Translation of Euler's paper E797

Recherches sur le probléme de quatre nombres positives et en proportion arithmétique tels, que la somme de deux quelconques soit toujours un nombre carré.

"Research into the problem of four positive numbers and such an arithmetic proportion of them, that the sum of any two of them is always a square number."

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NOTE: This translation is done by Kathryn Robertson who is an undergraduate student at Rowan University. Abdul Hassen is a professor of mathematics at Rowan and helped with mathematical aspect of the paper. It is a pleasant experience for both of us to have worked on this project. We have tried to keep Euler's notations in as much as possible. The few exceptions where we differed from his notations are in expressing decimals. For example, Euler uses 2,34 while we used 2.34. At the end of translation, we added notes by sections in which we summarized the main ideas of the sections and indicated what Euler has assumed the reader should know.

1. Let A, B, C, D be four numbers, arranged in the order of size, in particular, we assume that $A < B < C < D$. The six conditions that need to be satisfied are:

$$A + B = pp,$$
$$A + C = qq,$$
$$A + D = rr = B + C,$$
$$B + D = ss,$$
$$C + D = tt,$$

Then $2rr = pp + tt = qq + ss$, and the four numbers A, B, C, D can be expressed as

$$2A = pp + qq - rr$$
$$2B = pp + rr - qq$$
$$2C = qq + rr - pp$$
$$2D = 3rr - pp - qq$$

If the number A is positive, then $pp + qq > rr$. Note then that the numbers $B, C, D,$ are also positive. From the condition $2rr = pp + tt = qq + ss$, we conclude that $p < t$ and $q < s$.

2. Furthermore, from $2rr = pp + tt$, we conclude that r is equal to the sum of two squares: $r = xx + yy$. From this we get $rr = (xx - yy)^2 + (2xy)^2$ and hence

$$2rr = (\pm(xx - yy) - 2xy)^2 + (\pm(xx - yy) + 2xy)^2.$$

Here and throughout the rest of this paper, the signs $\pm$ are determined so that $xx - yy$ is positive. From the above equation, we see that $2rr = pp + tt$ and the condition $p < t$ will be satisfied if we take

$$p = \pm(xx - yy) - 2xy \text{ and } t = \pm(xx - yy) + 2xy.$$

Similarly the equation $2rr = qq + ss$ implies that r has another decomposition as a sum of two squares: $r = x'x' + y'y'$. As before we see that the conditions $2rr = qq + ss$ and $q < s$ will be satisfied if we let

$$q = \pm(x'x' - y'y') - 2x'y' \text{ and } s = \pm(x'x' - y'y') + 2x'y'.$$

From $p = \pm(xx - yy) - 2xy$ and $r = xx + yy$, we get

$$pp = (xx - yy)^2 \mp 4xy(xx - yy) + 4xxyy = (xx + yy)^2 \mp 4xy(xx - yy) = rr \mp 4xy(xx - yy).$$

Similarly, from the equations $q = \pm(x'x' - y'y') - 2x'y'$ and $r = x'x' + y'y'$, we get

$$qq = rr \mp 4x'y'(x'x' - y'y').$$

Using the last two expressions for pp and qq, we note that the condition $pp + qq > rr$ becomes

$$rr > \pm 4xy(xx - yy) \pm 4x'y'(x'x' - y'y').$$

The two double signs $\pm$ are determined so that $xx - yy$ and $x'x' - y'y'$ are positive.

3. Since r must be the sum of two squares of in two different ways, r must be the product of two factors of this same form. In other words, $r = (aa + bb)(cc + dd)$, where we may suppose $a > b, c > d$. For reducing the number of letters in our investigation, we let $\frac{a}{b} = f$ and $\frac{c}{d} = z$. Note that $f > 1$ and $z > 1$. Thus $r = bbdd(ff + 1)(zz + 1)$. Since the square factor $bbdd$ is common to all of the terms of the inequality $rr > \pm 4xy(xx - yy) \pm 4x'y'(x'x' - y'y')$ that we will be considering, we cancel this factor from both sides. Thus we may assume that

$$r = (ff + 1)(zz + 1).$$

From this and the equations $r = xx + yy$ and $r = x'x' + y'y'$, the values of x, y, x', y' can be given by

$$x = fz + 1, \quad x' = fz - 1, \quad y = z - f, \quad y' = z + f.$$

But then

$$xy(xx - yy) = (fz + 1)(z - f)((f + 1)z - f + 1)((f - 1)z + f + 1) = M,$$

$$x'y'(x'x' - y'y') = (fz - 1)(z + f)((f + 1)z + f - 1)((f - 1)z - f - 1) = N.$$

The inequality $rr > \pm 4xy(xx - yy) \pm 4x'y'(x'x' - y'y')$ becomes

$$rr > \pm 4M \pm 4N.$$

As before, one chooses signs $\pm$ so that $\pm M$ and $\pm N$ are positive.

