4.4 Best Approximation

Given a function $f(x)$ that is continuous on a given interval $[a, b]$, consider approximating it by some polynomial $p(x)$. To measure the error in $p(x)$ as an approximation, introduce

$$E(p) = \max_{a \le x \le b} |f(x) - p(x)|$$

This is called the *maximum error* or *uniform error* of approximation of $f(x)$ by $p(x)$ on $[a, b]$.

With an eye towards efficiency, we want to find the 'best' possible approximation of a given degree n. With this in mind, introduce the following:

$$\begin{aligned}\rho_n(f) &= \min_{\deg(p)\le n} E(p)\\ &= \min_{\deg(p)\le n}\left[\max_{a\le x\le b}|f(x)-p(x)|\right]\end{aligned}$$

The number $\rho_n(f)$ will be the smallest possible uniform error, or *minimax error*, when approximating $f(x)$ by polynomials of degree at most n. If there is a polynomial giving this smallest error, we denote it by $m_n(x)$; thus $E(m_n) = \rho_n(f)$.

$$E(p) = \max_{a\le x\le b}|f(x)-p(x)|$$

Example. Let $f(x) = e^x$ on $[-1, 1]$. In the following table, we give the values of $E(t_n)$, $t_n(x)$ the Taylor polynomial of degree n for e^x about $x = 0$, and $E(m_n)$.

	Maximum Error in:	
n	$t_n(x)$	$m_n(x)$
1	7.18E − 1	2.79E − 1
2	2.18E − 1	4.50E − 2
3	5.16E − 2	5.53E − 3
4	9.95E − 3	5.47E − 4
5	1.62E − 3	4.52E − 5
6	2.26E − 4	3.21E − 6
7	2.79E − 5	2.00E − 7
8	3.06E − 6	1.11E − 8
9	3.01E − 7	5.52E − 10

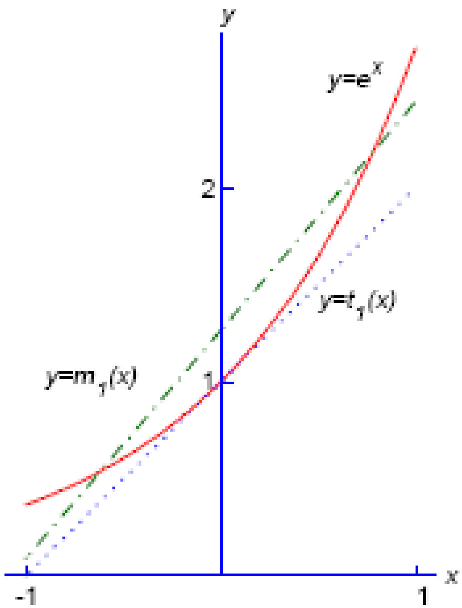
Consider graphically how we can improve on the Taylor polynomial

$$t_1(x) = 1 + x$$

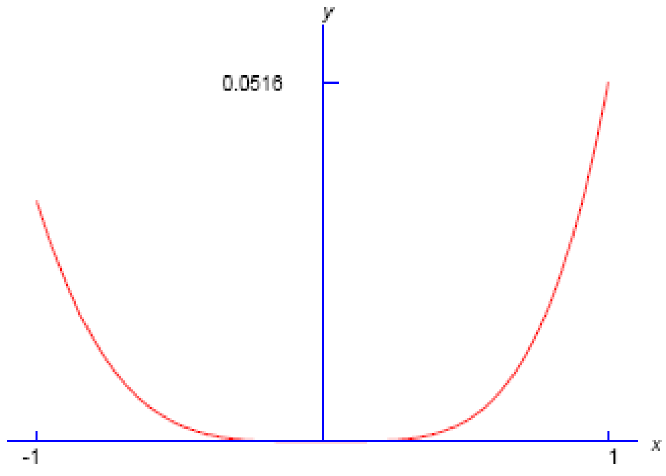
as a uniform approximation to e^x on the interval $[-1, 1]$.

The linear minimax approximation is

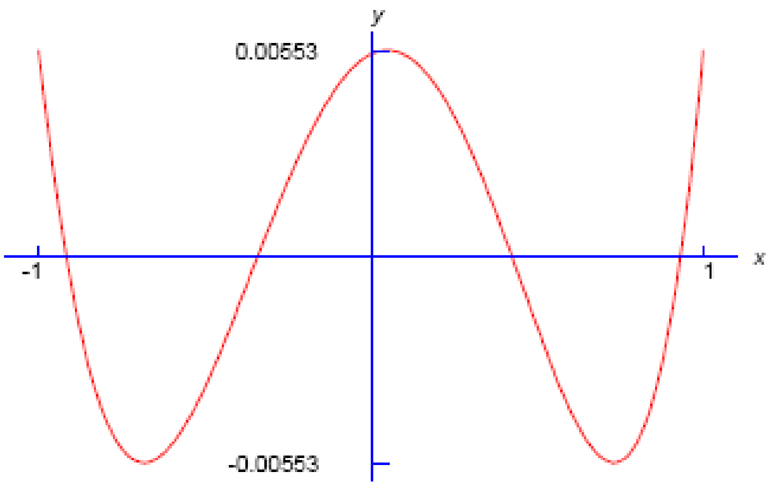
$$m_1(x) = 1.2643 + 1.1752x$$



Linear Taylor and minimax approximations to e^x



Error in cubic Taylor approximation to e^x



Error in cubic minimax approximation to e^x

Accuracy of the minimax approximation.

$$\rho_n(f) \leq \frac{[(b-a)/2]^{n+1}}{(n+1)!2^n} \max_{a \leq x \leq b} \left| f^{(n+1)}(x) \right|$$

This error bound does not always become smaller with increasing n, but it will give a fairly accurate bound for many common functions $f(x)$.

