Chapter 17 Fundamental Theorems of Vector Analysis

Useful Tip: If you are reading the electronic version of this publication formatted as a *Mathematica* Notebook, then it is possible to view 3-D plots generated by *Mathematica* from different perspectives. First, place your screen cursor over the plot. Then drag the mouse while pressing down on the left mouse button to rotate the plot.

NOTE: In order to perform the operations of curl and divergence on vector fields discussed in this section using *Mathematica*, it is necessary to first load the **VectorAnalysis** package:

In[641]:=
```
Needs["VectorAnalysis`"]
```

The Fundamental Theorem of Calculus for functions of a single variable states that the integral of a function $f(x)$ over an interval $[a, b]$ (domain) can be calculated as the difference of its anti-derivative $F(x)$ at the endpoints (boundary) of the interval:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

This integral relationship between domain and boundary can be generalized to vector fields involving the operations of curl and divergence and is made precise by three theorems that will be discussed in this chapter: Green's Theorem, Stoke's Theorem, and Divergence Theorem.

17.1 Green's Theorem

Students should read Section 17.1 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ a vector field continuous on an oriented curve C. Recall that the *line integral of* **F** along C is denoted by

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{s} = \int_C P\, dx + Q\, dy$$

If $\mathbf{c}\,(t) = \langle x(t),\ y(t),\ z(t)\rangle$ is the vector equation of the curve C, then

$$\oint_C P\, dx + Q\, dy = \int_a^b \left(P(x(t),\ y(t))\, \frac{dx}{dt} + Q(x(t),\ y(t))\, \frac{dy}{dt} \right) dt$$

The following is a generalization of the Fundamental Theorem of Calculus to two dimensions, which relates a double integral over a region with a corresponding line integral along its boundary.

Green's Theorem: If C is a simple closed curve oriented counterclockwise and D is the region enclosed, and if P and Q are differentiable and have continuous first partial derivatives, then

$$\oint_C P\, dx + Q\, dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Refer to your textbook for a detailed discussion and proof of Green's Theorem.

Example 17.1. Compute the line integral $\oint_C e^{2x+y}\, dx + e^{-y}\, dy$, where C is the boundary of the square with vertices (0, 0), (1, 0), (1, 1), (1, 0) oriented counterclockwise.

Solution: We will use Green's Theorem. Thus, we need to verify that the hypotheses of Green's Theorem hold. To this end, we define the function P and Q and compute their partial derivatives.

In[642]:=
```
Clear[x, y, P, Q]
P[x_, y_] = E^(2 x+y)
Q[x_, y_] = E^(-y)
```

Out[643]= e^{2x+y}

Out[644]= e^{-y}

In[645]:=
```
D[P[x, y], x]
D[P[x, y], y]
D[Q[x, y], x]
D[Q[x, y], y]
```

Out[645]= $2\,e^{2x+y}$

Out[646]= e^{2x+y}

Out[647]= 0

Out[648]= $-e^{-y}$

The partial derivatives are continuous inside the square and the curve is oriented counterclockwise. Thus, the hypotheses of Green's Theorem are satisfied. Note that the region D enclosed by C is given by $0 \le x \le 1$ and $0 \le y \le 1$.

In[649]:= $\int_0^1\int_0^1 (\texttt{D[Q[x, y], x] - D[P[x, y], y]})\,dy\,dx$

Out[649]= $-\frac{1}{2}(-1+e)^2(1+e)$

In[650]:=
```
N[%]
```

Out[650]= -5.4891

NOTE: If we were to solve this using the definition of line integral as discussed in Chapter 16 of this text, we would then need to consider four pieces of parametrization of C and then sum the four integrals. Toward this end, let us use C_1 for the lower edge, C_2 for the right edge, C_3 for the top edge, and C_4 for the left edge of the square. Here are the parametrizations followed by their line integrals.