4. To simplify the expressions for M and N, we introduce a new variable ρ: $\rho = \frac{f+1}{f-1}$. Note that $f > 1$ and $\rho > 1$. We now express $\frac{M}{(f-1)^2}$ and $\frac{N}{(f-1)^2}$ in terms of z, f, and ρ:

$$\frac{M}{(f-1)^2} = (fz+1)(z-f)(\rho z-1)(z+\rho) = P,$$
$$\frac{N}{(f-1)^2} = (fz-1)(z+f)(\rho z+1)(z-\rho) = Q.$$
Our inequality now becomes
$$\frac{rr}{(f-1)^2} > 4\ (\pm P \pm Q),$$
where we must always take the signs $\pm$ to make $\pm P \pm Q$ positive.

5. When considering this formula, we first note that f and ρ are permutable with each other, since when substituting one for the other, the value P changes to Q and vice versa. Also from $\rho = \frac{(f+1)}{(f-1),}$, we can solve for f to get $f = \frac{(\rho+1)}{(\rho-1)}$. Note then that we have $f\rho - \rho - f = 1$, or $(f-1)(\rho-1) = 2$.

Observe here that, in the case when $f = \rho$, we then have $f - 1 = \rho - 1 = \sqrt{2}$, and consequently, $f = \rho = 1+\sqrt{2}.$ In all the other cases, one of the numbers is smaller and the other is greater than the number $1+\sqrt{2}.$ Thus, if we suppose $\rho > f,$ then we have $f < 1+\sqrt{2}$. When $f = 1,$ the value of ρ becomes infinitely large.

6. From $r = (ff+1)(zz+1),$ we have $rr = (ff+1)^2(zz+1)^2$ and hence
$$\frac{rr}{(f-1)^2} = \frac{(ff+1)^2(zz+1)^2}{(f-1)^2}.$$
However, we also have $\frac{(ff+1)}{(f-1)} = \frac{(\rho\rho+1)}{(\rho-1)}$ and from this we obtain
$$\frac{rr}{(f-1)^2} = \frac{(ff+1)^2(zz+1)^2}{(f-1)^2} = \frac{(ff+1)(\rho\rho+1)}{(f-1)(\rho-1)}(zz+1)^2.$$
But the value of the product $(f-1)(\rho-1)$ is equal to 2, as we have shown above. Thus the above reduces to
$$\frac{rr}{(f-1)^2} = \frac{1}{2}(ff+1)(\rho\rho+1)(zz+1)^2.$$
Substituting this expression of $\frac{rr}{(f-1)^2}$ into the preceding inequality, we find that the our inequality becomes
$$(ff+1)(\rho\rho+1)(zz+1)^2 > 8\ (\pm P \pm Q).$$

7. Expanding the expressions for P and Q, we have
$$P = f\rho z^4 + (f\rho+1)(\rho-f)z^3 - \left((ff\rho\rho+1)-(\rho-f)^2\right)zz - (f\rho+1)(\rho-f)z + f\rho,$$
$$Q = f\rho z^4 - (f\rho+1)(\rho-f)z^3 - \left(ff\rho\rho+1-(\rho-f)^2\right)zz + (f\rho+1)(\rho-f)z + f\rho.$$
The coefficient of zz can be reduced to a very simple form. Since

$$(\rho - f)^2 = (\rho + f)^2 - 4f\rho,$$

the coefficient can be written as $ff\rho\rho + 1 + 4f\rho - (\rho + f)^2$. But we have seen (5) that $\rho + f = f\rho - 1$ and hence $(\rho + f)^2 = ff\rho\rho - 2f\rho + 1$. Consequently, the coefficient of zz reduces to $6f\rho$. Thus we have

$$P = f\rho z^4 + (f\rho + 1)(\rho - f)z^3 - 6f\rho zz - (f\rho + 1)(\rho - f)z + f\rho,$$
$$Q = f\rho z^4 - (f\rho + 1)(\rho - f)z^3 - 6f\rho zz + (f\rho + 1)(\rho - f)z + f\rho.$$

8. The values of P and Q given by the above equations must satisfy the inequality

$$(ff + 1)(\rho\rho + 1)(zz + 1)^2 > 8(\pm P \pm Q).$$

Thus we need to study the following problem:

9. **Problem**. Given the number f, and consequently also ρ, find all the values of z that satisfy the above inequality.

Observe then that from this we find the complete solution of the principal problem, since $f = \frac{a}{b}$ gives the numbers a and b, and $z = \frac{c}{d}$ likewise gives c and d, from which we get x, y, x', y'. These in turn lead to the values of p, q, r and finally to those of A, B, C, D.

