
Chapter 12 Vector Geometry

Useful Tip: If you are reading the electronic version of this publication formatted as a *Mathematica* Notebook, then it is possible to view 3-D plots generated by *Mathematica* from different perspectives. First, place your screen cursor over the plot. Then drag the mouse while pressing down on the left mouse button to rotate the plot.

■ 12.1 Vectors

Students should read Sections 12.1 - 12.3 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

A vector is an object that has magnitude and direction. In physics, these vectors are denoted by arrows, where the magnitude of the vector is represented by the length of the vector, and the way in which the arrow points indicates its direction. In mathematics, these vectors are represented by points in two or three dimensions, where the vector is the arrow that starts at the origin and ends at the point. For example, the point $(2, 1, 3)$ could be considered both as a point in 3-D space and as a vector from $(0, 0, 0)$ to $(2, 1, 3)$. To distinguish a point from a vector, we will use the angled brackets \langle and \rangle instead of parentheses. Thus, the point $(2, 1, 3)$ is denoted $(2, 1, 3)$ as usual, but the vector from the origin to that point is denoted $\langle 2, 1, 3 \rangle$.

The length or magnitude of a vector \mathbf{v} is denoted $\|\mathbf{v}\|$, and is read as "norm \mathbf{v} ." If $\mathbf{v} = \langle a, b, c \rangle$, then $\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$. In two dimensions, if $\mathbf{v} = \langle a, b \rangle$, then $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$.

Vectors and matrices, in *Mathematica*, are simply lists. A vector is a list of numbers within braces, with commas between numbers, while a matrix is a list of lists (vectors), with each vector list being a row of the matrix (for a complete description of lists in *Mathematica*, see Section 1.2.3 of this text). Of course, all rows must be the same size. For example, consider the vector \mathbf{a} below:

```
In[1]:= a = {1, 3, 5}
```

```
Out[1]= {1, 3, 5}
```

The i th component of the vector \mathbf{a} is denoted by a_i , or in *Mathematica*, by $\mathbf{a}[[i]]$. For instance the second component of \mathbf{a} , which is 3, would be obtained by:

```
In[2]:= a[[2]]
```

```
Out[2]= 3
```

All of the usual vector algebra operations are available to us:

Dot Product

The Dot Product of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined by $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$. For example:

```
In[3]:= a = {1, 3, 5}
      b = {1, -2, 3}
      a.b
```

```
Out[3]= {1, 3, 5}
```

```
Out[4]= {1, -2, 3}
```

```
Out[5]= 10
```

or

```
In[6]:= Dot[a, b]
```

```
Out[6]= 10
```

NOTE: We use the ordinary period symbol on the keyboard for the dot product.

Cross Product

The cross product of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, is defined as a vector perpendicular to both \mathbf{u} and \mathbf{v} , and calculated by the following "right-hand" rule:

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

This calculation can be done in *Mathematica* in two ways. The first is to use the **Cross** command:

```
In[7]:= Cross[a, b]
```

```
Out[7]= {19, 2, -5}
```

The second is by using the multiplication symbol " \times ". This special symbol can be entered on the **Basic Math Input Palette** or by pushing the escape key, followed by typing the word "cross" and hitting the escape key again: `[esc]cross[esc]`

```
In[8]:= a × b
```

```
Out[8]= {19, 2, -5}
```

Recall that the cross product of 2 vectors, \mathbf{a} and \mathbf{b} creates a vector perpendicular to the plane of the vectors \mathbf{a} and \mathbf{b} . In your Calculus text, the cross product is also defined as the determinant of a special matrix. We will look at this a little later.

Norm (Length) of a Vector

The norm or length of a vector can be calculated in *Mathematica* by the **Norm** command

```
In[9]:= Clear[x, y, z]
```

```
In[10]:= Norm[{x, y, z}]
```

```
Out[10]=  $\sqrt{\text{Abs}[x]^2 + \text{Abs}[y]^2 + \text{Abs}[z]^2}$ 
```

```
In[11]:= Norm[a]
```

```
Out[11]=  $\sqrt{35}$ 
```

```
In[12]:= Norm[2 a]
```

```
Out[12]=  $2 \sqrt{35}$ 
```

In[13]:=

Vector Addition

The sum of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined to be $\mathbf{u} + \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.

In[14]:= `2 a - 3 b + {1, 1, 1}`

Out[14]= `{0, 13, 2}`

Example 12.1. Let $\mathbf{a} = \langle 1, 2, 3 \rangle$. Show that $\frac{\mathbf{a}}{\|\mathbf{a}\|}$ is a unit vector.

In[15]:=

Solution:

In[16]:= `Norm[a / Norm[a]]`

Out[16]= `1`

Example 12.2. Find the equation of a line in 3-space passing through $P_0 = (3, -1, 4)$ in the direction of $\mathbf{v} = \langle 2, 7, 1 \rangle$ and graph it.