In[651]:=
```
Clear[x1, x2, x3, x4, y1, y2, y3, y4, t, F, c1, c2, c3, c4]
F[x_, y_] = {P[x, y], Q[x, y] }
x1[t_] = t
y1[t_] = 0
c1[t_] = {x1[t], y1[t]}

x2[t_] = 1
y2[t_] = t
c2[t_] = {x2[t], y2[t]}

x3[t_] = 1 - t
y3[t_] = 1
c3[t_] = {x3[t], y3[t]}

x4[t_] = 0
y4[t_] = 1 - t
c4[t_] = {x4[t], y4[t]}
```

Out[652]= $\{e^{2x+y}, e^{-y}\}$

Out[653]= t

Out[654]= 0

Out[655]= {t, 0}

Out[656]= 1

Out[657]= t

Out[658]= {1, t}

Out[659]= 1 - t

Out[660]= 1

Out[661]= {1 - t, 1}

Out[662]= 0

Out[663]= 1 - t

Out[664]= {0, 1 - t}

In[665]:=

$$\int_0^1 F[x1[t], y1[t]].c1'[t]\,dt + \int_0^1 F[x2[t], y2[t]].c2'[t]\,dt + \int_0^1 F[x3[t], y3[t]].c3'[t]\,dt + \int_0^1 F[x4[t], y4[t]].c4'[t]\,dt$$

Out[665]= $-1 + \frac{1}{e} + \frac{-1 + e}{e} + \frac{1}{2}\left(-1 + e^2\right) - \frac{1}{2}\, e\left(-1 + e^2\right)$

In[666]:= **N[%]**

Out[666]= -5.4891

Exercises

In Exercises 1 through 4, use Green's Theorem to evaluate the given line integral.

1. $\oint_C y^2 \sin x \, dx + x y \, dy$, where C is the boundary of the triangle with vertices (0, 0), (1, 0), (1, 1), oriented counterclockwise.

2. $\oint_C 2 x^2 y \, dx + x^3 \, dy$, where C is the circle $x^2 + y^2 = 4$, oriented counterclockwise.

3. $\oint_C (x^2 + y^2) \, dx + \; y e^x \, dy$, where C is the boundary of the region bounded between the parabola $y = 5 - x^2$ and the line $y = 2x - 3$, oriented clockwise.

4. $\oint_C \frac{x}{x^2+y^2} \, dx - \frac{y}{x^2+y^2} \, dy$, where C is the boundary of the quarter-annulus situated between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ in the first quadrant (see plot below), oriented counterclockwise.

5. Let $\mathbf{F}(x, y) = (2 x y + y^3, x^2 + 3 x y + 2 y)$. Use Green's Theorem to demonstrate that the line integral $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{s} = 0$ for every simple closed curve C. What kind of a vector field do we call $\mathbf{F}$?

17.2 Stokes's Theorem

Students should read Section 17.2 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this section.

Let $\mathbf{F}(x, y, z) = \langle F_1, F_2, F_3 \rangle$ be a vector field. The curl of $\mathbf{F}$, denoted by curl($\mathbf{F}$) or $\nabla \times \mathbf{F}$, is defined by

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Here, we are using the *del* or symbol ∇ (nabla) to denote the vector operator $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

The *Mathematica* command for computing the curl of a vector field **F** is **Curl[F,coordsys]**, where **coordsys** is the coordinate system of the vector field. This is demonstrated in the next example.

The following is a generalization of the Fundamental Theorem of Calculus three dimensions, which relates a surface integral involving curl with a corresponding line integral along its boundary.

Stokes's Theorem: If $\mathbf{F}(x, y, z)$ a vector field with continuous partial derivatives and if S is an oriented surface S with boundary ∂S, then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

If S is closed, then it has no boundary and hence both integrals are equal to 0.

NOTE: Recall that if the surface S is given by $G(u, v) = (x(u, v), \; y(u, v), \; z(u, v))$, where $(u, v) \in D$, then $\int_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$ is given by

$$\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_D \text{curl}(\mathbf{F}) \, (G(u, v)) \cdot \mathbf{n}(u, v) \, du \, dv$$

Refer to your textbook for a detailed discussion and proof of Stokes's Theorem.