10. **Solution.** We begin with the observation that the appropriate values of z are comprised in certain limits in accordance with the value of the number f which is always bigger than 1 and smaller than $1 + \sqrt{2}$, or, in accordance with the values $\rho = \frac{(f + 1)}{(f - 1)}$ which always surpasses $1 + \sqrt{2}$. This limitation can be easily be assigned, once we know the case in which the first member of our formula becomes equal to the other. In other words, we need to know the root of the equation

$$(ff + 1)(\rho\rho + 1)(zz + 1)^2 = 8(\pm P \pm Q).$$

We recall that $z \geq 1$.

11. To understand this equation, we use the equation that relates f and ρ, namely, $\frac{(ff + 1)}{(f - 1)} = \frac{(\rho\rho + 1)}{(\rho - 1)}$. We now introduce a new variable n:

$$\frac{(ff + 1)}{(f - 1)} = 2n, \qquad \frac{(\rho\rho + 1)}{(\rho - 1)} = 2n.$$

When we solve these equations for values of f and ρ, we get

$$n \pm \sqrt{(nn - 2n - 1)}.$$

Or, if f is smaller than ρ, then we have

$$f = n - \sqrt{(nn - 2n - 1)} \quad \text{and} \quad \rho = n + \sqrt{(nn - 2n - 1)}.$$

From this we get

$$f + \rho = 2n, \quad \rho - f = 2\sqrt{(nn - 2n - 1)} \text{ and } f\rho = 2n + 1.$$

Next we let $k=\sqrt{nn-2n-1}$ so that $f=n-k, \quad \rho=n+k, \quad \rho-f=2k.$ It is not difficult to eliminate f and ρ from our equation $(ff+1)(\rho\rho+1)(zz+1)^2=8\,(\pm P\pm Q)$ and express it in terms of n and z.

12. To this end, we start with the first member (left hand side) of our equation. Observe that

$$(ff+1)(\rho\rho+1)=(f\rho-1)^2+(f+\rho)^2$$

and that $f+\rho=2n$ and $f\rho-1=2n.$ Thus the left hand side becomes $8nn(zz+1)^{2,}$ and the equation to resolve is

$$nn(zz+1)^2=\pm P\pm Q.$$

We now consider P and $Q.$ From the fact that $f\rho=2n+1,$ and $\rho-f=2k,$ the expressions for P and Q become

$$P=(2n+1)z^4+4(n+1)kz^3-6(2n+1)zz-4(n+1)kz+2n+1,$$

$$Q=(2n+1)z^4-4(n+1)kz^3-6(2n+1)zz+4(m+1)kz+2n+1.$$

13. To find the values of z in our equation, we must consider with care of the signs of P and Q. First note that when $z>\rho$, both expression in (4) defining P and Q are positive (recall that $\rho>f$) ; thus one takes the sign +. But if z is smaller than f, then P and Q become negative and consequently we must give them the – sign.

Finally, if z is between f and ρ, the P will be positive and Q is negative. After these considerations we see that, in accordance with z being much bigger than ρ or much smaller than f, or finally content between ρ and f, we have three cases to consider. These are numbered as follows.

First Case: When the values of z bigger than ρ.

14. In this case P and Q having the sign + and we have

$$P+Q=2(2n+1)z \quad -12(2n+1)zz+2(2n+1).$$

Our equation becomes

$$nn(z^4+2zz+1)=2(2n+1)(z^4-6zz+1).$$

When $nn>2(2n+1),$ the left hand side always surpasses the right hand side, and consequently, all the values of $z,$ from ρ to the infinite, give rise to a solution to our problem and we always have

$$pp+qq>rr.$$

Note that $nn>2(2n+1)$ holds when $n>2+\sqrt{6}.$ In the case when $n=2+\sqrt{6},$ we have

$$k=\sqrt{nn-2n-1}=\sqrt{5+2\sqrt{6}}=\sqrt{3}+\sqrt{2},$$
$$\rho=n+k=\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}+1\right);$$
$$f=n-k=\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}-1\right).$$

Thus, $n>2+\sqrt{6}$ will take place, when $\rho>\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}+1\right)$ and $f<\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}-1\right)$ or, in decimals, when $\rho>7.59574$ and $f<1.3032254$.
Therefore, whenever $\rho>7.59574$ and $f<1.3032254$, the quantity z can be any number greater than ρ until the infinite.