Solution: The line through $P_0 = (x_0, y_0, z_0)$ in the direction of $\mathbf{v} = \langle a, b, c \rangle$ is described in vector or parametric form by:

Vector form: $\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

Parametric Form: $x = x_0 + a t$, $y = y_0 + b t$, $z = z_0 + c t$

Thus, the vector description of the line is

In[17]:= `Clear[r, t];`

`r[t_] = {3, -1, 4} + t {2, 7, 1}`

Out[18]= `{3 + 2 t, -1 + 7 t, 4 + t}`

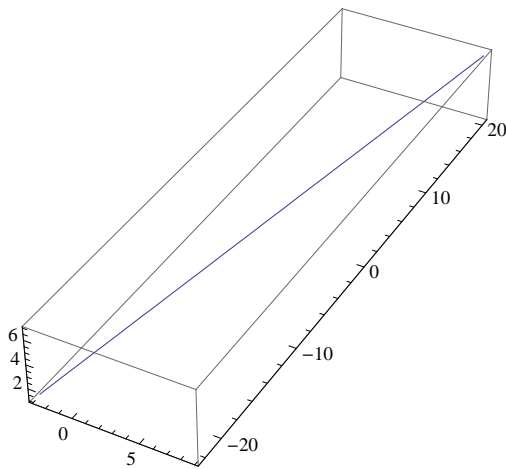
To graph this line we use the **ParametricPlot3D** command:

`ParametricPlot3D[{f_x, f_y, f_z}, {u, u_min, u_max}]`

produces a three-dimensional space curve parametrized by a variable u which runs from u_{min} to u_{max} .

```
In[19]:= ParametricPlot3D[r[t], {t, -3, 3},
          ImageSize -> {250}, ImagePadding -> {{15, 15}, {15, 15}}]
```

Out[19]=



NOTE: This plot command uses the option **ImageSize** to specify the size of graphics output. Settings include **Tiny**, **Small**, **Medium**, **Large**, or **{pt}**, where **pt** is the number of points.

Example 12.3. Give the description in vector form of the line that passes through the points $P = (1, 0, 4)$ and $Q = (3, 2, 1)$, then find the midpoint of the line segment \overline{PQ} and plot this line segment.

Solution: The line through points $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$ has vector form $\mathbf{r}(t) = (1 - t)\langle a_1, b_1, c_1 \rangle + t\langle a_2, b_2, c_2 \rangle$. In this parametrization, $\mathbf{r}(0) = P$ and $\mathbf{r}(1) = Q$. Thus,

```
In[20]:= r[t_] = (1 - t) {1, 0, 4} + t {3, 2, 1}
```

```
Out[20]= {1 + 2 t, 2 t, 4 (1 - t) + t}
```

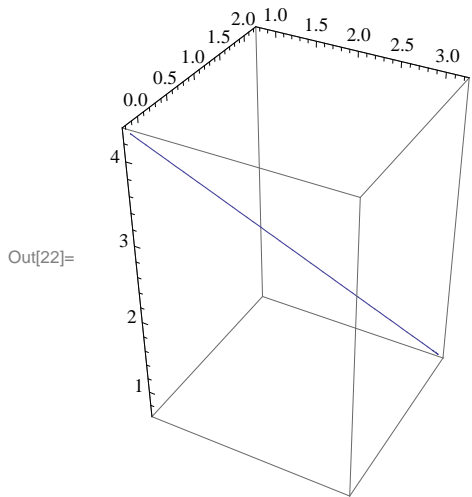
The midpoint of the line segment \overline{PQ} is

```
In[21]:= r[1/2]
```

```
Out[21]= {2, 1, 5/2}
```

The plot of the line segment is

```
In[22]:= ParametricPlot3D[r[t], {t, -0.1, 1.1},
ImageSize -> {250}, ImagePadding -> {{15, 15}, {15, 15}}]
```



Example 12.4. Find the angle between the vectors $\mathbf{v} = \langle 3, 6, 2 \rangle$ and $\mathbf{w} = \langle 6, 3, 6 \rangle$.

Solution: Remember that the angle between two vectors, \mathbf{v} and \mathbf{w} , is given by θ , which is defined by $\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}\right)$.

Therefore,

```
In[23]:= v = {3, 6, 2}
w = {6, 3, 6}
```

Out[23]= {3, 6, 2}

Out[24]= {6, 3, 6}

```
In[25]:= theta = ArcCos[ $\frac{\mathbf{v} \cdot \mathbf{w}}{\text{Norm}[\mathbf{v}] \text{Norm}[\mathbf{w}]}$ ]
```

Out[25]= ArcCos[$\frac{16}{21}$]

```
In[26]:= N[%]
```

Out[26]= 0.704547

Therefore, $\theta = .7045$ radians.

■ Exercises

1. Calculate the length of the vector $\mathbf{v} = \langle 1, 3, 4 \rangle$.

In Exercises 2 and 3, calculate the linear combinations.

2. $5 \langle 2, -2, 5 \rangle + 6 \langle 1, 3, 8 \rangle$ 3. $6 \langle 2, 0, -1 \rangle - 3 \langle 8, 6, 9 \rangle$

4. Find a vector parametrization for the line that passes through $P = (1, 2, -6)$ with direction vector $\mathbf{v} = \langle 2, 1, 5 \rangle$.

In Exercises 5 and 6, determine whether the two given vectors are orthogonal ($\mathbf{v} \perp \mathbf{w}$ iff $\mathbf{v} \cdot \mathbf{w} = 0$):

5. $\langle 1, 1, 1 \rangle, \langle 1, -2, 3 \rangle$ 6. $\langle 1, 1, 1 \rangle, \langle -3, 2, 1 \rangle$

In Exercises 7 and 8, find the angle between the vectors:

7. $\langle 1, 2 \rangle$, $\langle 5, 7 \rangle$ 8. $\langle 2, 4, 1 \rangle$, $\langle 1, -3, 5 \rangle$

■ 12.2 Matrices and the Cross Product

Students should read Section 12.4 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

In order to understand the alternate approach to the cross product alluded to above, we need to define the terms matrix and determinant.