Example 17.2. Find the curl of the vector field $\mathbf{F}(x, y, z) = \langle x \sin(y z), e^{x/y} z, y x^2 \rangle$.

Solution: We use the **Curl** command:

In[667]:=
```
Clear[F, F1, F2, F3, x, y, z]
F1 = x Sin[y z]
F2 = E^(x/y) z
F3 = x^2 y
F = {F1, F2, F3}
```

Out[668]= $x \,\mathrm{Sin}[y z]$

Out[669]= $e^{x/y} z$

Out[670]= $x^2 y$

Out[671]= $\{x \,\mathrm{Sin}[y z], e^{x/y} z, x^2 y\}$

In[672]:=
```
Curl[F, Cartesian[x, y, z]]
```

Out[672]= $\left\{-e^{x/y} + x^2, -2 x y + x y \,\mathrm{Cos}[y z], \frac{e^{x/y} z}{y} - x z \,\mathrm{Cos}[y z]\right\}$

NOTE: We obtain the same answer for the curl of **F** using the explicit formula:

In[673]:= $\mathrm{curl} = \{\partial_y F3 - \partial_z F2, \partial_z F1 - \partial_x F3, \partial_x F2 - \partial_y F1\}$

Out[673]= $\left\{-e^{x/y} + x^2, -2 x y + x y \,\mathrm{Cos}[y z], \frac{e^{x/y} z}{y} - x z \,\mathrm{Cos}[y z]\right\}$

Or equivalently,

In[674]:=
```
CurlF = {D[F3, y] - D[F2, z], D[F3, x] - D[F1, z], D[F2, x] - D[F1, y]}
```

Out[674]= $\left\{-e^{x/y} + x^2, 2 x y - x y \,\mathrm{Cos}[y z], \frac{e^{x/y} z}{y} - x z \,\mathrm{Cos}[y z]\right\}$

Example 17.3. Let $f(x, y, z)$ be a function of three variables with continuous first and second partial derivatives and let $\mathbf{F} = \nabla f$ be the gradient of f. Find the curl of the vector field **F**.

Solution:

In[675]:=
```
Clear[f, F1, F2, F3, x, y, z]
F1 = D[f[x, y, z], x]
F2 = D[f[x, y, z], y]
F3 = D[f[x, y, z], z]
F = {F1, F2, F3}
```

Out[676]= $f^{(1,0,0)}[x, y, z]$

Out[677]= $f^{(0,1,0)}[x, y, z]$

Out[678]= $f^{(0,0,1)}[x, y, z]$

Out[679]= $\{f^{(1,0,0)}[x, y, z], f^{(0,1,0)}[x, y, z], f^{(0,0,1)}[x, y, z]\}$

Then the curl of **F** is

```
In[680]:= Curl[F, Cartesian[x, y, z]]
Out[680]= {0, 0, 0}
```

To see why the curl is zero, let us examine each partial derivative used in computing the curl of **F**.

```
In[681]:= D[F3, y]
          D[F2, z]
```

Out[681]= $f^{(0,1,1)}[x, y, z]$

Out[682]= $f^{(0,1,1)}[x, y, z]$

NOTE: Here, $\mathbf{f}^{(0,1,1)}$**[x, y, z]** stands for the second partial derivative f_{yz}. Thus, the two partial derivatives that appear in the x-component of the curl of **F** are equal and hence their difference is zero. Similarly, we have

```
In[683]:= D[F3, x]
          D[F1, z]
```

Out[683]= $f^{(1,0,1)}[x, y, z]$

Out[684]= $f^{(1,0,1)}[x, y, z]$

and

```
In[685]:= D[F2, x]
          D[F1, y]
```

Out[685]= $f^{(1,1,0)}[x, y, z]$

Out[686]= $f^{(1,1,0)}[x, y, z]$

Example 17.4. Compute $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle x\,y\,z,\; z + 3\,x - 3\,y,\; y^2\,x \rangle$ and S is the upper hemisphere of radius 4.