Example. Let $f(x) = e^x$ for $-1 \leq x \leq 1$. Then

$$\rho_n(e^x) \leq \frac{e}{(n+1)!2^n} \qquad (*)$$

n	*Bound* (*)	$\rho_n(f)$
1	6.80E − 1	2.79E − 1
2	1.13E − 1	4.50E − 2
3	1.42E − 2	5.53E − 3
4	1.42E − 3	5.47E − 4
5	1.18E − 4	4.52E − 5
6	8.43E − 6	3.21E − 6
7	5.27E − 7	2.00E − 7

4.5 Chebyshev Polynomials

Chebyshev polynomials are used in many parts of numerical analysis, and more generally, in applications of mathematics. For an integer $n \geq 0$, define the function

$$T_n(x) = \cos\left(n \cos^{-1} x\right), \qquad -1 \leq x \leq 1 \qquad (1)$$

This may not appear to be a polynomial, but we will show it is a polynomial of degree n. To simplify the manipulation of (1), we introduce

$$\theta = \cos^{-1}(x) \quad \text{or} \quad x = \cos(\theta), \qquad 0 \leq \theta \leq \pi \quad (2)$$

$$\theta = \cos^{-1}(x) \quad \text{or} \quad x = \cos(\theta), \qquad 0 \leq \theta \leq \pi \quad (2)$$

Then

$$T_n(x) = \cos(n\theta) \qquad (3)$$

Example. $n = 0$

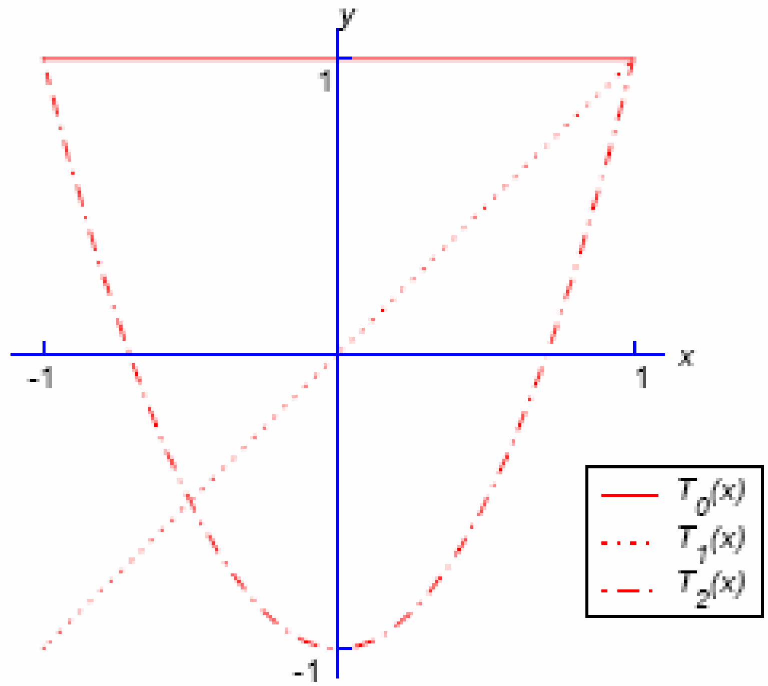
$$T_0(x) = \cos(0 \cdot \theta) = 1$$

$n = 1$

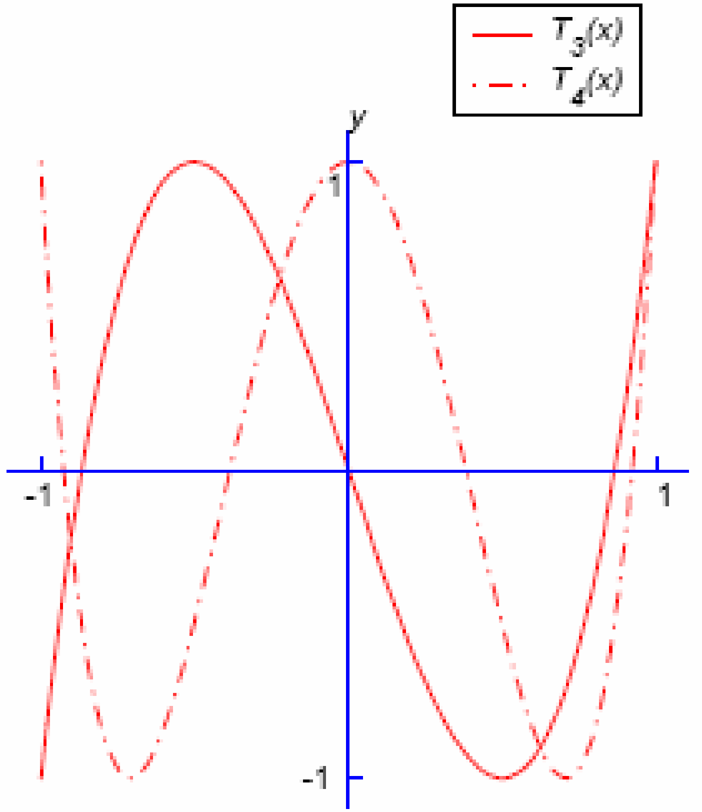
$$T_1(x) = \cos(\theta) = x$$

$n = 2$

$$T_2(x) = \cos(2\theta) = 2\cos^2(\theta) - 1 = 2x^2 - 1$$



y
1
-1
x
-1
1
$T_0(x)$
$T_1(x)$
$T_2(x)$



$T_3(x)$
$T_4(x)$
y
1
-1
x
-1
1

The triple recursion relation. Recall the trigonometric addition formulas,

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Let $n \geq 1$, and apply these identities to get

$$\begin{aligned} T_{n+1}(x) &= \cos[(n+1)\theta] = \cos(n\theta + \theta) \\ &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) \\ T_{n-1}(x) &= \cos[(n-1)\theta] = \cos(n\theta - \theta) \\ &= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta) \end{aligned}$$

Add these two equations, and then use (1) and (3) to obtain

$$T_{n+1}(x) + T_{n-1} = 2\cos(n\theta)\cos(\theta) = 2xT_n(x)$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1 \tag{4}$$

This is called the *triple recursion relation* for the Chebyshev polynomials. It is often used in evaluating them, rather than using the explicit formula (1).

Example. Recall

$$T_0(x) = 1, \qquad T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1$$

Let $n = 2$. Then

$$\begin{aligned} T_3(x) &= 2xT_2(x) - T_1(x) \\ &= 2x(2x^2 - 1) - x \\ &= 4x^3 - 3x \end{aligned}$$

Let $n = 3$. Then

$$\begin{aligned} T_4(x) &= 2xT_3(x) - T_2(x) \\ &= 2x(4x^3 - 3x) - (2x^2 - 1) \\ &= 8x^4 - 8x^2 + 1 \end{aligned}$$

The minimum size property. Note that

$$|T_n(x)| \le 1, \qquad -1 \le x \le 1 \tag{5}$$

for all $n \ge 0$. Also, note that

$$T_n(x) = 2^{n-1}x^n + \text{lower degree terms}, \qquad n \ge 1 \tag{6}$$

This can be proven using the triple recursion relation and mathematical induction.

