# FRACTIONAL HYPERGEOMETRIC ZETA FUNCTIONS

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Both authors would like to dedicate this in fond memory of Marvin Knopp. Knop was the most humble and exemplary teacher and mathematician the second author has known.

Abstract. In this paper we investigate a continuous version of the hypergeometric zeta functions for any positive rational number "$a$" and demonstrate the analytic continuation. The fractional hypergeometric zeta functions is shown to exhibit many properties analogous to its hypergeometric counter part, including its intimate connection to Bernoulli numbers.

## 1. Introduction

In [1] Abdul Hassen and Heiu D. Nguyen introduced and investigated a generalization of the Riemann zeta function by replacing $e^x - 1$ the denominator in the integral representation with arbitrary Taylor difference $e^x - T_{N-1}(x)$ where $N$ is a positive integer and $T_{N-1}(x)$ is the Taylor polynomial of $e^x$ at the origin having degree $N-1$. This defines a family of what Hassen and Nguyen called hypergeometric zeta functions denoted by $\zeta_N(s)$ defined by:

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx.$$

Observe that $\zeta_1(s) = \zeta(s)$ , i.e.; when $N = 1$, we get the classical zeta function. In the same paper, ([1]) they developed the analytic continuation of $\zeta_N(s)$ to the entire complex plane, except for $N$ simple poles at $s = 1, 0, -1, \cdots, 2-N$. and established many of the properties of the classical zeta function were established for the hypergeometric zeta functions including generalized Bernoulli numbers. The same authors also investigated a continuous version of the hypergeometric zeta function called the error zeta function in ([2]) by generalizing the definition involving a positive integer $N$ to all real positive $N$ but focused only on $N = \frac{1}{2}$. Regarding the continuous version they developed the analytic continuation to the entire complex plane except for infinite number of poles and discovered that, the error zeta function shares many of the same properties found in hypergeometric zeta functions of integer order.

In this paper we introduce what we call fractional hypergeometric zeta functions of order "$a$" where $a$ is a positive real number. This can be done allowing the natural number $N$ to be any positive real number $a$. The authors of [1]defined $\zeta_N(s)$ for $\Re(s) > 1$ as,

$$\zeta_N(s) = \frac{\Gamma(N+1)}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s-2}}{{}_1F_1(1, N+1, x)} dx$$

where ${}_1F_1(b, c, x)$ is the confluent hypergeometric function.

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We now take this as our definition of $\zeta_N(s)$ and replace $N$ by $a$ to reserve $N$ for integer. We begin with the observation that

$$\begin{aligned} {}_1F_1(1, a+1, x) &= e^x x^{-a}(\Gamma(a+1) - a\Gamma(a,x)) \\ &= ae^x x^{-a}(\Gamma(a) - \Gamma(a,x)) \\ &= ae^x x^{-a}\gamma(a,x) \end{aligned} \tag{1.1}$$

where

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$$

is the lower incomplete gamma function and

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$$

is the upper incomplete gamma function. Therefore, the fractional hypergeometric zeta function of order $a$ is given by

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)} \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx$$

It is discovered that this fractional hypergeometric zeta function of order "$a$" shares many of the same properties found in hypergeometric zeta functions of integer order. Observe that, if $a$ is a positive integer we have the hypergeometric zeta function. In section 2, we formally define fractional hypergeometric zeta functions, establish its convergence in a right half of the complex plane, and develop its series representation. In section 3, we reveal its analytic continuation to all complex plane except at infinite number of poles. We show this in two approaches: in the first approach we require strip-by-strip continuation to the left half of the complex plane for any positive real number $a$, and in the second approach we use contour integral for half odd integers "$a$" and express $\zeta_a(s)$ as a contour integral. In section 4 we demonstrate a pre-functional equation satisfied by this function.

## 2. Preliminaries

We begin this section with the following definition:

**Definition 2.1.** The fractional hypergeometric zeta function is defined for all positive real number $a$ as,

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)} \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx.$$

Observe that when $a = N$, a natural number we get the classical hypergeometric zeta function.