15. We now consider the case, when $nn<2(2n+1)$, which happens when $n<2+\sqrt{6}$ or when $f>\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}-1\right)$ and $\rho<\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}+1\right)$. If we subtract left hand side in $nn(z^4+2zz+1)=2(2n+1)(z^4-6zz+1)$ from the right hand side, we have
$$(4n+2-nn)z^4-(24n+12+2nn)zz+4n+2-2nn=0.$$
Define Δ by
$$\Delta=\frac{nn+12n+6}{4n+2-nn}.$$
If we divide the above equation by $4n+2-nn$, we get
$$z^4-2\Delta zz+1=0.$$
In resolving this equation, one has $zz=\Delta\pm(\Delta^2-1)$, or finally
$$z=\pm\sqrt{\frac{\Delta+1}{2}}\pm\sqrt{\frac{\Delta-1}{2}}.$$
Among the four roots of the equation $z^4-2\Delta zz+1=0$, the largest, $\sqrt{\frac{\Delta+1}{2}}+\sqrt{\frac{\Delta-1}{2}}$, is the only one that surpasses 1, and from that we conclude that all the values, from ρ until the end, generate the appropriate values for z.

16. Suppose $f=\frac{3}{2}$ and consequently $\rho=5$. We have $n=\frac{13}{4}$, $\Delta=12.52112$, and $\frac{\Delta+1}{2}=6.76056$, $\frac{\Delta-1}{2}=5.76056$, finally $\sqrt{\frac{(\Delta+1)}{2}}=2.60$, $\sqrt{\frac{(\Delta-1)}{2}}=2.40$, and hence $z=5$, it is to say z does not surpass ρ by even an extremely small fraction.

Second Case: When values of z that are less than f.

17. Here P and Q are negative, and our equation reduces to

$$nn(zz+1)^2 = -P-Q = -2(2n+1)z^4 + 12(2n+1)zz - 2(2n+1) = -2(2n+1)(z^4 - 6zz + 1).$$

This can be reduced to the preceding if we make change of variables $z = \dfrac{\sigma+1}{\sigma-1}$:

$$nn(\sigma\sigma+1)^2 = 2(2n+1)(\sigma^4 - 6\sigma\sigma + 1).$$

Observe here that the two letters σ and z depend on one another in the same manner that f and ρ do. For example, $\sigma z = \sigma + z + 1$.

18. From the first case considered above, the appropriate values of σ are from ρ to ∞, when $\rho > 7.5957541$ or $f > 1.3032254$, and consequently $z = \dfrac{\sigma+1}{\sigma-1}$ can be taken between 1 and f when $\rho > 7.5957541$ or $f > 1.3032254$.

19. For abbreviation purposes, we write 7.5 and 1.3 for the numbers 7.5957541 and 1.3032254, respectively. We arrive at this important conclusion: every time ρ is between the limits 7.5 and ∞, or f is between 1 and 1.3, one always takes the number z to be between the limits ρ and ∞, or between 1 and f .

20. Presently, we examine the case when $\rho < 7.5$, or $f > 1.3$. We start with the case when $\rho = f = 1+\sqrt{2}$. Since $f = \rho$, we see that $k = 0$, and $nn - 2n - 1 = 0$, or $nn = 2n+1$. Consequently

$$\Delta = \frac{nn + 6(2n+1)}{2(2n+1) - nn} = \frac{7nn}{nn} = 7.$$

Finally

$$\sigma = \sqrt{\frac{7+1}{2}} + \sqrt{\frac{7-1}{2}} = 2 + \sqrt{3} = 3.7320508, \quad z = \frac{\sigma+1}{\sigma-1} = \sqrt{3} = 1.7320508.$$

For simplicity, we replace the number 3.7320508 and 1.7320503 with 3.73 and 1.73, respectively, and we draw this conclusion: in this case, where $f = \rho + 1 + \sqrt{2}$, We always take z to be between the limits ρ and 3.73, or between that of f and 1.73. We cannot debate more from research the case where ρ finds itself between the limits of 7.5 and $1+\sqrt{2}$, or f between that of 1.3 and $1+\sqrt{2}$.

21. Let us take $f = \dfrac{3}{2}$ and hence $\rho = 5$. If $\sigma = 5$ then $z = \dfrac{\sigma+1}{\sigma-1} = \dfrac{3}{2}$; that is to say that, in this case, z does not differ from f by an extremely small fraction. It follows from this that when one reduces ρ to be less than 7.5 by the term 5, the value of z decreases more and more until it clearly becomes equal to f.

Third Case: When values of z who find themselves between f and ρ.

22. In this case, the value of P is positive and that of Q is negative, and the equation to solve is

$$nn(zz+1)^2 = P - Q$$

Or

$$nn(zz+1)^2 = 8(n+1)k(z^5 - z).$$

Suppose here

$$\frac{8k(n+1)}{nn} = 4\vartheta \qquad \text{or} \qquad \frac{2k(n+1)}{nn} = \vartheta.$$

We have the double-squared [quartic] equation $z^4 - 4\vartheta z^3 + 2zz + 4\vartheta z + 1 = 0$, which can be resolved without resorting to the cubic.