Introduce a modified version of $T_n(x)$,

$$\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x) = x^n + \text{lower degree terms} \quad (7)$$

From $|T_n(x)| \leq 1$, and $T_n(x) = 2^{n-1}x^n + \text{lower degree terms}$,

we get

$$\left|\tilde{T}_n(x)\right| \leq \frac{1}{2^{n-1}}, \qquad -1 \leq x \leq 1, \qquad n \geq 1 \quad (8)$$

Example.

$$\tilde{T}_4(x) = \frac{1}{8}\left(8x^4 - 8x^2 + 1\right) = x^4 - x^2 + \frac{1}{8}$$

A polynomial whose highest degree term has a coefficient of 1 is called a *monic polynomial*. Formula (8) says the monic polynomial $\widetilde{T}_n(x)$ has size $1/2^{n-1}$ on $-1 \leq x \leq 1$, and this becomes smaller as the degree n increases. In comparison,

$$\max_{-1\leq x\leq 1} |x^n| = 1$$

Thus x^n is a monic polynomial whose size does not change with increasing n.

Theorem. Let $n \geq 1$ be an integer, and consider all possible monic polynomials of degree n. Then the degree n monic polynomial with the smallest maximum on $[-1, 1]$ is the modified Chebyshev polynomial $\widetilde{T}_n(x)$, and its maximum value on $[-1, 1]$ is $1/2^{n-1}$.

This result is used in devising applications of Chebyshev polynomials. We apply it to obtain an improved interpolation scheme.

4.6 A Near-Minimax Approximation

Let $f(x)$ be continuous on $[a, b] = [-1, 1]$. Consider approximating f by an interpolatory polynomial of degree at most $n = 3$. Let x_0, x_1, x_2, x_3 be interpolation node points in $[-1, 1]$; let $c_3(x)$ be of degree ≤ 3 and interpolate $f(x)$ at $\{x_0, x_1, x_2, x_3\}$. The interpolation error is

$$f(x) - c_3(x) = \frac{\omega(x)}{4!} f^{(4)}(\xi_x), \quad -1 \leq x \leq 1 \qquad (1)$$

$$\omega(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3) \qquad (2)$$

with ξ_x in $[-1, 1]$. We want to choose the nodes $\{x_0, x_1, x_2, x_3\}$ so as to minimize the maximum value of $|f(x) - c_3(x)|$ on $[-1, 1]$.

$$f(x) - c_3(x) = \frac{\omega(x)}{4!} f^{(4)}(\xi_x), \quad -1 \le x \le 1 \quad (1)$$
$$\omega(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3) \quad (2)$$

From (1), the only general quantity, independent of f, is $\omega(x)$. Thus we choose $\{x_0, x_1, x_2, x_3\}$ to minimize

$$\max_{-1 \le x \le 1} |\omega(x)| \quad (3)$$

Expand to get

$$\omega(x) = x^4 + \text{lower degree terms}$$

This is a monic polynomial of degree 4. From the theorem in the preceding section, the smallest possible value for (3) is obtained with

$$\omega(x) = \widetilde{T}_4(x) = \frac{T_4(x)}{2^3} = \frac{1}{8}(8x^4 - 8x^2 + 1) \quad (4)$$

$$\max_{-1 \le x \le 1} |\omega(x)| \qquad (3)$$

and the smallest value of (3) is $1/2^3$ in this case. The equation (4) defines implicitly the nodes $\{x_0, x_1, x_2, x_3\}$: they are the roots of $T_4(x)$.

In our case this means solving

$$T_4(x) = \cos(4\theta) = 0, \qquad x = \cos(\theta)$$

$$4\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$$

$$\theta = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}, \pm\frac{5\pi}{8}, \pm\frac{7\pi}{8}, \ldots$$

$$x = \cos\left(\frac{\pi}{8}\right), \cos\left(\frac{3\pi}{8}\right), \cos\left(\frac{5\pi}{8}\right), \ldots \qquad (5)$$

using $\cos(-\theta) = \cos(\theta)$.

$$x = \cos\left(\frac{\pi}{8}\right), \cos\left(\frac{3\pi}{8}\right), \cos\left(\frac{5\pi}{8}\right), \cos\left(\frac{7\pi}{8}\right), \ldots$$

The first four values are distinct; the following ones are repetitive. For example,

$$\cos\left(\frac{9\pi}{8}\right) = \cos\left(\frac{7\pi}{8}\right)$$

The first four values are

$$\{x_0, x_1, x_2, x_3\} = \{\pm 0.382683, \pm 0.923880\} \qquad (6)$$

Example. Let $f(x) = e^x$ on $[-1, 1]$. Use these nodes to produce the interpolating polynomial $c_3(x)$ of degree 3. From the interpolation error formula and the bound of $1/2^3$ for $|\omega(x)|$ on $[-1, 1]$, we have

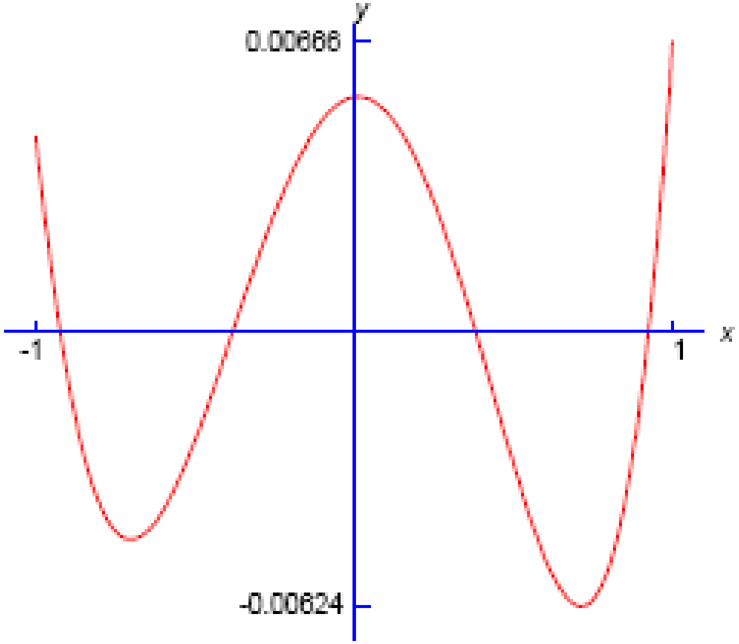
$$\begin{aligned}\max_{-1\le x\le 1} |f(x) - c_3(x)| &\le \frac{1/2^3}{4!} \max_{-1\le x\le 1} e^{\xi_x} \\ &\le \frac{e}{192} \doteq 0.014158\end{aligned}$$