**Lemma 2.2.** *$ae^x\gamma(a,x) \geq x^a$ for any positive real numbers $a$ and $x$*

*Proof.* The proof follows from the relation $ae^x x^{-a}\gamma(a,x) = {}_1F_1(1, a+1, x)$ and the fact that ${}_1F_1(1, a+1, x) > 1$ for positive real number $x$. □

**Lemma 2.3.** *For any real number $x$ in $[1, \infty)$ and any positive real number $a$ there is a constant $\delta > 0$ such that $\gamma(a,x) \geq \delta$.*

*Proof.* Since $\lim_{x\to\infty} \gamma(a,x) = \Gamma(a)$, there is a number $M$ such that $\gamma(a,x) > \frac{\Gamma(a)}{2}$. Since $\gamma(a,x)$ is continuous on $[1, M]$ it has a minimum value on $[1, M]$ say $m$. Then the lemma follows if we let $\delta$ to be the minimum of $m$ and $\frac{\Gamma(a)}{2}$. □

**Theorem 2.4.** *$\zeta_a(s)$ converges absolutely for $\sigma > 1$, where $s = \sigma + it$ and both $\sigma$ and $t$ are real numbers.*

*Proof.* Since $ae^x\gamma(a,x) \geq x^a$ on $[0,\infty)$, it holds on $(0,1]$. Let $s$ be a complex number with $\sigma \geq \sigma_0 > 1$, then on $(0,1]$, we have

$$\left|\frac{x^{s+a-2}}{ae^x\gamma(a,x)}\right| \leq \left|\frac{x^{s+a-2}}{x^a}\right| \leq x^{\sigma_0-2}$$

Since $\sigma_0 > 1$, we have

$$\int_0^1 x^{\sigma_0-2}dx = \frac{1}{\sigma_0-1}$$

Hence,

$$\left|\int_0^1 \frac{x^{s+a-2}}{ae^x\gamma(a,x)}dx\right| \leq \int_0^1 \left|\frac{x^{s+a-2}}{ae^x\gamma(a,x)}dx\right| \leq \frac{1}{\sigma_0-1}.$$

On the other hand, let $s$ be a complex number with $\sigma > 1$ and $x$ be a real number in $[1,\infty)$, then there is a constant $C$ such that $x^{\sigma+a-2} < Ce^{\frac{x}{2}}$. Thus it follows that

$$\left|\frac{x^{s+a-2}}{ae^x\gamma(a,x)}\right| \leq \frac{Ce^{\frac{x}{2}}}{ae^x\gamma(a,x)} \leq \frac{Ce^{\frac{-x}{2}}}{a\delta}$$

This holds true since from the previous lemma we have,$\gamma(a,x) \geq \delta$ and,

$$\int_1^\infty \frac{Ce^{\frac{-x}{2}}}{a\delta}dx = \frac{2Ce^{\frac{-1}{2}}}{a\delta}$$

$$\left|\int_1^\infty \frac{x^{s+a-2}}{ae^x\gamma(a,x)}dx\right| \leq \int_1^\infty \left|\frac{x^{s+a-2}}{ae^x\gamma(a,x)}dx\right| \leq \frac{2Ce^{\frac{-1}{2}}}{a\delta}.$$

Therefore the theorem follows from the following inequality.

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)}\left[\int_0^1 \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx + \int_1^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx\right]$$

thus it follows that,

$$|\zeta_a(s)| \leq \left|\frac{\Gamma(a+1)}{\Gamma(s+a-1)}\right|\left[\int_0^1 \left|\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}\right|dx + \int_1^\infty \left|\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}\right|dx\right].$$

Here the right hand side of the integrals are both finite. □

Thus we have shown that the fractional hypergeometric zeta function converges absolutely for $\sigma > 1$ and converges uniformly for $\sigma \geq \sigma_0 > 1$. Therefore, the fractional hypergeometric zeta function is analytic on $\{s : \sigma > 1\}$ where $s = \sigma + it$ for $\sigma$ and $t$ real numbers. We now develop series representation using its expansion as follows:

**Lemma 2.5.** *For $\Re(s) = \sigma > 1$, we have,*

$$\zeta_a(s) = \sum_{n=1}^{\infty} f_n(a,s)$$

*where*

$$f_n(a,s) = \int_0^\infty \frac{x^{s+a-2}e^{-x}}{\Gamma(s+a-1)}\left(\frac{\Gamma(a,x)}{\Gamma(a)}\right)^n dx.$$