23. Suppose

$$z^4 - 4\vartheta z^3 + 2zz + 4\vartheta z + 1 = (zz - \alpha z - 1)(zz - \beta z - 1).$$

Expanding the right hand side we get

$$(zz - \alpha z - 1)(zz - \beta z - 1) = z^4 - (\alpha + \beta)z^3 + (\alpha\beta - 2)zz + (\alpha + \beta)z + 1.$$

Comparing the coefficient with the left hand side of the previous equation, we get $\alpha + \beta = 4\vartheta$ and $\alpha\beta - 2 = 2$. Note that the latter yields $\alpha\beta = 4$. From these we get

$$\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = 4\sqrt{\vartheta\vartheta - 1}.$$

This shows the impossibility of solving the equation $nn(z^4 + 2zz + 1) = 8(n+1)k(z^3 - z)$ in the case when $\vartheta < 1$. Thus, for $\vartheta < 1$, we always have $nn(z^4 + 2zz + 1) > 8(n+1)k(z^3 - z)$, and consequently all the values of z between f and ρ will satisfy the inequality mentioned in (8).

24. To find the values of n for which $\vartheta < 1$, we take the expression of $\vartheta = \dfrac{2k(n+1)}{nn}$, where $k = \sqrt{nn - 2n - 1}$. Thus, for the determination of these values of n, we have the condition,

$$\frac{2(n+1)\sqrt{nn - 2n - 1}}{nn} < 1$$

Or

$$n^4 - \frac{16}{3}nn - \frac{16}{3}n - \frac{4}{3} < 0,$$

which reduces to the inequality $n^4 < \dfrac{4}{3}(2n+1)^2$. We rewrite this as $n^2 < \dfrac{2(2n+1)}{\sqrt{3}} < \dfrac{4n}{\sqrt{3}} + \dfrac{2}{\sqrt{3}}$,

and finally $n < \dfrac{2 + \sqrt{4 + 2\sqrt{3}}}{\sqrt{3}} = 1 + \sqrt{3} = 2.7320508.$

It follows from this that, as long as n is smaller than 2.7320508, ϑ will be smaller than 1, and all the values of z from f until ρ satisfy our goal.

25. Now we must consider the case when $\vartheta > 1$. From $\alpha + \beta = 4\vartheta$ and $\alpha - \beta = 4\sqrt{\vartheta\vartheta - 1}$, we have $\alpha = 2\vartheta + 2\sqrt{\vartheta\vartheta - 1}, \quad \beta = 2\vartheta - 2\sqrt{\vartheta\vartheta - 1}$. Thus the two factors of our double-square [quartic] will be

$$zz - 2(\vartheta + \sqrt{(\vartheta\vartheta - 1)})z - 1 \qquad \text{and} \qquad zz - 2(\vartheta - \sqrt{(\vartheta\vartheta - 1)})z - 1.$$

Setting these equal to 0, we get the four roots of our equation:

$$\vartheta + \sqrt{\vartheta\vartheta - 1} + \sqrt{2\vartheta\left(\vartheta + \sqrt{\vartheta\vartheta - 1}\right)},$$
$$\vartheta + \sqrt{\vartheta\vartheta - 1} - \sqrt{2\vartheta\left(\vartheta + \sqrt{\vartheta\vartheta - 1}\right)},$$
$$\vartheta - \sqrt{\vartheta\vartheta - 1} + \sqrt{2\vartheta\left(\vartheta - \sqrt{\vartheta\vartheta - 1}\right)},$$
$$\vartheta - \sqrt{\vartheta\vartheta - 1} - \sqrt{2\vartheta\left(\vartheta - \sqrt{\vartheta\vartheta - 1}\right)}.$$

But it is not difficult to remark that $\vartheta \pm \sqrt{\vartheta\vartheta - 1} = \left(\sqrt{\dfrac{\vartheta + 1}{2}} \pm \sqrt{\dfrac{\vartheta - 1}{2}}\right)^2$. Hence, the expressions found for the roots of our equation reduce to the following:

$$\vartheta + \sqrt{\vartheta\vartheta - 1} + \sqrt{\vartheta\left(\vartheta\vartheta - 1\right)} + \sqrt{\vartheta\left(\vartheta - 1\right)},$$
$$\vartheta + \sqrt{\left(\vartheta\vartheta - 1\right)} - \sqrt{\vartheta\left(\vartheta\vartheta - 1\right)} - \sqrt{\vartheta\left(\vartheta - 1\right)},$$
$$\vartheta - \sqrt{\vartheta\vartheta - 1} + \sqrt{\vartheta\left(\vartheta\vartheta - 1\right)} + \sqrt{\vartheta\left(\vartheta - 1\right)},$$
$$\vartheta - \sqrt{\vartheta\vartheta - 1} - \sqrt{\vartheta\left(\vartheta\vartheta - 1\right)} - \sqrt{\vartheta\left(\vartheta - 1\right)}.$$

