx$
4. $\int_{-\infty}^{\infty} x e^{-x^2} dx$
5. $\int_0^3 \frac{1}{x-1} dx$
6. $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx$
7. $\int_1^{\infty} \frac{1}{x^{999}} dx$
8. $\int_1^{\infty} \frac{1}{x^{1.003}} dx$

11. Find the volume of the solid obtained by rotating the region below the graph of $y = e^{-x}$ about the x -axis for $0 \leq x < \infty$.

12. Determine how large the number b has to be in order that $\int_b^{\infty} \frac{1}{x^2+1} dx < .0001$.

13. Evaluate the improper integral $\int_{-1}^1 \frac{1}{\sqrt[3]{|x|}} dx$.

14. Determine how large the number b should be so that $\int_b^{\infty} \frac{1}{x^2+1} dx < .0001$.

15. Consider the function defined by

$$G(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

- a) Evaluate $G(n)$ for $n = 0, 2, 3, 4, \dots, 10$. Make a conjecture about these values. Verify your conjecture.
- b) Evaluate $G((2n-1)/2)$, for $n = 1, 2, 3, \dots, 10$. Make a conjecture about these values. Verify your conjecture.
- c) Plot the graph of $G(x)$ on the interval $[0, 5]$.

NOTE: The function G is called the *gamma* function and is denoted by $\Gamma[x]$. In *Mathematica* it is denoted by **Gamma[x]**. The gamma function was first introduced by Euler as a generalization of the factorial function.

■ 7.4 Hyperbolic and Inverse Hyperbolic Functions

Students should read Section 7.5 of Rogawski's Calculus [1] for a detailed discussion of the material presented in this section.

■ 7.4.1. Hyperbolic Functions

The *hyperbolic* functions are defined in terms of the exponential functions. They have a direct connection to engineering mathematics, including bridge construction. For example, cables from suspension bridges typically form a curve called a *catenary* (derived from the Latin word *catena*, which means chain) that is described by these functions.

The six hyperbolic functions are denoted and defined as follows:

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2}, & \cosh x &= \frac{e^x + e^{-x}}{2}, & \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ \coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}, & \operatorname{sech} x &= \frac{2}{e^x + e^{-x}}, & \operatorname{csch} x &= \frac{2}{e^x - e^{-x}} \end{aligned}$$

The reason these functions are called hyperbolic functions is due to their connection with the *equilateral hyperbola* $x^2 - y^2 = 1$. Here, one defines $x = \cosh t$ and $y = \sinh t$. Hence, one obtains the basic hyperbolic identity $\cosh^2 t - \sinh^2 t = 1$, much the same manner as the corresponding trigonometric identity $\cos^2 t + \sin^2 t = 1$, when one considers the unit circle $x^2 + y^2 = 1$ with $x = \cos t$ and $y = \sin t$.

In *Mathematica*, we use the same notation with the obvious convention that the first letter of each function is capitalized and square brackets must be used in place of parentheses. Thus, $\sinh x$ will be entered as **Sinh[x]**.

Example 7.9. Consider the hyperbolic sine function $f(x) = \sinh x$.

- Plot the graph of f .
- From the graphs deduce the domain and range of the function.
- Is f bounded?
- Does f attain an absolute minimum? Maximum?
- Repeat a) through d) for the hyperbolic function $g(x) = \cosh x$
- Repeat a) through d) for the hyperbolic function $h(x) = \tanh x$.

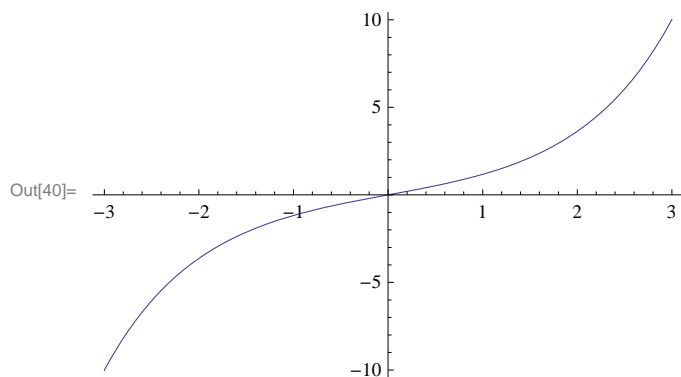
Solution: We begin by defining f in *Mathematica*:

```
In[38]:= Clear[f, x]
         f[x_] = Sinh[x]
```

```
Out[39]= Sinh[x]
```