*Proof.* From integral representation of $\zeta_a(s)$, we have,

$$\begin{aligned}\zeta_a(s) &= \frac{\Gamma(a+1)}{\Gamma(s+a-1)}\int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx \\ &= \frac{\Gamma(a)}{\Gamma(s+a-1)}\int_0^\infty \frac{x^{s+a-2}e^{-x}}{\gamma(a,x)}dx. \qquad (2.1)\end{aligned}$$

Since,$\gamma(a,x) > 0$, $\Gamma(a,x) > 0$ for $x > 0$ and$\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$, we have,

$$\gamma(a,x) = \Gamma(a)\left(1 - \frac{\Gamma(a,x)}{\Gamma(a)}\right)$$

Since, $0 < \frac{\Gamma(a,x)}{\Gamma(a)} < 1$, using geometric series and reversing the order of summation and integration we have the required result. □

**Lemma 2.6.** *For each $n = 0, 1, 2, \cdots$, we have,*

$$f_n(a,1) = \frac{1}{n+1}.$$

*Proof.* Substituting 1 for $s$ in the definition of $f_n(a,s)$, we have,

$$f_n(a,1) = \int_0^\infty \frac{x^{a-1}e^{-x}}{\Gamma(a)}\left(\frac{\Gamma(a,x)}{\Gamma(a)}\right)^n dx.$$

Using integration by substitution, $u = \frac{\Gamma(a,x)}{\Gamma(a)}$ and integrating we have the required result. □

The lemma reveals that

$$\zeta_a(1) = \sum_{n=1}^{\infty}\frac{1}{n}$$

formally generates the harmonic series.

## 3. Analytic Continuation

We now consider analytic continuation of the $\zeta_a(s)$. As in the classical case, to do this first we use strip by strip and then using Cauchy theory. As in the hypergeometric zeta case, both methods have their advantages.

3.1. **Method I-Strip-by-Strip.** From the definition of fractional hypergeometric zeta function of order $a$ for $\Re(s) = \sigma > 1$, we have,

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) = \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx.$$

So using series representation of $\frac{1}{a\gamma(a,x)}$, we can add and subtract each of the terms in the series to the integrand without violating convergence. This can be shown in the following theorem:

**Theorem 3.1.** *For $0 < \Re(s) = \sigma < 1$,*

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)}\left[\Gamma(s-1) + \int_0^\infty (\frac{1}{a\gamma(a,x)} - \frac{1}{x^a})x^{s+a-2}e^{-x}dx\right].$$

*Proof.* For $\Re(s) = \sigma > 1$, we have,

$$\begin{aligned}\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) &= \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx \\ &= \int_0^1 \left[\frac{1}{a\gamma(a,x)} - \frac{1}{x^a}\right]x^{s+a-2}e^{-x}dx + \int_0^1 x^{s-2}e^{-x}dx + \int_1^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx \qquad (3.1)\end{aligned}$$

But,

$$\begin{aligned}\int_0^1 x^{s-2}e^{-x}dx &= \Gamma(s-1)-\Gamma(s-1,1)\\ &= \Gamma(s-1)-\int_1^\infty \frac{x^{s+a-2}e^{-x}}{x^a}dx. \end{aligned} \tag{3.2}$$

So we have,

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)=\Gamma(s-1)+\int_0^\infty\left(\frac{1}{a\gamma(a,x)}-\frac{1}{x^a}\right)x^{s+a-2}e^{-x}dx.$$

Hence the theorem follows. □

From this theorem observe that $\zeta_a(s)$ has a zero at $s=1-a$ and a pole at $s=1$. Moreover, this process of analytic continuation can be repeated to extend the domain of $\zeta_a(s)$ to $\Re(s)>-1$. To this end we write

$$\begin{aligned}\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) &= \Gamma(s-1)+\int_0^1\left[\frac{1}{a\gamma(a,x)}-\frac{1}{x^a}-\frac{a}{(1+a)x^{a-1}}\right]x^{s+a-2}e^{-x}dx\\ &+ \frac{a}{1+a}\int_0^1 x^{s-1}e^{-x}dx+\int_1^\infty\left[\frac{1}{a\gamma(a,x)}-\frac{1}{x^a}\right]x^{s+a-2}e^{-x}dx. \end{aligned} \tag{3.3}$$