Solution: Note that ∂S is a circle of radius 4 lying on the xy-plane. Hence, ∂S can be parametrized by the curve $c(t) = (x(t), y(t), z(t))$ where

$$x = 4\cos t,\; y = 4\sin t,\; z = 0, \text{ where } 0 \leq t \leq 2\pi$$

We then use this parametrization to evaluate the line integral $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \mathbf{F}(x(t), y(t), z(t)) \cdot c'(t)\, dt$:

```
In[687]:= Clear[F, x, y, z, t, c, curlF]
          F[x_, y_, z_] = {x y z, z + 3 x - 3 y, y^2 x}
          x[t_] = 4 Cos[t]
          y[t_] = 4 Sin[t]
          z[t_] = 0
          c[t_] = {x[t], y[t], z[t]}

Out[688]= {x y z, 3 x - 3 y + z, x y^2}

Out[689]= 4 Cos[t]

Out[690]= 4 Sin[t]

Out[691]= 0

Out[692]= {4 Cos[t], 4 Sin[t], 0}
```

In[693]:= $\int_0^{2\,Pi}$ `F[x[t], y[t], z[t]].c'[t]` dt

Out[693]= $48\,\pi$

Next, we use Stokes's Theorem to obtain the same answer via the corresponding surface integral. The parametrization of the upper hemisphere of radius 4 is given by $S(u,\ v) = \{x(u,\ v),\ y(u,\ v),\ z(u,\ v)\}$, where

$$x = 4\cos u \sin v,\ \ y = 4\sin u\sin v, \text{ and } z = 4\cos v, \quad \text{where} \quad 0 \le u \le 2\,\pi, 0 \le v \le \pi/2$$

We now compute the normal of the upper hemisphere:

```
In[694]:= Clear[S, u, v, Tu, Tv, n]
          S[u_, v_] := { 4 Cos[u] Sin[v], 4 Sin[u] Sin[v], 4 Cos[v] }
          Tu[u_, v_] := D[S[u, v], u]
          Tv[u_, v_] := D[S[u, v], v]
          n[u_, v_] = Cross[Tv[u, v], Tu[u, v]]

Out[698]= {16 Cos[u] Sin[v]^2, 16 Sin[u] Sin[v]^2,
           16 Cos[u]^2 Cos[v] Sin[v] + 16 Cos[v] Sin[u]^2 Sin[v]}
```

The curl of **F** is

```
In[699]:= curlF[x_, y_, z_] = Curl[F[x, y, z], Cartesian[x, y, z]]

Out[699]= {-1 + 2 x y, x y - y^2, 3 - x z}
```

Thus, the surface integral is given by

In[700]:= $\int_0^{Pi/2}\int_0^{2\,Pi}$ `curlF[S[u, v][[1]], S[u, v][[2]], S[u, v][[3]]] .n[u, v]` $du\, dv$

Out[700]= $48\,\pi$

This answer agrees with the one obtained using the line integral definition.

Example 17.5. Find the flux of the curl of the vector field $\mathbf{F}(x,\ y,\ z) = \langle x^2,\ z^2,\ y + x^2\rangle$ across S, where S is the part of the cone $z^2 = x^2 + y^2$ for which $1 \le z \le 4$ with outward normal orientation.

Solution: First, we will need the following parametric equations to describe the cone S: $x = u\cos v$, $y = u\sin v$, and $z = u$, where $0 \le v \le 2\pi$ and $1 \le u \le 4$.