- We next plot its graph on the interval $[-3, 3]$.

```
In[40]:= Plot[f[x], {x, -3, 3}]
```



- The preceding graph indicates that the domain and range of $\sinh x$ is $(-\infty, \infty)$. To convince yourself, you should plot the graph over wider intervals. We should also expect this from the definition of $\sinh x$ itself. Can you explain why?

- The function $\sinh x$ is not bounded. The graph earlier should not be used as a proof of this. However, we can evaluate its limit at $-\infty$ and ∞ to see that this is indeed true.

```
In[41]:= Limit[f[x], x -> -∞]
         Limit[f[x], x -> ∞]
```

```
Out[41]= -∞
```

```
Out[42]= ∞
```

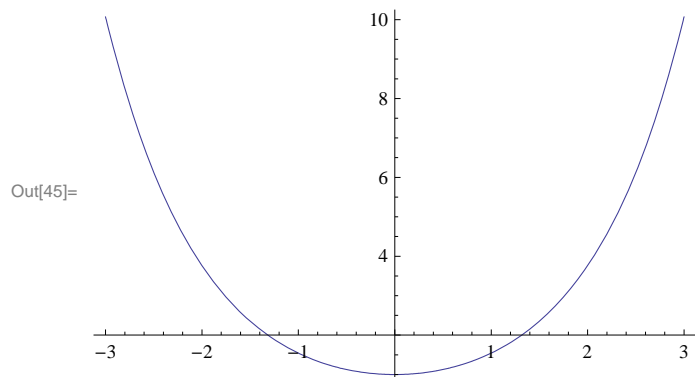
- The limits just computed show that $\sinh x$ has no absolute maximum or minimum since it is unbounded.

- Next, we consider the hyperbolic cosine function denoted by $\cosh x$.

```
In[43]:= Clear[g, x]
          g[x_] = Cosh[x]
```

```
Out[44]= Cosh[x]
```

```
In[45]:= Plot[g[x], {x, -3, 3}]
```



The preceding graph indicates that the domain of $\cosh x$ is $(-\infty, \infty)$. The range appears to be $[1, \infty)$. Can you prove this?

The hyperbolic cosine function, $\cosh x$, is not bounded from above. This can be seen from the following limits:

```
In[46]:= Limit[Cosh[x], x -> -∞]
          Limit[Cosh[x], x -> ∞]
```

```
Out[46]= ∞
```

```
Out[47]= ∞
```

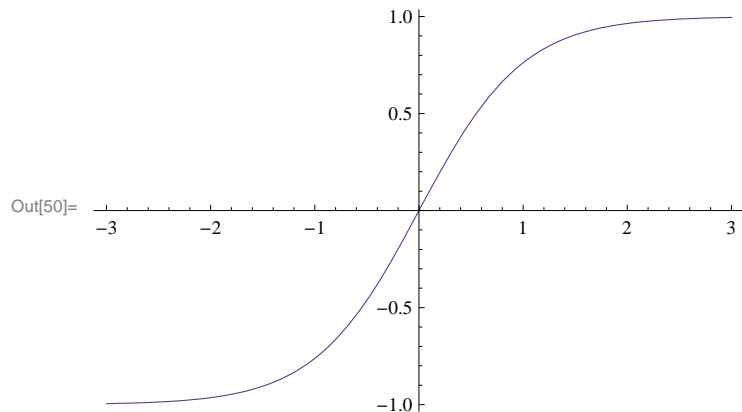
Again, since $\cosh x$ is not bounded from above, it follows that $\cosh x$ has no absolute maximum. As we have observed in part b) of this example, $\cosh x$ has absolute minimum value 1, attained at $x = 0$.

f) Finally, we consider the hyperbolic tangent function, $\tanh x$:

```
In[48]:= Clear[h, x]
          h[x_] = Tanh[x]
```

```
Out[49]= Tanh[x]
```

```
In[50]:= Plot[h[x], {x, -3, 3}]
```