But then,

$$\frac{a}{1+a}\int_0^1 x^{s-1}e^{-x}dx=\frac{a}{1+a}\Gamma(s)-\frac{a}{1+a}\int_1^\infty x^{s-1}e^{-x}dx.$$

Therefore,

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s)=\Gamma(s-1)+\frac{a}{1+a}\Gamma(s)+\int_0^\infty\left[\frac{1}{a\gamma(a,x)}-\frac{1}{x^a}-\frac{a}{(1+a)x^{a-1}}\right]x^{s+a-2}e^{-x}dx.$$

From this, observe that $\zeta_a(s)$ has a zero at $s=-a$ and a pole at $s=0$.
Continuing in this fashion, we observe that:$\zeta_a(s)$ has a zeros at $s=1-a,-a,-(1+a),-(2+a),\cdots$, and poles at $s=1,0,-1,-2,-3,\cdots$. These zeros are called the trivial zeros of $\zeta_a(s)$. For example, if $a=\frac{3}{2}$, the trivial zeros of $\zeta_{\frac{3}{2}}(s)$ are at $s=-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},\cdots$.

The main advantage of analytic continuation using the method of strip-by-strip is that it reveals the behavior of $\zeta_a(s)$ near the distinguished pole $s=1$. The following theorem is valid for all positive real values of $a$ and generalizes the corresponding result on hypergeometric zeta functions of positive integer order. This is the content of the next theorem.

**Theorem 3.2.**

$$\lim_{s\to1}[\zeta_a(s)-\frac{a}{s-1}]=\log(\Gamma(a+1))-a\frac{\Gamma'(a)}{\Gamma(a)}$$

*Proof.*

$$\begin{aligned}\lim_{s\to1}\left[\zeta_a(s)-\frac{a}{s-1}\right] &= \frac{\Gamma(a+1)}{\Gamma(a)}\int_0^\infty\left(\frac{1}{{}_1F_1(1,a+1;x)}-e^{-x}\right)\frac{dx}{x}\\ &= \frac{\Gamma(a+1)}{\Gamma(a)}\left(\gamma+\frac{\log(\Gamma(a+1))}{a}\right)\\ &= a\gamma+\log(\Gamma(a+1)) \end{aligned} \tag{3.4}$$

where $\gamma$ is Euler's constant. But on the other hand we have,

$$\begin{aligned}\lim_{s\to 1}\left[\zeta_a(s)-\frac{a}{s-1}\right] &= \lim_{s\to 1}\left[\zeta_a(s)-\frac{\Gamma(a+1)\Gamma(s-1)}{\Gamma(s+a-1)}\right]-\lim_{s\to 1}\left[\frac{a}{s-1}-\frac{\Gamma(a+1)\Gamma(s-1)}{\Gamma(s+a-1)}\right]\\ &= a\gamma+\log(\Gamma(a+1))-\left(a\gamma+a\frac{\Gamma'(a)}{\Gamma(a)}\right)\\ &= \log(\Gamma(a+1))-a\frac{\Gamma'(a)}{\Gamma(a)} \qquad (3.3)\end{aligned}$$

as desired. □

Observe that this result is analogous to the classic result for $\zeta(s)$

$$\lim_{s\to 1}\left[\zeta(s)-\frac{1}{s-1}\right]=\frac{\Gamma'(1)}{\Gamma(1)}=\gamma\approx 0.577.$$