By direct calculation,

$$\max_{-1\le x\le 1} |e^x - c_3(x)| \doteq 0.00666$$

Interpolation Data: $f(x) = e^x$

i	x_i	$f(x_i)$	$f[x_0, \ldots, x_i]$
0	0.923880	2.5190442	2.5190442
1	0.382683	1.4662138	1.9453769
2	−0.382683	0.6820288	0.7047420
3	−0.923880	0.3969760	0.1751757



The error $e^x - c_3(x)$

For comparison, $E(t_3) \doteq 0.0142$ and $\rho_3(e^x) \doteq 0.00553$.

For $f(x) = e^x$, find $c_1(x)$.

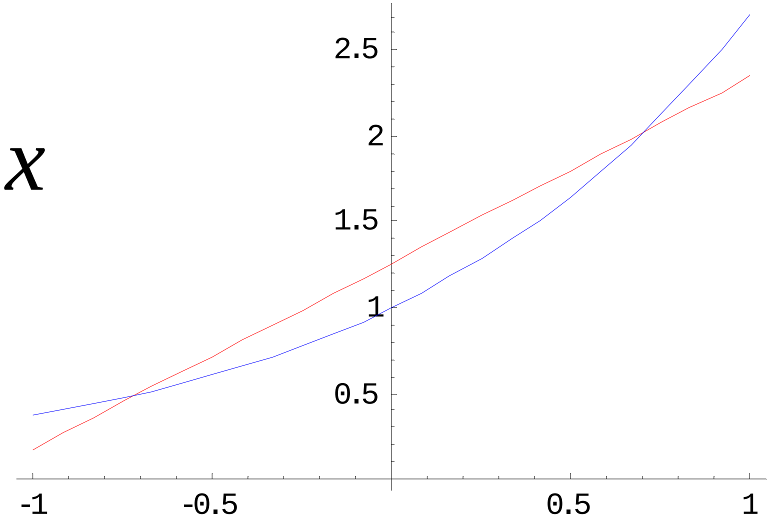
In this case we have

$$w(x) = (x - x_0)(x - x_1) = \frac{T_2(x)}{2} = x^2 - \frac{1}{2}$$

Hence $x_0 = -1/\sqrt{2} = -0.707107$ and $x_1 = 1/\sqrt{2} = 0.707107$

But then $y_0 = e^{-0.707107} = 0.493069$ and $y_1 = e^{0.707107} = 2.02811$

$$c_1(x) = 1.26059 + 1.08544x$$



THE GENERAL CASE

Consider interpolating $f(x)$ on $[-1,1]$ by a polynomial of degree $\leq n$, with the interpolation nodes $\{x_0, \ldots, x_n\}$ in $[-1,1]$. Denote the interpolation polynomial by $c_n(x)$. The interpolation error on $[-1,1]$ is given by

$$\begin{aligned} f(x) - c_n(x) &= \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\xi_x) \qquad (7) \\ \omega(x) &= (x - x_0) \cdots (x - x_n) \end{aligned}$$

with ξ_x and unknown point in $[-1,1]$. In order to minimize the interpolation error, we seek to minimize

$$\max_{-1 \leq x \leq 1} |\omega(x)| \qquad (8)$$

The polynomial being minimized is monic of degree $n+1$,

$$\omega(x) = x^{n+1} + \text{ lower degree terms}$$

From the theorem of the preceding section, this minimum is attained by the monic polynomial

$$\widetilde{T}_{n+1}(x) = \frac{1}{2^n} T_{n+1}(x)$$

Thus the interpolation nodes are the zeros of $T_{n+1}(x)$; and by the procedure that led to (5), they are given by

$$x_j = \cos\left(\frac{2j+1}{2n+2}\pi\right), \qquad j = 0, 1, \ldots, n \qquad (9)$$

The near-minimax approximation $c_n(x)$ of degree n is obtained by interpolating to $f(x)$ at these $n+1$ nodes on $[-1, 1]$.

The polynomial $c_n(x)$ is sometimes called a *Chebyshev approximation*.

Example. Let $f(x) = e^x$. the following table contains the maximum errors in $c_n(x)$ on $[-1, 1]$ for varying n. For comparison, we also include the corresponding minimax errors. These figures illustrate that for practical purposes, $c_n(x)$ is a satisfactory replacement for the minimax approximation $m_n(x)$.

n	$\max\|e^x - c_n(x)\|$	$\rho_n(e^x)$
1	3.72E − 1	2.79E − 1
2	5.65E − 2	4.50E − 2
3	6.66E − 3	5.53E − 3
4	6.40E − 4	5.47E − 4
5	5.18E − 5	4.52E − 5
6	3.80E − 6	3.21E − 6

THEORETICAL INTERPOLATION ERROR

For the error

$$f(x) - c_n(x) = \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\xi_x)$$

we have

$$\max_{-1 \le x \le 1} |f(x) - c_n(x)| \le \frac{\max\limits_{-1 \le x \le 1} |\omega(x)|}{(n+1)!} \max_{-1 \le \xi \le 1} |f(\xi)|$$

From the theorem of the preceding section,

$$\max_{-1 \le x \le 1} \left|\widetilde{T}_{n+1}(x)\right| = \max_{-1 \le x \le 1} |\omega(x)| = \frac{1}{2^n}$$

in this case. Thus

$$\max_{-1 \le x \le 1} |f(x) - c_n(x)| \le \frac{1}{(n+1)!2^n} \max_{-1 \le \xi \le 1} |f(\xi)|$$

OTHER INTERVALS

Consider approximating $f(x)$ on the finite interval $[a, b]$. Introduce the linear change of variables

$$x = \frac{1}{2}\left[(1-t)\,a + (1+t)\,b\right] \tag{10}$$

$$t = \frac{2}{b-a}\left[x - \frac{b+a}{2}\right] \tag{11}$$

Introduce

$$F(t) = f\left(\frac{1}{2}\left[(1-t)\,a + (1+t)\,b\right]\right), \quad -1 \le t \le 1$$

The function $F(t)$ on $[-1, 1]$ is equivalent to $f(x)$ on $[a, b]$, and we can move between them via (10)-(11). We can now proceed to approximate $f(x)$ on $[a, b]$ by instead approximating $F(t)$ on $[-1, 1]$.