Matrices

A *matrix* is a rectangular array of numbers consisting of n rows and m columns (denoted $n \times m$). We are especially interested in square matrices where $m = n$ and, in particular, $m = 2$ or $m = 3$. For example: A 3×3 matrix would be

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

but *Mathematica* would show this matrix as:

```
In[27]:= A = Table[10 i + j, {i, 3}, {j, 3}]
```

```
Out[27]= {{11, 12, 13}, {21, 22, 23}, {31, 32, 33}}
```

```
In[28]:= B = Table[i + j, {i, 2}, {j, 2}]
```

```
Out[28]= {{2, 3}, {3, 4}}
```

To have *Mathematica* display a matrix in the traditional way, use the **MatrixForm** command:

```
In[29]:= MatrixForm[A]
```

```
MatrixForm[B]
```

```
Out[29]/MatrixForm=
```

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix}$$

```
Out[30]/MatrixForm=
```

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

Note that in the definition of the matrices **A** and **B**, *Mathematica* treats them as lists and when we use the command **MatrixForm**, we can see the matrices presented in the traditional way.

Determinants

The *determinant* is a function, **Det**, which assigns to each square matrix a number which is defined for 2×2 and 3×3 matrices as follows:

```
In[31]:= Clear[a, b];
         F = {{a, b}, {c, d}}
         MatrixForm[F]
```

```
Out[32]= {{a, b}, {c, d}}
```

```
Out[33]//MatrixForm=
  ( a  b )
  ( c  d )
```

```
In[34]:= Det[F]
```

```
Out[34]= -b c + a d
```

```
In[35]:= G = {{a1, a2, a3}, {b1, b2, b3}, {c1, c2, c3}}
         MatrixForm[G]
```

```
Out[35]= {{a1, a2, a3}, {b1, b2, b3}, {c1, c2, c3}}
```

```
Out[36]//MatrixForm=
  ( a1 a2 a3 )
  ( b1 b2 b3 )
  ( c1 c2 c3 )
```

```
In[37]:= Det[G]
```

```
Out[37]= -a3 b2 c1 + a2 b3 c1 + a3 b1 c2 - a1 b3 c2 - a2 b1 c3 + a1 b2 c3
```

Using these definitions, we can now define the cross product of two vectors by the formula

$$\langle b_1, b_2, b_3 \rangle \times \langle c_1, c_2, c_3 \rangle = \text{Det} \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

where $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Example 12.5. Calculate the cross product of $\mathbf{v} = \langle 1, 3, 6 \rangle$ and $\mathbf{w} = \langle -2, 8, 5 \rangle$.

```
In[38]:=
```

Solution:

```
In[39]:= Clear[i, j, k]
         g = {i, j, k}
         v = {1, 3, 6}
         w = {-2, 8, 5}
         A = {g, v, w}
```

```
Out[40]= {i, j, k}
```

```
Out[41]= {1, 3, 6}
```

```
Out[42]= {-2, 8, 5}
```

```
Out[43]= {{i, j, k}, {1, 3, 6}, {-2, 8, 5}}
```

In[44]:= **MatrixForm[A]**

Out[44]//MatrixForm=

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 6 \\ -2 & 8 & 5 \end{pmatrix}$$

In[45]:= **v × w**

Det[A]

Out[45]= { -33, -17, 14 }

Out[46]= -33 i - 17 j + 14 k

Observe that the two previous outputs are equivalent.

■ Exercises

1. Calculate the determinants of $\begin{pmatrix} 0 & 5 & 0 \\ 1 & 3 & 6 \\ 2 & 5 & 5 \end{pmatrix}$ and of $\begin{pmatrix} 3 & 5 \\ 6 & 2 \end{pmatrix}$.

2. Calculate the cross product of $\mathbf{v} = \langle 2, 0, 0 \rangle$ and $\mathbf{w} = \langle -1, 0, 1 \rangle$. Do this using the **Cross** command as well as by the determinant approach.

3. Calculate the area of the parallelogram spanned by the vectors \mathbf{v} and \mathbf{w} above. (Hint: look up the formula for this in your calculus textbook.)

4. Calculate the volume of the parallelepiped spanned by:

$$\mathbf{u} = \langle 2, 2, 1 \rangle, \mathbf{v} = \langle 1, 0, 3 \rangle, \text{ and } \mathbf{w} = \langle 0, -4, 2 \rangle$$

5. Show that $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ and that $\mathbf{v} \times \mathbf{v} = \mathbf{0}$.

■ 12.3 Planes in 3-Space

Students should read Section 12.5 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

Note that a plane in 3-D space is defined as all points $P(x, y, z)$ such that the line segment $\overrightarrow{P_0P}$ is perpendicular to a given vector \mathbf{n} , called the normal vector, where the initial point of \mathbf{n} is $P_0 = (x_0, y_0, z_0)$. In vector notation, this is described by the equation $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$, where $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$. Therefore, the equation of the plane through $P_0 = (x_0, y_0, z_0)$ with nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be denoted by either of the following:

Vector form: $\mathbf{n} \cdot \langle x, y, z \rangle = d$

Scalar form: $ax + by + cz = d$

Here, $d = ax_0 + by_0 + cz_0 = \mathbf{n} \cdot \langle x_0, y_0, z_0 \rangle$.

Example 12.6. Find an equation of the plane determined by the points $P = (1, 0, -1)$, $Q = (2, 2, 1)$, and $R = (4, 2, 5)$. Then plot the graph of the plane.