For the cone to have outward orientation, we set $\mathbf{n} = \mathbf{T}_v \times \mathbf{T}_u$ (right-hand rule) since $\mathbf{T}_v$ points in the horizontal direction around the cone and $\mathbf{T}_u$ points in the direction along the length of the cone.

```
In[701]:= Clear[F, S, u, v, Tu, Tv, n]
          F[x_, y_, z_] = {x^2 + y^2, x + z^2, 0}
          S[u_, v_] := { u Cos[v] , u Sin[v] , u }
          Tu[u_, v_] := D[S[u, v], u]
          Tv[u_, v_] := D[S[u, v], v]
          n[u_, v_] = Cross[Tv[u, v], Tu[u, v]]
```

Out[702]= $\{x^2 + y^2, x + z^2, 0\}$

Out[706]= $\{u \operatorname{Cos}[v], u \operatorname{Sin}[v], -u \operatorname{Cos}[v]^2 - u \operatorname{Sin}[v]^2\}$

We now compute the flux of curl ($\mathbf{F}$) across S through the following steps.

```
In[707]:= curlF[x_, y_, z_] = Curl[F[x, y, z], Cartesian[x, y, z]]
```

Out[707]= $\{-2z, 0, 1 - 2y\}$

In[708]:= $\text{Flux} = \int_1^4 \int_0^{2\,\text{Pi}} \text{curlF}[S[u, v][[1]], S[u, v][[2]], S[u, v][[3]]].n[u, v]\, dv\, du$

Out[708]= -15π

Exercises

NOTE: In order to perform the curl operation in *Mathematica*, it is necessary to first load the **VectorAnalysis** package. See instructions given at the beginning of this chapter.

In Exercises 1 and 2, find the curl of the given vector field.

1. $\mathbf{F}(x, y, z) = \left\langle \ln\left(x^2 + y^2 + z^2\right), x / z, e^x \sin(y z) \right\rangle$

2. $\mathbf{F}(x, y, z) = \left\langle -\frac{x}{(x^2+y^2+z^2)^{3/2}}, -\frac{y}{(x^2+y^2+z^2)^{3/2}}, -\frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$

In Exercises 3 and 4, verify Stokes's Theorem for the given vector field $\mathbf{F}$ and surface S.

3. $\mathbf{F}(x, y, z) = \left\langle x^3 e - 3xy + z^3, 2z^3 - xz^2 + y^4, 6y + 2z^3 x^2 \right\rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ for which $z \le 9$ and with outward normal orientation.

4. $\mathbf{F}(x, y, z) = \langle x y z, x y, x + y + z \rangle$ and S is the elliptical region in the plane $y + z = 2$ whose boundary is the intersection of the plane with the cylinder $x^2 + y^2 = 1$ and with upward normal orientation.

In Exercises 5 and 6, use Stokes's Theorem to compute the flux of the curl of the vector field $\mathbf{F}$ across the surface S.

5. $\mathbf{F}(x, y, z) = \left\langle \tan(x y z), e^{y - x z}, \sec\left(y^2 x\right) \right\rangle$ and S is the upper hemisphere of radius 4.

6. $\mathbf{F}(x, y, z) = \left\langle x^2 z, x y^2, z^2 \right\rangle$ and S consists of the top and four sides of the cube (excluding the bottom) with vertices at (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1).

17.3 Divergence Theorem

Students should read Section 17.3 of Rogawski's *Calculus* [1] for a detailed discussion of the material presented in this

section.

Let $\mathbf{F}(x, y, z) = \langle F_1, F_2, F_3 \rangle$ be a vector field. The divergence of **F**, denoted by div(**F**) or $\nabla \cdot \mathbf{F}$, is defined by

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

where $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

The *Mathematica* command for computing the divergence of a vector field **F** is **Div[F,coordsys]**, where **coordsys** is the coordinate system of the vector field. This is demonstrated in the next example.

The following is another generalization of the Fundamental Theorem of Calculus three dimensions, which relates a triple integral of a solid object involving divergence with a corresponding surface integral along its boundary.