Example. Approximating $f(x) = \cos x$ on $[0, \pi/2]$ is equivalent to approximating

$$F(t) = \cos\left(\frac{1+t}{4}\pi\right), \quad -1 \le t \le 1$$

4.7 Least Square Approximation

Another approach to approximating a function $f(x)$ on an interval $a \leq x \leq b$ is to seek an approximation $p(x)$ with a small 'average error' over the interval of approximation. A convenient definition of the average error of the approximation is given by

$$E(p;f) \equiv \left[\frac{1}{b-a}\int_a^b [f(x)-p(x)]^2\,dx\right]^{\frac{1}{2}} \qquad (1)$$

This is also called the *root-mean-square-error* (denoted subsequently by *RMSE*) in the approximation of $f(x)$ by $p(x)$. Note first that choosing $p(x)$ to minimize $E(p;f)$ is equivalent to minimizing

$$E(p;f) \equiv \left[\frac{1}{b-a}\int_a^b [f(x)-p(x)]^2\,dx\right]^{\frac{1}{2}} \qquad (1)$$

Note first that choosing $p(x)$ to minimize $E(p;f)$ is equivalent to minimizing

$$\int_a^b [f(x)-p(x)]^2\,dx$$

thus dispensing with the square root and multiplying fraction (although the minimums are generally different). The minimizing of (1) is called the *least squares approximation problem*.

Example. Let $f(x) = e^x$, let $p(x) = \alpha_0 + \alpha_1 x$, α_0, α_1 unknown. Approximate $f(x)$ over $[-1, 1]$.

Choose α_0, α_1 to minimize

$$g(\alpha_0, \alpha_1) \equiv \int_{-1}^{1} \left[e^x - \alpha_0 - \alpha_1 x\right]^2 dx \qquad (2)$$

$$g(\alpha_0, \alpha_1) = \int_{-1}^{1} \left\{ \begin{array}{c} e^{2x} + \alpha_0^2 + \alpha_1^2 x^2 - 2\alpha_0 e^x \\ -2\alpha_1 x e^x + 2\alpha_0 \alpha_1 x \end{array} \right\} dx$$

Integrating,

$$g(\alpha_0, \alpha_1) = c_1\alpha_0^2 + c_2\alpha_1^2 + c_3\alpha_0\alpha_1 + c_4\alpha_0 + c_5\alpha_1 + c_6$$

with constants $\{c_1, \ldots, c_6\}$, e.g.

$$c_1 = 2, \qquad c_6 = \left(e^1 - e^{-1}\right)/2.$$

$$g(\alpha_0, \alpha_1) = c_1\alpha_0^2 + c_2\alpha_1^2 + c_3\alpha_0\alpha_1 + c_4\alpha_0 + c_5\alpha_1 + c_6$$

g is a quadratic polynomial in the two variables α_0 α_1.

To find its minimum, solve the system

$$\frac{\partial g}{\partial \alpha_0} = 0, \qquad \frac{\partial g}{\partial \alpha_1} = 0$$

It is simpler to return to (2) to differentiate,

$$g(\alpha_0, \alpha_1) \equiv \int_{-1}^{1} [e^x - \alpha_0 - \alpha_1 x]^2 \, dx \qquad (2)$$

obtaining

$$2\int_{-1}^{1} [e^x - \alpha_0 - \alpha_1 x]\,(-1)\; dx = 0$$
$$2\int_{-1}^{1} [e^x - \alpha_0 - \alpha_1 x]\,(-x)\; dx = 0$$

This simplifies to

$$2\alpha_0 = \int_{-1}^{1} e^x dx = e - e^{-1}$$

$$\frac{2}{3}\alpha_1 = \int_{-1}^{1} x e^x dx = 2e^{-1}$$

$$\alpha_0 = \frac{e - e^{-1}}{2} \doteq 1.1752$$

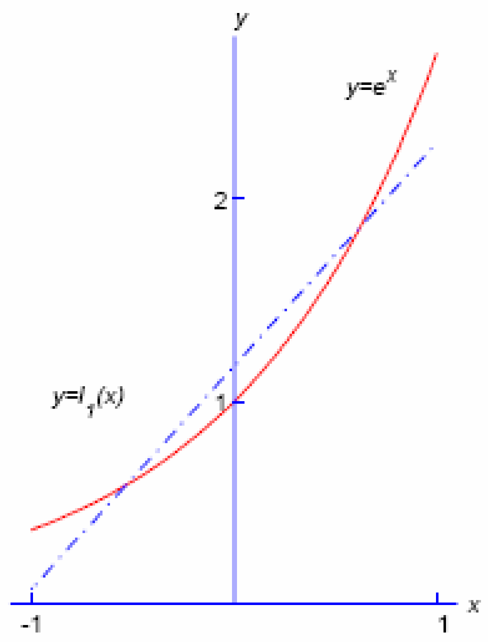
$$\alpha_1 = 3e^{-1} \doteq 1.1036$$

Using these values for α_0 and α_1, we denote the resulting linear approximation by

$$\ell_1(x) = \alpha_0 + \alpha_1 x$$

It is called the best linear approximation to e^x in the *sense of least squares*. For the error,