3.2. **Method II-Contour Integral.** We now take a different approach and follow Riemann by using contour integration to develop the analytic continuation of the fractional hypergeometric zeta function of order $a$. This will not only allow us to make precise our earlier statements about $\zeta_a(s)$ having an infinite number of poles but also make explicit the role of the zeros of incomplete gamma function in determining the values of $\zeta_a(s)$ at negative integers. To this end we consider the contour integral for half odd integers $a$. That is $a=\frac{2N+1}{2}$, where $N$ is a nonnegative integer, so that,

$$\zeta_{\frac{2N+1}{2}}(s)=\frac{\Gamma(\frac{2N+1}{2}+1)}{\Gamma(s+\frac{2N+1}{2}-1)}\int_0^\infty\left(\frac{1}{\frac{2N+1}{2}\gamma(\frac{2N+1}{2},x^2)}\right)x^{2(s+\frac{2N+1}{2}-2)}e^{-x^2}2xdx$$

Simplifying this we get,

$$\zeta_{\frac{2N+1}{2}}(s)=\frac{(2N+1)\Gamma\left(\frac{2N+1}{2}\right)}{\Gamma(s+\frac{2N-1}{2})}\int_0^\infty\frac{2x^{2(s+N-1)+1}e^{-x^2}}{(2N+1)\gamma(\frac{2N+1}{2},x^2)}\frac{dx}{x}$$

We now consider the contour integral,

$$I(s)=\frac{1}{2\pi i}\int_C\frac{2(-w)^{2(s+N-1)+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2},w^2)}\frac{dw}{w}$$

where the contour $C$ is given as follows:

$$C=C_-+C_\delta+C_+$$

$C_-$ is taken along the real axis from $\infty$ to $\delta>0$. $C_\delta$ is taken counterclockwise around a circle of radius $\delta>0$. $C_+$ is taken along the real axis from $\delta>0$ to $\infty$. Moreover, we let $-w$ have argument $-\pi$ backwards along $\infty$ to $\delta>0$, and argument $\pi$ when going from $\delta>0$ to $\infty$. Also we choose the radius $\delta>0$ to be sufficiently small so that there are no roots of $\gamma(\frac{2N+1}{2},w^2)$ inside the circle of radius $\delta>0$ besides the trivial zeros at $w_0=0$. This follows from the fact that $w_0=0$ is an isolated zero. It is then clear from this assumption that $I(s)$ must converge for all complex $s$ and therefore defines an entire function. Now we begin by evaluating $I(s)$ at integer values. To this end we decompose the integral as follows:

$$\begin{aligned}I(s) &= \frac{1}{2\pi i}\int_{C_-}\frac{2(-x)^{2(s+N-1)+1}e^{-x^2}}{(2N+1)\gamma(\frac{2N+1}{2},x^2)}\frac{dx}{x}+\frac{1}{2\pi i}\int_{C_\delta}\frac{2(-w)^{2(s+N-1)+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2},w^2)}\frac{dw}{w}\\ &\quad+\frac{1}{2\pi i}\int_{C_+}\frac{2(-x)^{2(s+N-1)+1}e^{-x^2}}{(2N+1)\gamma(\frac{2N+1}{2},x^2)}\frac{dx}{x} \qquad (3.4)\end{aligned}$$

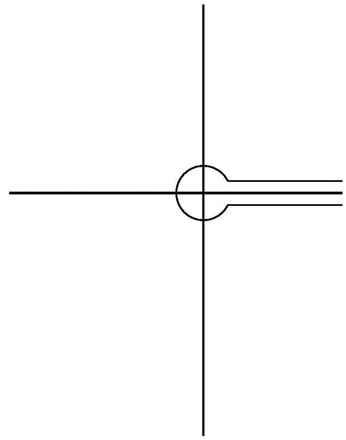

FIGURE 1. Contour C

Now if $2(s+N-1)+1$ is an integer the two integrals along $C_-$ and $C_+$ cancel each other and we have only

$$I(s)=\frac{1}{2\pi i}\int_{C_\delta}\frac{2(-w)^{2(s+N-1)+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2},w^2)}\frac{dw}{w}$$

But the function defined by,

$$f(w)=\frac{2w^{2N+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2},w^2)}$$

in the integrand inside $C_\delta$ has a removable singularity at the origin, hence by Cauchy's theorem we have $I(s)=0$ for integer values $2s-3\geq 0$; that is for $s=\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2},\cdots$. For integers $2s-3\leq -1$; that is for $s=1,\frac{1}{2},0,\frac{-1}{2},-1,\frac{-3}{2},\cdots$, we consider power series expansion at the origin for $f(w)$,

$$f(w)=\frac{2w^{2N+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2},w^2)}=\sum_{m=0}^{\infty}B_{\frac{2N+1}{2},m}\frac{w^m}{m!}$$