Solution: The vectors $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$ lie in the plane, so the cross product $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is normal to the plane:


```
In[47]:= Clear[a, b, n]
         a = {2, 2, 1} - {1, 0, -1}
         b = {4, 2, 5} - {1, 0, -1}
         n = a × b
         n . {x, y, z} == d
```

```
Out[48]= {1, 2, 2}
```

```
Out[49]= {3, 2, 6}
```

```
Out[50]= {8, 0, -4}
```

```
Out[51]= 8 x - 4 z == d
```

To compute the value of d , we choose any point on the plane, that is, we can choose either P , Q , or R , and then compute $d = \mathbf{n} \cdot \mathbf{P}$, $d = \mathbf{n} \cdot \mathbf{Q}$, or $d = \mathbf{n} \cdot \mathbf{R}$. Let us choose $\mathbf{P} = \langle 1, 0, -1 \rangle$.

```
In[52]:= d = n . {1, 0, -1}
```

```
Out[52]= 12
```

Therefore, the plane we want has equation $8x - 4z = 12$ and the graph is obtained by using the **ContourPlot3D** command which has the form:

```
ContourPlot3D[f, {x, xmin, xmax}, {y, ymin, ymax}, {z, zmin, zmax}]
```

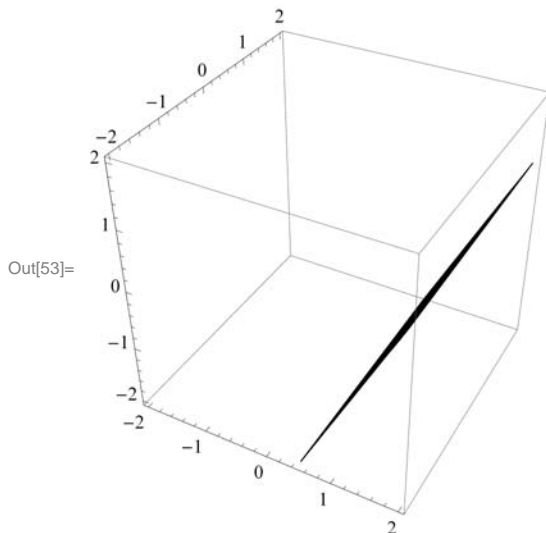
which produces a three-dimensional contour plot of f as a function of x , y and z .

or

```
ContourPlot3D[f == g, {x, xmin, xmax}, {y, ymin, ymax}, {z, zmin, zmax}]
```

which plots the contour surface for which $f = g$.

```
In[53]:= ContourPlot3D[8 x - 4 z == 12, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}, ImageSize -> {250}]
```



In order to see this plane more clearly from a different perspective, move your cursor over the plot. Then drag the mouse while pressing and holding the left mouse button to rotate the plot.

■ Exercises

1. Let PL be the plane with equation $7x - 4y + 2z = 10$. Find an equation of the plane QL parallel to PL and passing through $Q = (2, 1, 3)$ and graph it.
2. Find the equation of the plane through the points $P = (1, 5, 5)$, $Q = (0, 1, 1)$, and $R = (2, 0, 1)$ and graph it.
3. Find the angle between the two planes: $x + 2y + z = 3$ and $4x + y + 3z = 2$. (Hint: The angle between two planes is the angle between their normal vectors.)

■ 12.4 A Survey of Quadric Surfaces

Students should read Section 12.6 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

A quadric surface is the three-dimensional equivalent of a conic section (i.e., ellipses, hyperbolas, and parabolas). The basic types of quadric surfaces are ellipsoids, hyperboloids (of one or two sheets), paraboloids (elliptic or hyperbolic), and cones.

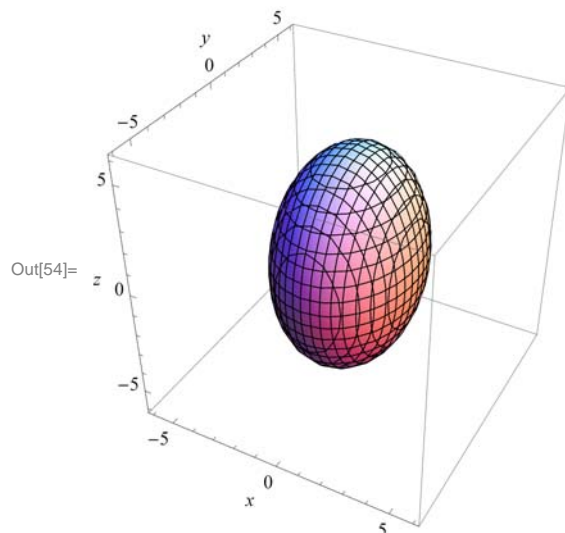
■ 12.4.1 Ellipsoids

The *standard ellipsoid* is described by $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. To help us visualize it, we are often interested in the mesh of curves called *traces*, obtained by intersecting our quadric surface with planes parallel to one of the coordinate planes. In the plot below, you can see that mesh, and also see that the traces of an ellipsoid are themselves ellipses.

Example 12.7. Graph the ellipsoid above, with $a = 3$, $b = 4$, and $c = 5$, and describe the traces of this ellipsoid.