$$\max_{-1 \le x \le 1} \left| e^x - \ell_1(x) \right| \doteq 0.439$$



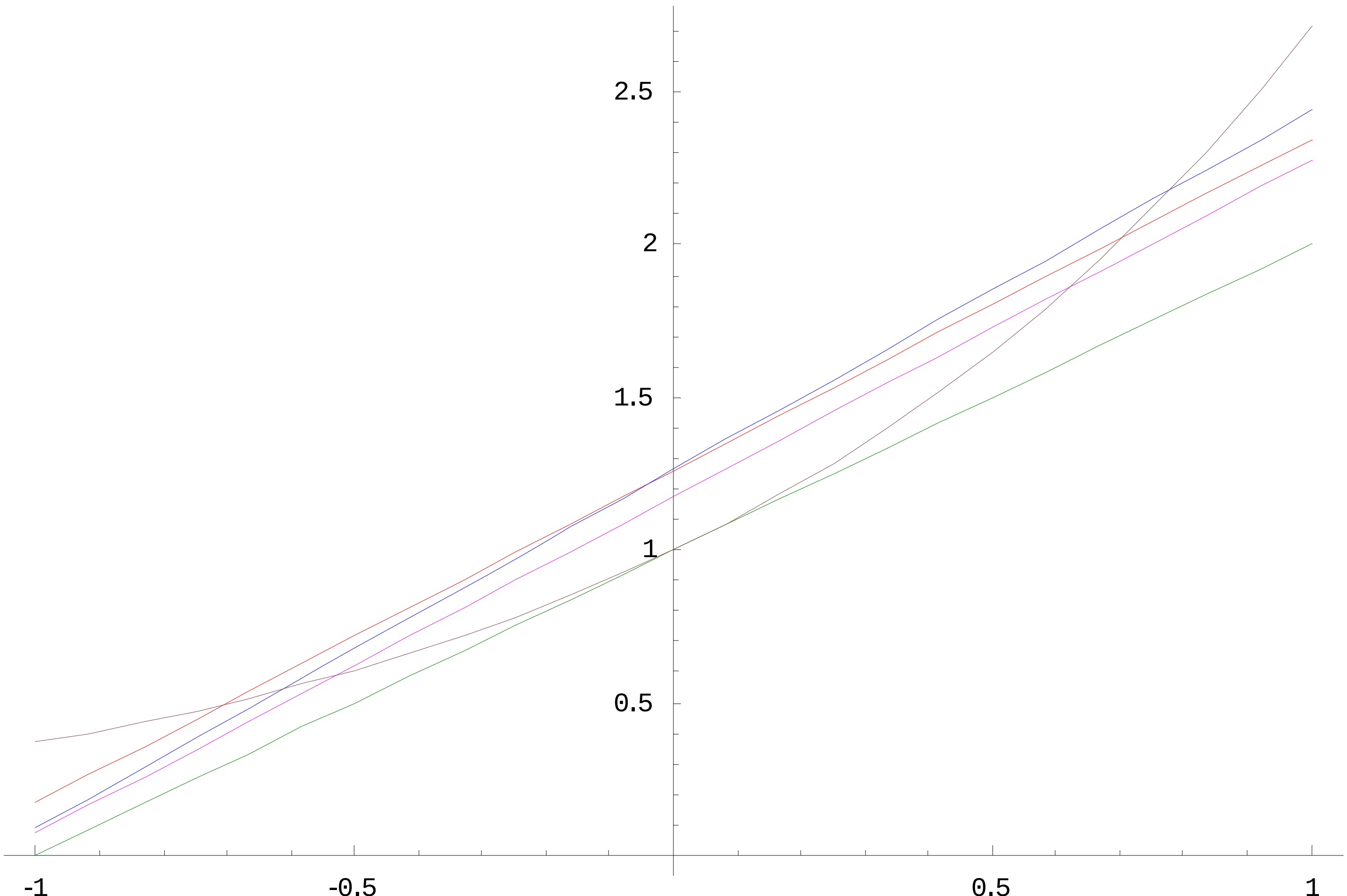
The linear least squares approximation to e^x

$$t_1(x) = 1 + x$$

$$m_1(x) = 1.2643 + 1.1752x$$

$$c_1(x) = 1.26059 + 1.08544x$$

$$l_1(x) = 1.1752 + 1.1036x$$



Errors in linear approximations of e^x:

Approximation	*Max Error*	*RMSE*
Taylor $t_1(x)$	0.718	0.246
Least squares $\ell_1(x)$	0.439	0.162
Chebyshev $c_1(x)$	0.372	0.184
Minimax $m_1(x)$	0.279	0.190

RMSE= Root Mean Square Error

$$E(p;f) = \sqrt{\frac{1}{b-a}\int_a^b \left[f(x) - p(x)\right]^2 dx}$$

THE GENERAL CASE

Approximate $f(x)$ on $[a, b]$, and let $n \geq 0$. Seek $p(x)$ to minimize the *RMSE*. Write

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$$g(\alpha_0, \alpha_1, \ldots, \alpha_n) \equiv \int_{-1}^{1} \begin{bmatrix} f(x) - \alpha_0 - \alpha_1 x \\ - \cdots - \alpha_n x^n \end{bmatrix}^2 dx$$

Find coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n$ to minimize this integral. The integral $g(\alpha_0, \alpha_1, \ldots, \alpha_n)$ is a quadratic polynomial in the $n + 1$ variables $\alpha_0, \alpha_1, \ldots, \alpha_n$.

To minimize $g(\alpha_0, \alpha_1, \ldots, \alpha_n)$, invoke the conditions

$$\frac{\partial g}{\partial \alpha_i} = 0, \qquad i = 0, 1, \ldots, n$$

This yields a set of $n+1$ equations that must be satisfied by a minimizing set $\alpha_0, \alpha_1, \ldots, \alpha_n$ for g. Manipulating this set of conditions leads to a simultaneous linear system.

To better understand the form of the linear system, consider the special case of $[a,b]=[0,1]$. Differentiating g with respect to each α_i, we obtain

$$2\int_{-1}^{1}[e^x-\alpha_0-\cdots-\alpha_n x^n](-1)\,dx=0$$

$$2\int_{-1}^{1}[e^x-\alpha_0-\cdots-\alpha_n x^n](-x)\,dx=0$$

$$\vdots$$

$$2\int_{-1}^{1}[e^x-\alpha_0-\cdots-\alpha_n x^n](-x^n)\,dx=0$$

Then the linear system is

$$\sum_{j=0}^{n}\frac{\alpha_j}{i+j+1}=\int_0^1 x^i f(x)\,dx, \qquad i=0,1,\ldots,n$$

LEGENDRE POLYNOMIALS

Define the *Legendre polynomials* as follows.