Again, the preceding graph indicates that the domain of $\tanh x$ is $(-\infty, \infty)$. The range appears to be $(-1, 1)$. This can be seen from the following limits:

```
In[51]:= Limit[Tanh[x], x -> -∞]
          Limit[Tanh[x], x -> ∞]
```

```
Out[51]= -1
```

```
Out[52]= 1
```

The graph of $\tanh x$ also indicates that it is strictly increasing on its domain. This can be proven by showing that its derivative, which we will calculate later, is strictly positive. It is clear that $\tanh x$ has no absolute extrema.

NOTE: The reader will notice some similarities between the hyperbolic functions and the associated trigonometric functions. Moreover, if one studies the theory of functions of a complex variable, the relationship between these classes of transcendental functions becomes even more transparent; for numerous identities exist between the classes of functions.

■ 7.4.2 Identities Involving Hyperbolic Functions

It is immediate that the ratio and reciprocal identities for the hyperbolic functions coincide with their trigonometric counterparts. In fact, for each trigonometric identity, there is a corresponding (not necessarily the same) hyperbolic identity. Following are some examples.

Example 7.10. Show that the following identities hold true.

a) $1 - \tanh^2 x = \operatorname{sech}^2 x$ b) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

Solution:

a) We use the definitions for $\tanh x$ and $\operatorname{sech} x$ to express each side of the identity in terms of exponentials:

```
In[53]:= Simplify[(1 - Tanh[x]^2) /. Tanh[x] -> (E^x - E^(-x)) / (E^x + E^(-x))]
```

$$\text{Out[53]} = \frac{4e^{2x}}{(1 + e^{2x})^2}$$

```
In[54]:= Simplify[Sech[x]^2 /. Sech[x] -> 2 / (E^x + E^(-x))]
```

$$\text{Out[54]} = \frac{4}{(e^{-x} + e^x)^2}$$

We leave it for the reader to verify that both of these outputs agree, that is, $\frac{4e^{2x}}{(1+e^{2x})^2} = \frac{4}{(e^{-x}+e^x)^2}$ (cross-multiply and then simplify).

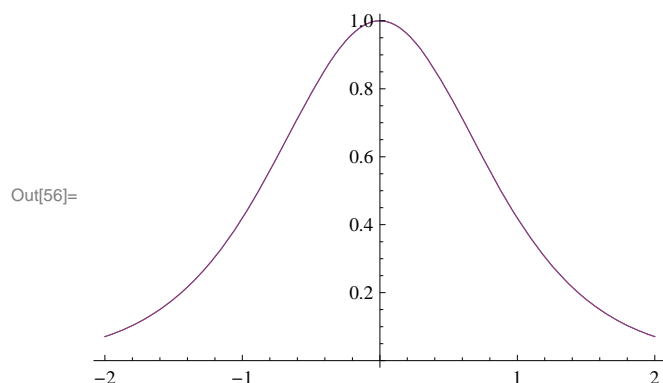
The identity can also be confirmed in *Mathematica* by evaluating the difference between its left- and right-hand sides, which should equal zero:

```
In[55]:= Simplify[1 - Tanh[x]^2 - Sech[x]^2]
```

```
Out[55]= 0
```

NOTE: We can also confirm the identity graphically by plotting the graphs of each side of the identity, which should coincide.

```
In[56]:= Plot[{1 - Tanh[x]^2, Sech[x]^2}, {x, -2, 2}]
```



b) We again evaluate the difference between the left- and right-hand sides of the identity:

```
In[57]:= Simplify[Cosh[x + y] - (Cosh[x] Cosh[y] + Sinh[x] Sinh[y])]
```

Out[57]= 0

■ 7.4.3 Derivatives of Hyperbolic Functions

We next contrast the formulas for the derivatives of the trigonometric functions versus the formulas for the derivatives of the companion hyperbolic functions.