Solution: The correct *Mathematica* command to use is `ContourPlot3D`. This is shown following:

```
In[54]:= ContourPlot3D[(x/3)^2 + (y/4)^2 + (z/5)^2 == 1, {x, -6, 6},
  {y, -6, 6}, {z, -6, 6}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



Again, note that the ellipsoid can be manually rotated to look at it from different perspectives. First, place your screen cursor over the plot. Then drag the mouse while pressing down on the left mouse button to rotate the plot. When you do this, you will note that, indeed, all of the traces are ellipses.

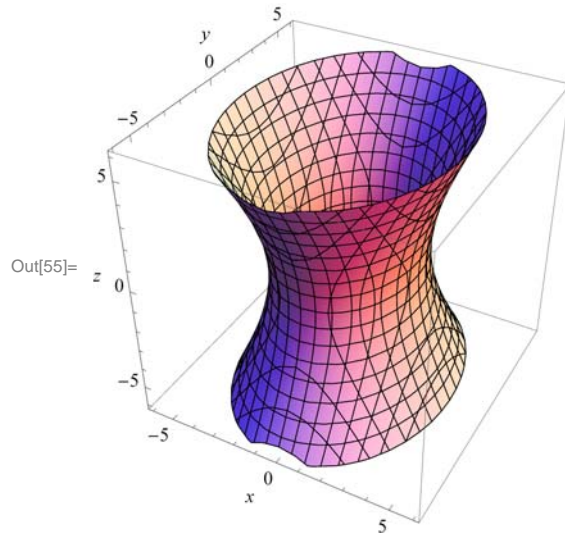
■ 12.4.2 Hyperboloids

The three-dimensional hyperbolas are called *hyperboloids*, and come in two types: the hyperboloid of one sheet, with standard form $(x/a)^2 + (y/b)^2 = (z/c)^2 + 1$, and the hyperboloid of two sheets, with standard form $(x/a)^2 + (y/b)^2 = (z/c)^2 - 1$. A limiting case of the hyperboloid is the *elliptic cone*, defined by the equation $(x/a)^2 + (y/b)^2 = (z/c)^2$.

Example 12.8. Describe the traces of the two hyperboloids: $(x/3)^2 + (y/4)^2 = (z/5)^2 + 1$ and $(x/3)^2 + (y/4)^2 = (z/5)^2 - 1$.

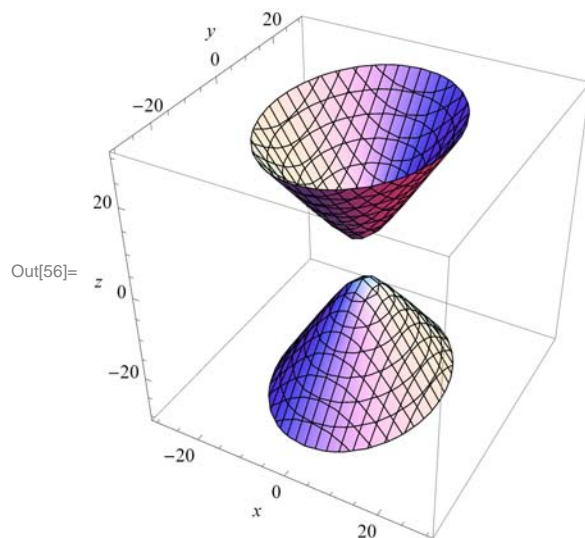
Solution: First we graph the hyperboloids:

```
In[55]= ContourPlot3D[(x / 3) ^ 2 + (y / 4) ^ 2 == (z / 5) ^ 2 + 1, {x, -6, 6},
           {y, -6, 6}, {z, -6, 6}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



In this case, the traces parallel with the xy -axis are all ellipses, and the traces parallel with the xz - and yz -axes are hyperbolas.

```
In[56]= ContourPlot3D[(x / 3) ^ 2 + (y / 4) ^ 2 == (z / 5) ^ 2 - 1, {x, -30, 30},
           {y, -30, 30}, {z, -30, 30}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```

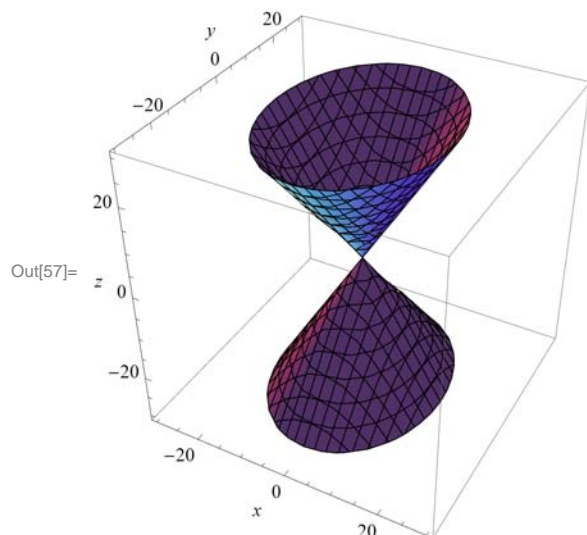


When we look at this plot, we see that the traces are the same as for the previous hyperboloid of one sheet.

Example 12.9. Graph the cone with $a = 3$, $b = 4$, and $c = 5$, and define its relationship to the hyperboloid of one sheet.

Solution: We get the graph by using the `ContourPlot3D` Command:

```
In[57]:= ContourPlot3D[(x/3)^2 + (y/4)^2 == (z/5)^2, {x, -30, 30},
  {y, -30, 30}, {z, -30, 30}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



When we compare this plot with that of the hyperboloid of one sheet (see previous example), we can see clearly that this cone can be thought of as a limiting case of the hyperboloid of one sheet in which we pinch the waist down to a point.

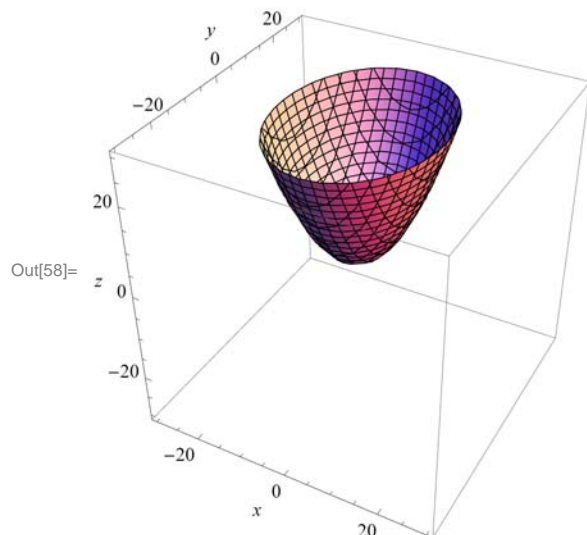
■ 12.4.3 Paraboloids

The final family of quadric surfaces that we want to consider are the *paraboloids*, of which there are two types: elliptic and hyperbolic. Their standard equations are $z = (x/a)^2 + (y/b)^2$ (elliptic paraboloid) and $z = (x/a)^2 - (y/b)^2$ (hyperbolic paraboloid).