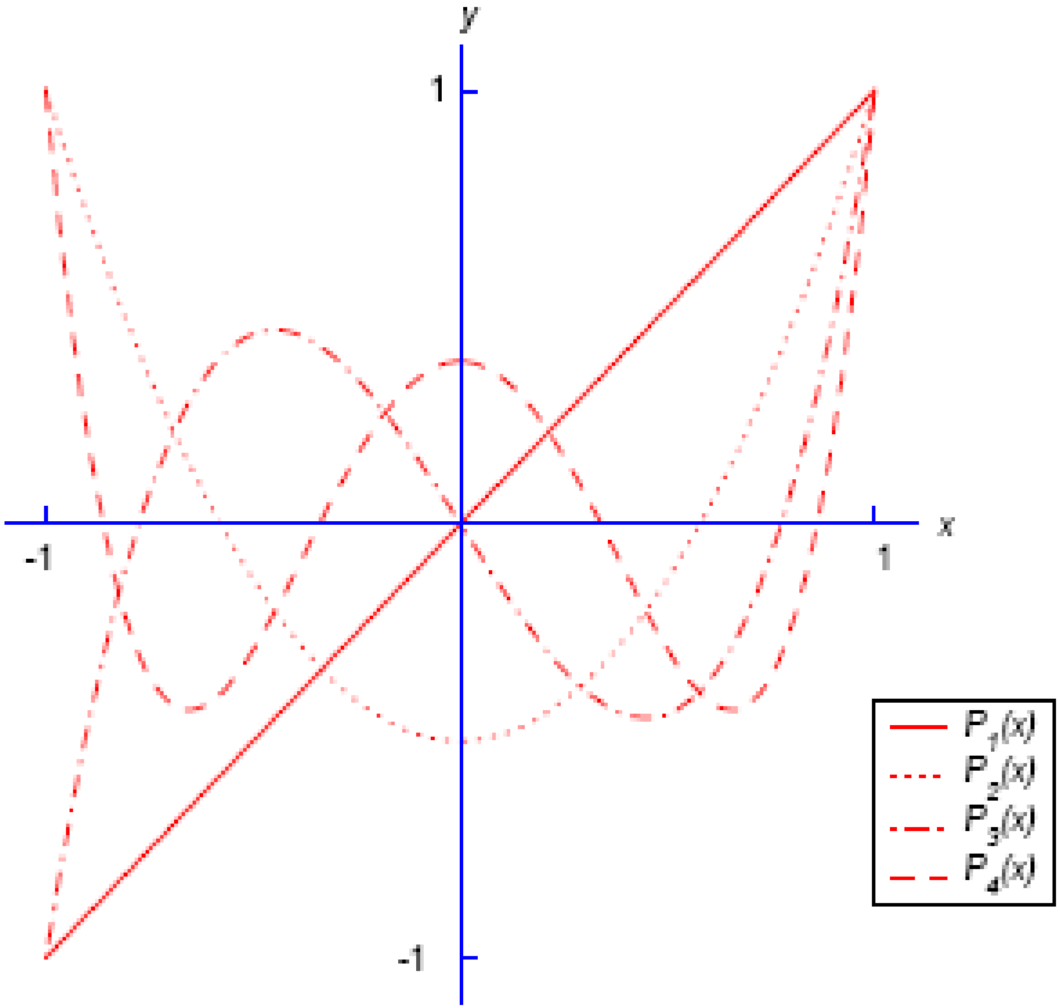
$$P_0(x) = 1$$

$$P_n(x) = \frac{1}{n!2^n} \cdot \frac{d^n}{dx^n}\left[\left(x^2 - 1\right)^n\right], \qquad n = 1, 2, \ldots$$

For example,

$$P_1(x) = x \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \qquad P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$



Legendre polynomials of degrees $1, 2, 3, 4$

PROPERTIES

The Legendre polynomials have many special properties, and they are widely used in numerical analysis and applied mathematics.

Introduce the special notation

$$(f, g) = \int_a^b f(x)g(x)\,dx$$

for general functions $f(x)$ and $g(x)$.

- *Degree and normalization*:

$$\deg P_n = n, \qquad P_n(1) = 1, \qquad n \geq 0$$

- *Triple recursion relation*: For $n \geq 1$,

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

- *Orthogonality and size*:

$$\left(P_i, P_j\right) = \begin{cases} 0, & i \neq j \\ \dfrac{2}{2j+1}, & i = j \end{cases}$$

- *Zeroes*:

  All zeroes of $P_n(x)$ are located in $[-1, 1]$;
  all zeroes are simple roots of $P_n(x)$

- *Basis*: Every polynomial $p(x)$ of degree $\leq n$ can be written in the form

$$p(x) = \sum_{j=0}^{n} \beta_j P_j(x)$$

with the choice of $\beta_0, \beta_1, \ldots, \beta_n$ uniquely determined from $p(x)$:

$$\beta_j = \frac{(p, P_j)}{(P_j, P_j)}, \qquad j = 0, 1, \ldots, n$$

FINDING THE LEAST SQUARES APPROXIMATION

We solve the least squares approximation problem on only the interval $[-1, 1]$. Approximation problems on other intervals $[a, b]$ can be accomplished using a linear change of variable.

We seek to find a polynomial $p(x)$ of degree n that minimizes

$$\int_a^b [f(x) - p(x)]^2 \, dx$$

This is equivalent to minimizing

$$(f - p, f - p) \tag{3}$$

We begin by writing $p(x)$ in the form

$$p(x) = \sum_{j=0}^{n} \beta_j P_j(x)$$

Substitute into (3) $\quad (f - p, f - p)$

$$\begin{aligned}\tilde{g}\,(\beta_0, \beta_1, \ldots, \beta_n) &\equiv (f - p, f - p) \\ &= \left(f - \sum_{j=0}^{n} \beta_j P_j, f - \sum_{i=0}^{n} \beta_i P_i \right)\end{aligned}$$

$$\tilde{g}(\beta_0, \beta_1, \ldots, \beta_n) \equiv (f - p, f - p)$$
$$= \left(f - \sum_{j=0}^{n} \beta_j P_j, f - \sum_{i=0}^{n} \beta_i P_i \right)$$

Expand this into the following:

$$\tilde{g} = (f, f) - \sum_{j=0}^{n} \frac{(f, P_j)^2}{(P_j, P_j)}$$
$$+ \sum_{j=0}^{n} \left(P_j, P_j\right) \left[\beta_j - \frac{(f, P_j)}{(P_j, P_j)} \right]^2$$