Example 7.11. Compare the derivatives of the given pair of functions.

a) $\sinh x$ and $\sin x$ b) $\cosh x$ and $\cos x$ c) $\tanh x$ and $\tan x$

Solution: We use the derivative command, **D**, to evaluate derivatives of each pair.

a)

```
In[58]:= D[Sinh[x], x]
          D[Sin[x], x]
```

Out[58]= Cosh[x]

Out[59]= Cos[x]

b)

```
In[60]:= D[Cosh[x], x]
          D[Cos[x], x]
```

Out[60]= Sinh[x]

Out[61]= -Sin[x]

b)

```
In[62]:= D[Tanh[x], x]
          D[Tan[x], x]
```

Out[62]= Sech[x]^2

Out[63]= Sec[x]^2

It is clear that derivatives of hyperbolic and trigonometric functions are quite similar.

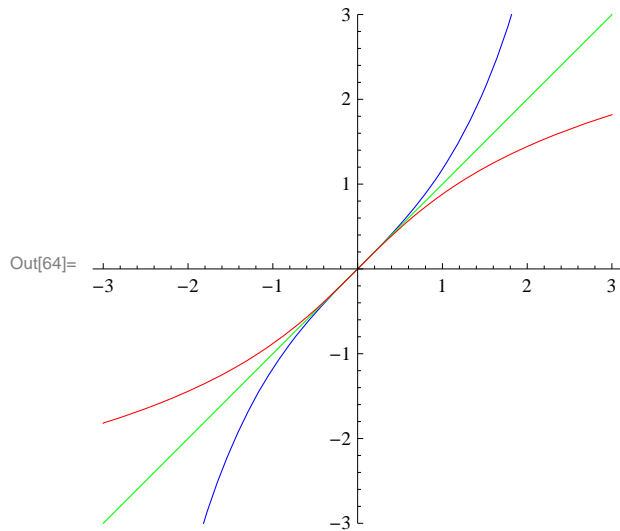
■ 7.4.4 Inverse Hyperbolic Functions

In light of the fact that hyperbolic functions are defined in terms of the exponential functions, it is readily apparent that the inverse hyperbolic functions are defined in terms of the natural logarithmic function. The inverses of the hyperbolic functions have notation similar to those of inverse trigonometric functions. Thus, the inverse of $\sinh x$ is denoted by $\operatorname{arcsinh} x$ or $\sinh^{-1} x$. In *Mathematica*, the notation is $\sinh^{-1} x$ is **ArcSinh[x]**.

Example 7.12. Plot the graphs of $\sinh^{-1} x$ and $\sinh x$ on the same axis.

Solution: Recall that the graph of a function and the graph of its inverse are reflections of each other across the line $y = x$. This is confirmed by the following plot of $\sinh^{-1} x$ (in blue) and $\sinh x$ (in red).

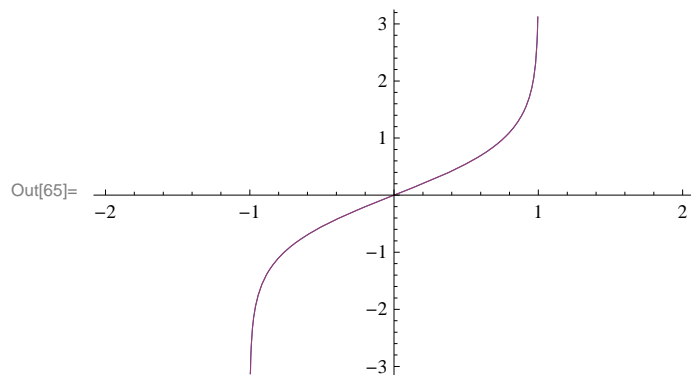
```
In[64]:= Plot[{Sinh[x], x, ArcSinh[x]}, {x, -3, 3},
  PlotStyle -> {Blue, Green, Red}, AspectRatio -> Automatic, PlotRange -> {-3, 3}]
```



Example 7.13. Show that $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ for $-1 < x < 1$.