Example 12.10. Graph the two types of paraboloids for $a = 3$ and $b = 4$ and describe their traces.

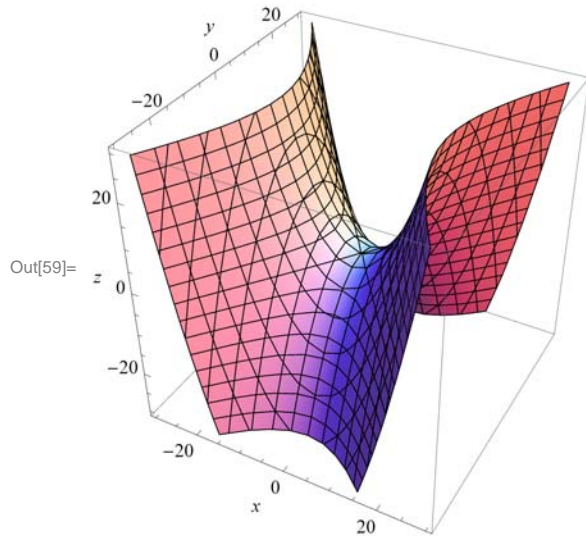
Solution: Here is the graph of the elliptic paraboloid:

```
In[58]:= ContourPlot3D[(x/3)^2 + (y/4)^2 == z, {x, -30, 30},
  {y, -30, 30}, {z, -30, 30}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



Observe that the traces in the direction of the xz - and yz -axes are both parabolas while those in the xy -direction are ellipses, which can be seen by dragging the plot in the appropriate directions. Similarly, for the hyperbolic paraboloid:

```
In[59]:= ContourPlot3D[(x/3)^2 - (y/4)^2 == z, {x, -30, 30},
  {y, -30, 30}, {z, -30, 30}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



Again, by dragging the plot above, we see that the traces in the yz -direction are parabolas while those in the xz -direction are hyperbolas.

■ 12.4.4 Quadratic Cylinders

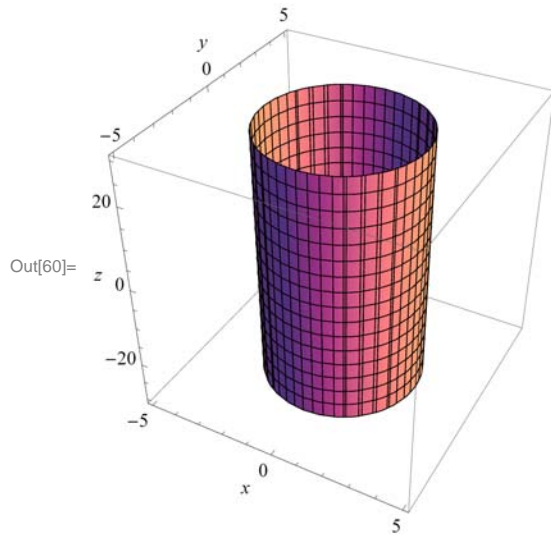
The last group of quadric surfaces we will look at are the quadratic cylinders. These are surfaces formed from a two-dimensional curve (in the xy -plane) along with all vertical lines passing through the curve:

Example 12.11. Graph a selection of quadratic cylinders.

Solution:

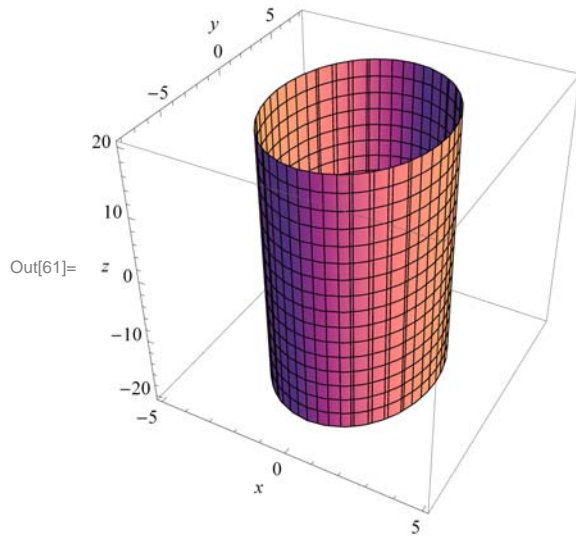
a) A circular cylinder of radius r : $x^2 + y^2 = r^2$. For the graph, we will use $r = 3$.

```
In[60]:= ContourPlot3D[x^2 + y^2 == 3^2, {x, -5, 5}, {y, -5, 5},
  {z, -30, 30}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



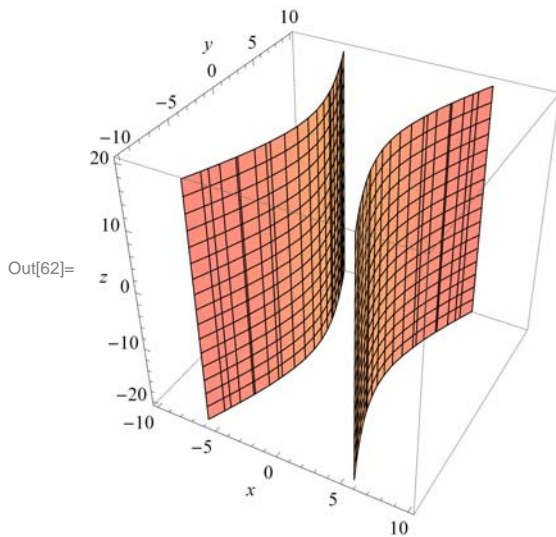
b) An elliptic with equation $(x/a)^2 + (y/b)^2 = 1$. We will use $a = 3$ and $b = 6$.