Looking at this carefully, we see that it is smallest when

$$\beta_j = \frac{(f, P_j)}{(P_j, P_j)}, \qquad j = 0, 1, \ldots, n$$

$$p(x) = \sum_{j=0}^{n} \beta_j P_j(x) \quad \text{minimizes} \quad \int_a^b [f(x) - p(x)]^2 \, dx$$

$$\Longrightarrow \quad \tilde{g} = (f, f) - \sum_{j=0}^{n} \frac{(f, P_j)^2}{(P_j, P_j)} + \sum_{j=0}^{n} \left(P_j, P_j\right) \left[\beta_j - \frac{(f, P_j)}{(P_j, P_j)}\right]^2$$

$$\Longrightarrow \quad \beta_j = \frac{(f, P_j)}{(P_j, P_j)}, \qquad j = 0, 1, \ldots, n$$

$$\Longrightarrow \quad \tilde{g} = (f, f) - \sum_{j=0}^{n} \frac{(f, P_j)^2}{(P_j, P_j)}$$

We call

$$\ell_n(x) = \sum_{j=0}^{n} \frac{(f, P_j)}{(P_j, P_j)} P_j(x) \tag{4}$$

the *least squares approximation of degree* n to $f(x)$ on $[-1, 1]$.

$$\beta_j = \frac{(f, P_j)}{(P_j, P_j)}, \qquad j = 0, 1, \ldots, n$$

If $\beta_n = 0$, then its actual degree is less than n.

Example. Approximate $f(x) = e^x$ on $[-1, 1]$.

We use $\ell_n(x) = \sum_{j=0}^{n} \frac{(f, P_j)}{(P_j, P_j)} P_j(x)$ with $n = 3$:

$$\ell_3(x) = \sum_{j=0}^{3} \beta_j P_j(x), \quad \beta_j = \frac{(f, P_j)}{(P_j, P_j)}$$

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

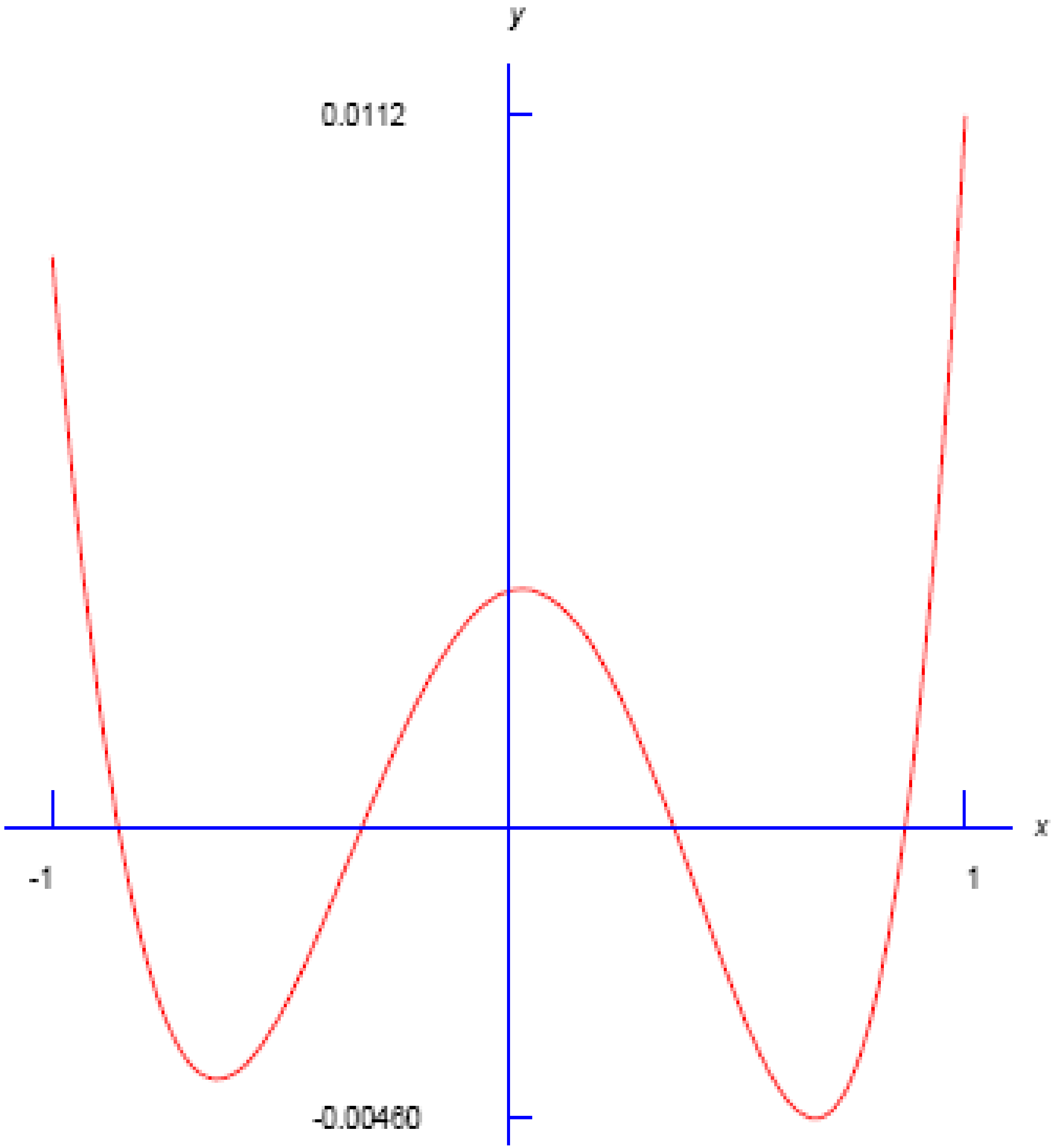
$$(f, g) = \int_a^b f(x) g(x)\, dx$$

The coefficients $\{\beta_0, \beta_1, \beta_2, \beta_3\}$ are as follows.

j	0	1	2	3
β_j	2.35040	0.73576	0.14313	0.02013

$$\ell_3(x) = \sum_{j=0}^{3} \beta_j P_j(x), \quad \beta_j = \frac{(f, P_j)}{(P_j, P_j)}$$

$$\ell_3(x) = .996294 + .997955x + .536722x^2 + .176139x^3$$



Error in the cubic least squares approximation to e^x

The error in various cubic approximations:

Approximation	*Max Error*	*RMSE*
Taylor $t_3(x)$	.0516	.0145
Least squares $\ell_3(x)$	.0112	.00334
Chebyshev $c_3(x)$	.00666	.00384
Minimax $m_3(x)$	.00553	.00388