From these four roots we only need to consider $\vartheta + \sqrt{\vartheta\vartheta - 1} + \sqrt{\vartheta\left(\vartheta + 1\right)} + \sqrt{\vartheta\left(\vartheta - 1\right)}$ and $\vartheta - \sqrt{\vartheta\vartheta - 1} + \sqrt{\vartheta\left(\vartheta + 1\right)} - \sqrt{\vartheta\left(\vartheta - 1\right)}$, since the other two are smaller than 1. Using these two values of z, which give $nn(zz+1)^2 - 8(n+1)k\left(z^3 - z\right) = 0,$ it is not difficult to find the solution of the inequality $nn(zz+1)^2 > 8(n+1)k\left(z^3 - z\right)$.

For this, we remark that the larger value of z, which gives $nn(zz+1)^2 - 8(n+1)k(z^3 - z) = 0,$ is $\vartheta + \sqrt{\vartheta\vartheta - 1} + \sqrt{\vartheta\left(\vartheta + 1\right)} + \sqrt{\vartheta\left(\vartheta - 1\right)}$; then all the values that surpass this limit give $nn(zz+1)^2 - 8(n+1)k(z^3 - z) > 0$ and consequently filling the condition $nn(zz+1)^2 > 8(n+1)k\left(z^3 - z\right)$. All the values of z that are below that of $\vartheta + \sqrt{\vartheta\vartheta - 1} + \sqrt{\vartheta\left(\vartheta + 1\right)} + \sqrt{\vartheta\left(\vartheta - 1\right)}$, and which are not inferior to the other root of the equation $nn(zz+1)^2 - 8(n+1)k\left(z^3 - z\right) = 0,$ which is $\vartheta - \sqrt{\vartheta\vartheta - 1} + \sqrt{\vartheta\left(\vartheta + 1\right)} - \sqrt{\vartheta\left(\vartheta - 1\right)}$ and consequently does not verify the condition $nn(zz+1)^2 > 8(n+1)k\left(z^3 - z\right)$. But, passing this limit, all the values of z give $nn(zz+1)^2 - 8(n+1)k\left(z^3 - z\right)$ positive, and consequently satisfy the condition $nn(zz+1)^2 > 8(n+1)k\left(z^3 - z\right)$.

Hence, this condition will not be fulfilled for the values of z comprised between the limits $\vartheta+\sqrt{\vartheta\vartheta-1}+\sqrt{\vartheta(\vartheta+1)}+\sqrt{\vartheta(\vartheta-1)}$ and $\vartheta-\sqrt{\vartheta\vartheta-1}+\sqrt{\vartheta(\vartheta+1)}-\sqrt{\vartheta(\vartheta-1)}$.

26. We are now going to assign, for each proposed value of f or of $\rho=\frac{f+1}{f-1}$, some appropriate values of z between f and ρ. First we compute $2n=\frac{ff+1}{f-1}$, or $n=\frac{\rho\rho+1}{\rho-1}$, and then $k=\sqrt{nn-2n-1}$. (We could also get k from $k=n-f=\rho-n$.) Finally, we use $\vartheta=\frac{2(n+1)k}{nn}$ and determine the values of z from the formulas:

$\vartheta+\sqrt{\vartheta\vartheta-1}+\sqrt{\vartheta(\vartheta+1)}+\sqrt{\vartheta(\vartheta-1)}$ and $\vartheta-\sqrt{\vartheta\vartheta-1}+\sqrt{\vartheta(\vartheta+1)}-\sqrt{\vartheta(\vartheta-1)}$.

27. Having determined, after these formulas, the values of z for several numbers f or ρ, and having the jointed values of z taken out from two previous researches, we have constructed a table which gives, for certain numbers of ρ, the limits of values of z, which fulfill the condition (8).

This is thusly how we reached the complete solution to the principal problem. If the fraction $\frac{a}{b}$ is much bigger than $1+\sqrt{2}$, we use this value for ρ it in the first column of the table below and determine the limits that yield $z=\frac{c}{d}$ from the second column. When $f=\frac{a}{b}<1+\sqrt{2}$, we take $\frac{a}{b}=f$, and compute $\rho=\frac{f+1}{f-1}$. We then find the value of $z=\frac{c}{d}$ corresponding to ρ.