Divergence Theorem: Let W be a region in $\mathbb{R}^3$ whose boundary ∂W is a piecewise smooth surface, oriented so that the normal vectors to ∂W point outside of W, and $\mathbf{F}(x, y, z)$ be a vector field with continuous partial derivatives whose domain contains W. Then

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div}(\mathbf{F})\, dV$$

Refer to your textbook for a detailed discussion and proof of the Divergence Theorem.

Example 17.8. Find the divergence of the vector field $\mathbf{F}(x, y, z) = \left\langle x \sin(yz), e^{x/y} z, yx^2 \right\rangle$.

Solution:

```
In[709]:= Clear[F1, F2, F3, x, y, z]
          F1 = x Sin[y z]
          F2 = E^(x/y) z
          F3 = x^2 y
          F = {F1, F2, F3}
```

Out[710]= $x \operatorname{Sin}[y z]$

Out[711]= $e^{x/y} z$

Out[712]= $x^2 y$

Out[713]= $\{x \operatorname{Sin}[y z], e^{x/y} z, x^2 y\}$

Then the divergence of **F** is

```
In[714]:= Div[F, Cartesian[x, y, z]]
```

Out[714]= $-\frac{e^{x/y} x z}{y^2} + \operatorname{Sin}[y z]$

NOTE: Again we obtain the same answer for the divergence of **F** using the explicit formula:

```
In[715]:= D[F1, x] + D[F2, y] + D[F3, z]
```

Out[715]= $-\frac{e^{x/y} x z}{y^2} + \operatorname{Sin}[y z]$

Example 17.9. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left\langle x, y^2, y + z \right\rangle$ and $S = \partial W$ is the boundary of the region W contained in the cylinder $x^2 + y^2 = 4$ between the plane $z = x$ and $z = 8$.

Solution: If S is the boundary of the solid W, then W is given by

$$W = \left\{(x, y, z) : -2 \leq x \leq 2, \ -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \ x \leq z \leq 8\right\}$$

In[716]:=
```
Clear[F, divF, x, y, z]
F[x_, y_, z_] = {x, y^2, y + z}
divF = Div[F[x, y, z], Cartesian[x, y, z]]
```

Out[717]= $\{x, y^2, y + z\}$

Out[718]= $2 + 2\,y$

By the Divergence Theorem, we see that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is given by

In[719]:= $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x}^{8} \text{divF} \, dz \, dy \, dx$

Out[719]= $64\,\pi$

Exercises

NOTE: In order to perform the divergence operation in *Mathematica*, it is necessary to first load the **VectorAnalysis** package. See instructions given at the beginning of this chapter.

In Exercises 1 and 2, find the divergence of the given vector field **F**.

1. $\mathbf{F}(x, y, z) = \langle x\,y\,z, x^2 + y^2 + z^2, x\,y + y\,z + x\,z \rangle$
2. $\mathbf{F}(x, y, z) = \langle e^{x\,y} \cos z, e^{y\,z} \sin z, \ z^2 \rangle$

In Exercises 3 and 4, verify the Divergence Theorem for the given vector field **F** and solid region W.

3. $\mathbf{F}(x, y, z) = \langle x^2\,y, y^2\,z, z^2\,x \rangle$ and $W = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}$ is the unit ball.
4. $\mathbf{F}(x, y, z) = \langle e^x \cos y, e^x \sin y, x\,y\,z \rangle$ and W is the region bounded by the paraboloid $z = x^2 + y^2$ and $z = 4$.

In Exercises 5 and 6, use the Divergence Theorem to calculate the flux of the vector field **F** across the surface S.

5. $\mathbf{F}(x, y, z) = \langle x\,e^z, y^2, y + z\,x \rangle$ and S is tetrahedron bounded by the plane $3\,x + 4\,y + 5\,z = 15$ and the coordinate planes in the first octant.
6. $\mathbf{F}(x, y, z) = \langle x\,y\,z, x^2 + y^2 + z^2, x\,y + y\,z + x\,z \rangle$ and S is the unit cube with vertices at (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1).