```
In[61]:= ContourPlot3D[(x / 3)^2 + (y / 6)^2 == 1, {x, -5, 5},
  {y, -8, 8}, {z, -20, 20}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



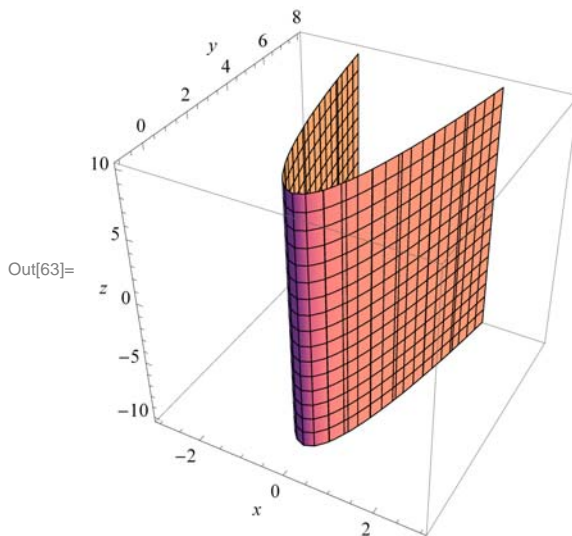
c) A hyperbolic cylinder with equation $(x/a)^2 - (y/b)^2 = 1$. We will use $a = 3$ and $b = 6$.

```
In[62]:= ContourPlot3D[(x/3)^2 - (y/6)^2 == 1, {x, -10, 10},
  {y, -10, 10}, {z, -20, 20}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



d) A parabolic cylinder with equation $y = ax^2$ with $a = 3$.

```
In[63]:= ContourPlot3D[y == 3 x^2, {x, -3, 3}, {y, -1, 8},
  {z, -10, 10}, AxesLabel -> {x, y, z}, ImageSize -> {250}]
```



■ Exercises

In Exercises 1 through 5, state whether the given equation defines an ellipsoid, hyperboloid, or paraboloid, and of which type. Then confirm your answer by graphing the quadric surface.

1. $(x/5)^2 + (y/7)^2 + (z/9)^2 = 1$
2. $(x/5)^2 - (y/7)^2 + (z/9)^2 = 1$
3. $x^2 + 5y^2 - 6z^2 = 1$
4. $z = (x/5)^2 + (y/7)^2$
5. $z = (x/5)^2 - (y/7)^2$

In Exercises 6 through 8, state the type of the quadric surface and graph it, and then describe the trace obtained by intersecting it with the given plane.

6. $(x/5)^2 + y^2 + (z/9)^2 = 1, z = 1/4$

7. $y = 2x^2, z = 25$

8. $(x/5)^2 - (y/7)^2 + (z/9)^2 = 1, y = 4$

■ 12.5 Cylindrical and Spherical Coordinates

Students should read Section 12.7 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

■ 12.5.1 Cylindrical Coordinates

In cylindrical coordinates, the point $P = (x, y, z)$ is expressed as (r, θ, z) where r and θ are the polar coordinates of x and y . The formulas for converting from (x, y, z) to (r, θ, z) are:

Cylindrical to Rectangular	Rectangular to Cylindrical
$x = r \cos \theta$	$r = \sqrt{x^2 + y^2}$
$y = r \sin \theta$	$\tan \theta = y/x$
$z = z$	$z = z$

The commands in *Mathematica* to do these conversions must first be loaded into *Mathematica* from the "Vector Analysis" external package:

```
In[64]:= << VectorAnalysis`
```

Example 12.12. Convert $(r, \theta, z) = (2, 3\pi/4, 5)$ to rectangular coordinates.

Solution: We use the `CoordinatesToCartesian` command to convert from cylindrical to rectangular coordinates:

```
In[65]:= CoordinatesToCartesian[{2, 3 Pi / 4, 5}, Cylindrical]
```

```
Out[65]:= {-√2, √2, 5}
```

```
In[66]:= N[%]
```

```
Out[66]:= {-1.41421, 1.41421, 5.}
```

Example 12.13. Convert $(x, y, z) = (2, 3, 5)$ to cylindrical coordinates.

Solution: We use the `CoordinatesFromCartesian` command to convert from rectangular to cylindrical coordinates:

```
In[67]:= CoordinatesFromCartesian[{2, 3, 5}, Cylindrical]
```

```
Out[67]:= {√13, ArcTan[3/2], 5}
```

```
In[68]:= N[%]
```

```
Out[68]:= {3.60555, 0.982794, 5.}
```

Of course, one very strong point for *Mathematica* is its graphing ability. It will easily graph functions described in cylindrical coordinates. The command to do this is `RevolutionPlot3D`.

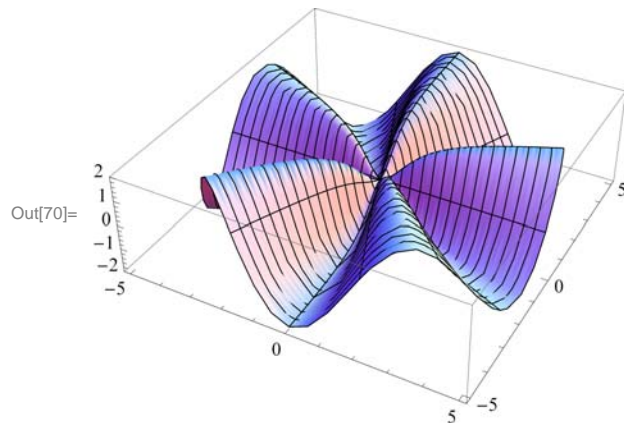
`RevolutionPlot3D[fz, {t, tmin, tmax}, {θ, θmin, θmax}`
 takes the azimuthal angle θ to vary between θ_{min} and θ_{max} .

Example 12.14. Graph the cylindrical coordinate function $z = \frac{2r^2 \sin(5\theta)}{1+r^2}$.

Solution:

In[69]:= `Clear[r, θ];`

`RevolutionPlot3D` $\left[\frac{2 r^2 \sin[5 \theta]}{1+r^2}, \{r, 0, 5\}, \{\theta, 0, 2 \pi\}, \text{ImageSize} \rightarrow \{250\}\right]$



■ 12.5.2 Spherical Coordinates

A point $P = (x, y, z)$ is described in spherical coordinates by a triple (ρ, θ, ϕ) where ρ is the distance of P from the origin, θ is the polar angle of the projection $(x, y, 0)$, and ϕ is the angle between the z -axis and the ray from the origin through P . The formulas for converting between rectangular and spherical coordinates are:

Spherical to Rectangular

$$x = \rho \cos \theta \sin \phi$$

$$y = \rho \sin \theta \sin \phi$$

$$z = \rho \cos \phi$$

Rectangular to Spherical

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\tan \theta = y/x$$

$$\cos \phi = z/\rho$$

These conversions are done in *Mathematica* using the same commands as with cylindrical coordinates, but with the word spherical replacing cylindrical.

Example 12.15. Convert $(\rho, \theta, \phi) = (2, 3\pi/4, \pi/5)$ to rectangular coordinates.

Solution:

In[71]:= `CoordinatesToCartesian[{2, 3 Pi / 4, π / 5}, Spherical]`

Out[71]= $\left\{ \frac{1 + \sqrt{5}}{2\sqrt{2}}, \sqrt{2 \left(\frac{5}{8} - \frac{\sqrt{5}}{8} \right)}, -\sqrt{2} \right\}$

In[72]:= `N[%]`

Out[72]= $\{1.14412, 0.831254, -1.41421\}$

Example 12.16. Convert $(x, y, z) = (2, 3, 5)$ to spherical coordinates.

Solution:

```
In[73]:= CoordinatesFromCartesian[{2, 3, 5}, Spherical]
```

```
Out[73]= { $\sqrt{38}$ , ArcCos[ $\frac{5}{\sqrt{38}}$ ], ArcTan[ $\frac{3}{2}$ ]}
```

```
In[74]:= N[%]
```

```
Out[74]= {6.16441, 0.624754, 0.982794}
```

Again, the main use here of *Mathematica* is its graphing ability. It will easily graph functions described in spherical coordinates. The command to do this is the **SphericalPlot3D** command.

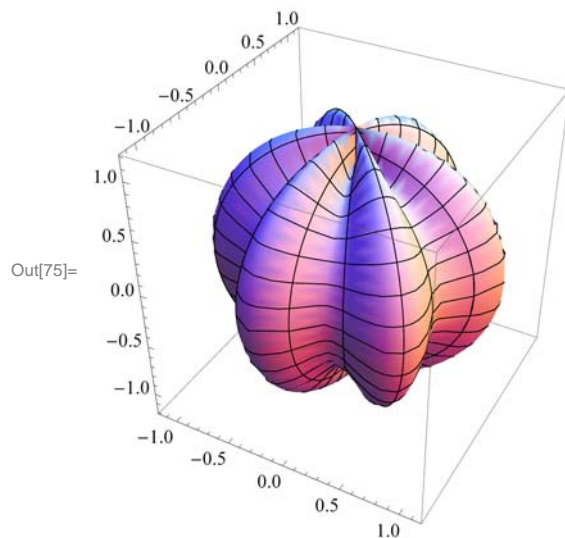
```
SphericalPlot3D[r, { $\theta$ ,  $\theta_{min}$ ,  $\theta_{max}$ }, { $\phi$ ,  $\phi_{min}$ ,  $\phi_{max}$ }]
```

generates a 3D plot with a spherical radius r as a function of spherical coordinates θ and ϕ .

Example 12.17. Graph the spherical coordinate function $\rho = 1 + \sin(6\phi)/6$.

Solution:

```
In[75]:= SphericalPlot3D[ $\rho = 1 + \text{Sin}[6\phi] / 6$ , { $\theta$ , 0, Pi}, { $\phi$ , 0, 2 Pi}, ImageSize -> {250}]
```



■ Exercises

Convert from cylindrical to rectangular:

1. $(2, \pi/3, -4)$
2. $(1, \pi/2, 3)$

Convert from rectangular to cylindrical:

3. $(2, 2, 5)$
4. $(4, \sqrt{3}, 8)$

5. Plot the surface $z^2 + r^2 = 25\theta$ and describe it.

Convert from spherical to rectangular:

6. $(2, \pi/5, \pi/3)$
7. $(4, \pi/6, 5\pi/6)$

Convert from rectangular to spherical:

8. $(\sqrt{2}, 2, 3)$

9. $(4, \sqrt{3}/2, \sqrt{8})$

10. Plot the surface $\rho \sin \phi = 5$ and describe it.