Table

That represents, for certain numbers ρ, the limits for z.

ρ	**Limits of** z	
2.41	1.73…3.73	
3.0	1.71…3.81	
3.5	1.66…4.00	
3.75	1.64…2.21	2.65…4.11
4.0	1.61…1.80	3.49…4.25
4.5	1.55…1.58	4.40…4.58
5.0	1.50…1.50	5.00…5.00
5.5	1.43…1.45	5.44…5.59
6.0	1.37…1.42	5.78…6.43
6.5	1.29…1.40	6.04…7.82
7.0	1.21…1.38	6.25…10.71
7.5	1.00…1.36	6.50…∞
8.0	1.00…1.36	6.56…∞
9.0	1.00…1.35	6.68…∞
10	1.00…1.34	6.92…∞
11	1.00…1.33	7.04…∞
13	1.00…1.32	7.21…∞
15	1.00…1.32	7.30…∞
∞	1.00…1.30	7.59…∞

28. Reviewing this table, we cannot at all assign the appropriate value of $z = \frac{c}{d}$, when $\rho = \frac{a}{b} = 5.$ In the case when $\rho = 5,$ the limits of z, to a near hundredth, combine the one with the other. But, moreover, when we remove from this singular case, we can intend the limits to be between that which the fraction $\frac{c}{d}$ can be well-proportioned.

For clarifying our method for an example, if we take $\frac{a}{b} = 4$, or $\frac{a}{b} = \frac{5}{3}$; the other fraction $\frac{c}{d}$ can be taken to be between the limits 1.61 and 1.80, or between 3.49 and 4.25. Thus if we take $\frac{a}{b} = \frac{4}{1}$ and $\frac{c}{d} = \frac{7}{2}$, we have

$a = 4$ $c = 7$	$b = 1$ $d = 2$
$x = 4 \cdot 7 + 1 \cdot 2 = 30$ $y = 4 \cdot 2 - 7 \cdot 1 = 1$	$x' = 4 \cdot 7 - 1 \cdot 2 = 26$ $y' = 4 \cdot 2 + 7 \cdot 1 = 15$
$xx - yy = 899$ $2xy = 60$	$x'x' - y'y' = 451$ $2x'y' = 780$
$xx - yy - 2xy = 839$	$x'x' - y'y' - 2x'y' = -329$

Thus $p = 329$, $q = 839$, and r will be $= 30^2 + 1^2 = 26^2 + 15^2 = 901$. With these values of p, q, and r the numbers A, B, C, D can be expressed thusly:

$$A = \frac{pp + qq - rr}{2} = \frac{361}{2},\quad B = \frac{pp + rr - qq}{2} = \frac{216121}{2},$$
$$C = \frac{qq + rr - pp}{2} = \frac{1407481}{2},\quad D = \frac{3rr - pp - qq}{2} = \frac{1623241}{2}.$$

These numbers when multiplied by 4, given as the solution to our problem of four whole numbers, follows:

$$A = 722,\quad B = 432242,\quad C = 2814962,\quad D = 3246482..$$

Notes:

<u>*Sections 1 to 3*</u> Here Euler states the problem of finding four positive integers $A < B < C < D$, with $A + D = B + C$ such that $A + B = pp, A + C = qq, A + D = rr = B + C, B + D = ss, C + D = tt.$

He then expresses the numbers in terms of the squares and observes that the solution of the original problem reduces to finding p, q, and r such that $r^2 > p^2 + q^2$. He observes that $2r^2 = p^2 + t^2 = q^2 + s^2$ and hence r is the sum of two squares in two different ways. Here we note that this follows from the fact that p and t must be of the same party and multiplying the first of the equations by 2 we obtain $4r^2 = 2p^2 + 2t^2 = (t-p)^2 + (t+p)^2$ and hence, after dividing by 4, we have $r^2 = \left(\frac{t-p}{2}\right)^2 + \left(\frac{t+p}{2}\right)^2$. Thus the numbers $\left(\frac{t-p}{2}, \frac{t+p}{2}, r\right)$ is a Pythagorean triple and hence there exist integers $x \geq y$ such that, after solving for p and t,

$$r = x^2 + y^2,\ p = x^2 - y^2 - 2xy,\ t = x^2 - y^2 + 2xy$$

Using the second equation in $2r^2 = p^2 + t^2 = q^2 + s^2$, he gets another representation for r as a sum of two squares:

$$r = x'^2 + y'^2,\ q = x'^2 - y'^2 - 2x'y',\ t = x'^2 - y'^2 + 2x'y'.$$

In section 3, Euler uses a fact that he was aware of (known today as Euler's Factorization Method): Any number that can be written as sum of two squares in two different ways is a product of numbers each of which is a sum of two squares. Thus $r = (a^2 + b^2)(c^2 + d^2)$ where he assumes $a \geq b$ and $c \geq d$. Next he expresses the inequality $r^2 > p^2 + q^2$ in terms of r, x, y, x', and y' as $r^2 > 4xy|x^2 - y^2| + 4x'y'|x'^2 - y'^2|$, since he did not made no assumption about x, y, x', y', he needs the absolute value.