Solution: We plot the graphs of $y = \tanh^{-1} x$ and $y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ on the same axes. Note that *Mathematica*'s notation of $\tanh^{-1} x$ is **ArcTanh[x]** and $\ln y$ is entered as **Log[y]**:

```
In[65]:= Plot[{ArcTanh[x], 1/2 Log[(1+x)/(1-x)]}, {x, -2, 2}]
```



The fact that there is only one graph indicates that the functions are the same. We prove this by letting $y = \tanh^{-1} x$ and solving for y as follows. From $y = \tanh^{-1} x$ we get $x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$. Now solving this last equation for y in *Mathematica* yields:

```
In[66]:= Solve[x == (E^y - E^(-y)) / (E^y + E^(-y)), y]
```

Solve::ifun : Inverse functions are being used by Solve, so
some solutions may not be found; use Reduce for complete solution information. >>

```
Out[66]= {{y -> Log[-(sqrt(-1-x)/sqrt(-1+x))]}, {y -> Log[(sqrt(-1-x)/sqrt(-1+x))]}}
```

The first solution in the preceding output is imaginary, which we ignore, and consider only the second solution. Hence,

$$\tanh^{-1} x = y = \ln \frac{\sqrt{-1-x}}{\sqrt{-1+x}} = \ln \sqrt{\frac{1+x}{1-x}} = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right).$$

NOTE: The message in the previous output refers to the fact that when solving equations involving inverse functions, not all solutions are necessarily found by *Mathematica* since there may be infinitely many of them or they depend on the domain of definition. For example, the equation $\sin x = 1$ has infinitely many solutions, in particular all values of the form $x = \pi/2 + 2\pi n$, where n is any integer. On the other hand, solving this equation in *Mathematica* yields only the solution in its principal domain, that is, $x = \pi/2$:

```
In[67]:= Solve[Sin[x] == 1, x]
```

Solve::ifun : Inverse functions are being used by Solve, so
some solutions may not be found; use Reduce for complete solution information. >>

```
Out[67]= {{x -> Pi/2}}
```

■ Exercises

In Exercises 1 through 5, verify the given hyperbolic identities using the **Simplify** command. Also state the corresponding trigonometric identity.

1. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

2. $\cosh 2x = \cosh^2 x + \sinh^2 x$

3. $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$

4. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

5. $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

6. Determine the first few positive integral powers of $\cosh x + \sinh x$. Can you form a general conjecture for the n th case, namely $(\cosh x + \sinh x)^n$, where n is any natural number? Then justify your conclusion via mathematical induction.

In Exercises 7 through 12, determine the derivatives of the given functions and simplify your answers where possible. Compare your solution via paper and pencil methods with the one generated by *Mathematica*.

7. $f(x) = \tanh(1 + x^2)$

8. $f(x) = x \sinh x - \cosh x$

9. $f(x) = \sqrt{\frac{1+\tanh x}{1-\tanh x}}$

10. $f(x) = x^2 \sinh^{-1}(2x)$

11. $f(x) = x \tanh^{-1} x + \ln(\sqrt{1-x^2})$

12. $f(x) = x \coth x - \operatorname{sech} x$

13. The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh(0.03291765 x)$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leq 91.20$.

- Plot the graph of the central curve.
- What is the height of the arch at its center?
- At what points is the arch 100 meters in height?
- What is the slope of the arch at the points in part (c)?

14. A flexible cable always hangs in the shape of a catenary $y = c + a \cosh(x/a)$, where c and a are constants and $a > 0$. Plot several members of the family of functions $y = a \cosh(x/a)$ for various values of a . How does the graph change as a varies?

In Exercises 15 through 17, evaluate each of the given integrals:

15. $\int \sinh x \cosh^n x \, dx$

16. $\int \frac{\cosh x}{\cosh^2 x - 1} \, dx$

17. $\int \frac{\operatorname{sech}^2 x}{2 + \tanh x} \, dx$

18. Let $t = \ln\left(\frac{1+\sqrt{5}}{2}\right)$ and define

$$f(n) = \begin{cases} \frac{2}{\sqrt{5}} \cosh(tn), & \text{if } n \text{ is odd} \\ \frac{2}{\sqrt{5}} \sinh(tn), & \text{if } n \text{ is even} \end{cases}$$

Evaluate $f(n)$ for $n = 1, 2, 3, \dots, 20$. Do these values seem familiar? If not, we highly recommend the interesting article by Thomas Osler, *Vieta-like products of nested radicals with Fibonacci and Lucas numbers*, to appear in the journal *Fibonacci Quarterly*.