He now uses $r = (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2 = (ac - bd)^2 + (ad + bc)^2$ and factors out bd and lets $f = \frac{a}{b}, z = \frac{c}{d}$.

Sections 4 to 8 In section 4, Euler introduces a new variable $\rho = \dfrac{f+1}{f-1}$ and expresses the inequality $r^2 > 4xy\left|x^2 - y^2\right| + 4x'y'\left|x'^2 - y'^2\right|$ in terms of $f, z,$ and ρ. He makes use of the symmetry between f and ρ, namely that $f = \dfrac{\rho+1}{\rho-1}$ and obtains identities that are useful for the inequality to be considered.

Sections 9 to 13 Here Euler reviews what he has achieved and where he is going in his research for the solution of the original problem. In particular, he emphasizes the relationship between f and ρ: $\dfrac{f^2+1}{f-1} = \dfrac{\rho^2+1}{\rho-1}$ He assumes then that $f \le \rho$ and introduces a new variable $2n$ by setting $\dfrac{f^2+1}{f-1} = \dfrac{\rho^2+1}{\rho-1} = 2n$ Form this he gets $f = n-k$ and $\rho = n+k$, where $k = \sqrt{n^2 - 2n - 1}$. He then expresses $r^2 > p^2 + q^2$ as $n^2\left(z^2+1\right)^2 \ge |P| + |Q|,$ where

$$P = (2n+1)z^4 + 4(n+1)kz^3 - 6(2n+1)z^2 - 4(n+1)kz + 2n+1,$$
$$Q = (2n+1)z^4 - 4(n+1)kz^3 - 6(2n+1)z^2 + 4(m+1)kz + 2n+1.$$

Sections 14 to 16 Euler now considers three different cases for the values of z given the value of f and ρ . (Note that f determines ρ.) In these sections he considers the case when $z \ge \rho$. He notes that both P and Q are positive and solves the inequality. His conclusion is that all values of $z \ge \rho$ is the solution of $n^2(z^2+1)^2 \ge P+Q,$ where $P + Q = 2(2n+1)\left(z^4 - 6z^2 + 1\right)$ are as given above. In section 16 he gives an example.

Sections 17to 21 Here Euler considers the case when $z \le f$ and use the symmetry between f and ρ to deduce any z between 1 and f is a solution of the inequality. Note that P and Q are both negative in this case.

Sections 12to 26 The last case Euler considers is $f \le z \le \rho$. In this case $P > 0$ and $Q < 0$ and hence the inequality becomes $n^2(z^2+1)^2 \ge P - Q,$ But $P - Q = 8(n+1)k\left(z^3 - z\right)$ and Euler has to deal with an inequality that involves a cubic term.

Sections 27and 28 In section 27, Euler gives a table of values in which he uses ρ as a free variable and determines the possible range of values of z. Note however that if the value of f is larger than $1+\sqrt{2}$, then we use f itself instead of $\rho = \dfrac{f+1}{f-1}$. But if $1 < f < 1+\sqrt{2}$, then we compute $\rho = \dfrac{f+1}{f-1}$ and then find the corresponding value of z.

In the last section Euler gives an example: Take $f = 4 = \dfrac{4}{1} = \dfrac{a}{b}$ and then use the table to find the possible range for z, in this case between 1.61 and 1.80 or between 3.49 and 4.25. Thus we can choose $z = \dfrac{7}{2} = \dfrac{c}{d}$ and gets four numbers

$$A = \frac{361}{2}, B = \frac{216121}{2}, C = \frac{1407481}{2}, D = \frac{1623241}{2}.$$

all of which are rational numbers. (It is interesting to note that these rational numbers do satisfy the required property: The sum of any two is a perfect square.) Euler remarks that multiplying each by 4, the smallest perfect square, yields four integers with the required property. Note also that if we take $\frac{a}{b} = \frac{5}{3} = 1.667$ (which Euler indicates as a possible value for $\frac{a}{b}$), which is less than $1+\sqrt{2}$, then we need to compute ρ, which will be $\rho = \frac{5/3+1}{5/3-1} = 4$, and pick z form the same interval. In this case, if we pick $z = \frac{7}{2} = \frac{c}{d}$ the table of section 28 will give the numbers 722, 432242, 2814962, and 3246482 but in different order.