From Cauchy integral formula we have,

$$B_{\frac{2N+1}{2},m}=\frac{m!}{2\pi i}\int_{C_\delta}\frac{f(w)}{w^{m+1}}dw$$

So we have,

$$\begin{aligned}I(s) &= (-1)^{2(s+N-1)+1}\frac{1}{2\pi i}\int_{C_\delta}\left(\frac{2(-w)^{2N+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2},w^2)}\right)\frac{dw}{w^{2(1-s)+1}}\\ &= (-1)^{2(s+N-1)+1}\frac{1}{2\pi i}\int_{C_\delta}\frac{f(w)}{w^{2(1-s)+1}}dw\\ &= (-1)^{2(s+N-1)+1}\frac{B_{\frac{2N+1}{2},2(1-s)}}{(2(1-s))!} \qquad (3.5)\end{aligned}$$

But at $s=\frac{1}{2},\frac{-1}{2},\frac{-3}{2},\cdots$, it can be easily shown that

$$\frac{B_{\frac{2N+1}{2},2(1-s)}}{(2(1-s))!}=0$$

Thus $I(s) = 0$ for $s = \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \cdots,$.
Therefore, the zeros of $I(s)$ can be summarized as follows:
$s = \frac{n}{2}$, $n$ is an integer and $n \geq 3$ , or $s = \frac{2n+1}{2}$, $n$ is an integer and $n \leq 0$.

3.3. **Relations Between $I(s)$ and $\zeta_{\frac{2N+1}{2}}(s)$.** We now express $\zeta_{\frac{2N+1}{2}}(s)$ in terms of $I(s)$. For $\Re(s) > 1$, the integral over $C_\delta$ goes to zero as $\delta$ goes to zero. Hence the integral over $C_+$ and $C_-$ yields that,

$$\begin{aligned} I(s) &= \frac{1}{2\pi i}\left[e^{2s+2N-1)\pi i} - e^{-(2s+2N-1)\pi i}\right]\int_{C_-}\left(\frac{2(x)^{2(s+N-1)}e^{-x^2}}{(2N+1)\gamma(\frac{2N+1}{2}, x^2)}\right)dx \\ &= \frac{\sin((2s+2N-1)\pi)}{\pi}\frac{\Gamma(s+\frac{2N-1}{2})}{(2N+1)\Gamma(\frac{2N+1}{2})}\zeta_{\frac{2N+1}{2}}(s) \end{aligned} \tag{3.6}$$

Thus we have,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{(2N+1)\Gamma\left(\frac{2N+1}{2}\right)}{\Gamma\left(s+\frac{2N-1}{2}\right)}\frac{\pi}{\sin(2(s+N)-1)\pi)}I(s)$$

Now using trigonometric identity, the double angle formula,

$$\sin(2(s+N)-1)\pi = 2\cos(s+N-\frac{1}{2})\pi\sin(s+N-\frac{1}{2})\pi$$

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2}\frac{\Gamma\left(\frac{2N+1}{2}\right)}{\Gamma\left(s+\frac{2N-1}{2}\right)}\frac{\pi}{2\cos(s+N-\frac{1}{2})\pi\sin(s+N-\frac{1}{2})\pi}I(s)$$

By using the functional equation of the gamma function,

$$\frac{\pi}{\sin((s+\frac{1}{2})\pi)\Gamma(s+\frac{1}{2})} = \Gamma(\frac{1}{2}-s)$$

Therefore, for $\Re(s) > 1$, we have the relation,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\frac{\Gamma\left(\frac{1}{2}-(s+N-1)\right)}{\cos(s+N-\frac{1}{2})}I(s)$$

From this relation we observe that the zeros of $I(s)$ at $\frac{n}{2}$ for $n \geq 3$ are simple, since we know by definition that $\zeta_{\frac{2N+1}{2}}(n) > 0$ for $n > 1$. We have also the following results as a consequence of the above relations:

**Theorem 3.3.** *$\zeta_{\frac{2N+1}{2}}(s)$ is analytic on the entire complex plane except for simple poles at $s = n+1-N$ where $n = N, N-1, N-2, N-3, \cdots$, with residue,*

$$Res(\zeta_{\frac{2N+1}{2}}(s), s=n) = \frac{2N+1}{2}\frac{\Gamma\left(\frac{2N+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right)}{\pi\sin\left(n+\frac{1}{2}\right)}\frac{B_{\frac{2N+1}{2},(2(N-n))}}{(2N-2n)!}$$

*Furthermore,$\zeta_{\frac{2N+1}{2}}(s) = 0$ at $s = \frac{-(2n+1)}{2}$ for $n = 0, 1, 2, 3, \cdots$.*

*Proof.* Since $\Gamma(\frac{1}{2}-(s+N-1)$ has only simple poles at $s = \frac{2(n-N)+1}{2}$ for $n = 1, 2, 3, \cdots$, which are canceled by the zeros of $I(s)$ at $s = \frac{n}{2}$ for integer $n \geq 3$, and $\cos((s+N-\frac{1}{2})\pi)$ has simple zeros at the integers, it follows that $\zeta_{\frac{2N+1}{2}}(s)$ is analytic on the entire complex plane except for simple poles at $s = n+N-1$ where

$n = N, N-1, N-2, N-3, \cdots$. The residue at these simple poles is given by

$$\begin{aligned}
Res(\zeta_{\frac{2N+1}{2}}(s), s = n+N-1) &= \lim_{s\to n+N-1}(s-(n+N-1))\zeta_{\frac{2N+1}{2}}(s) \\
&= \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right)I(n+1-N)\lim_{s\to n+1-N}\frac{s-n-1+N}{\cos((s+N-\frac{1}{2})\pi)} \\
&= \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right)I(n+1-N)\frac{-1}{\pi\sin(n+\frac{1}{2}\pi} \\
&= \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right)\frac{B_{\frac{2N+1}{2},2(N-1)}}{(2N-2n)!\pi\sin(n+\frac{1}{2}\pi}
\end{aligned} \tag{3.7}$$

Since, $I(s) = 0$ for $s = \frac{-(2n+1)}{2}$ for $n = 0, 1, 2, 3, \cdots$, we have that $\zeta_{\frac{2N+1}{2}}(s) = 0$ at $s = \frac{-(2n+1)}{2}$ for $n = 0, 1, 2, 3, \cdots$. □

## 4. Pre-Functional Equation

In this section we discuss on a pre-functional equation satisfied by $\zeta_{\frac{2N+1}{2}}(s)$. For this we need some properties of the lower incomplete gamma function. From the definition of the lower incomplete gamma function:

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$$

it follows that $\gamma(a,x)$ tends to $\Gamma(a)$ as $x$ tends to infinity for real parameter $x$. This also holds true if we replace $x$ by a complex parameter $z$ and let the modulus of $z$ tends to infinity for the positive real part of $z$. This can be shown as follows:

**Theorem 4.1.** *Let $z$ be a complex number such that its real part is positive, then*

$$\lim_{|z|\to\infty}\gamma(a,z) = \gamma(a,|z|)$$

*Proof.* Let $|argz| < \delta < \frac{\pi}{2}$ with fixed $\delta$ and $\gamma(a,z)$ be in the principal branch of this sector and $u$ be any complex number from the sector. Then

$$|\gamma(a,u) - \gamma(a,|u|)| = \left|\int_u^{|u|} z^{a-1}e^{-z}dz\right|$$

Now if we integrate along the arc with radius $R = |u|$ around zero connecting $u$ and $|u|$, then we have

$$|\gamma(a,u) - \gamma(a,|u|)| \le \int_u^{|u|}\left|z^{a-1}e^{-z}dz\right|$$

Since $|argz| < \delta$ and hence $\cos\delta \le \cos|argu|$ the integrand has a maximum value in this sector. Thus considering the length of the arc connecting $u$ and $|u|$, we have

$$|\gamma(a,u) - \gamma(a,|u|)| \le R^a\delta e^{-R\cos\delta}$$

Thus letting $R$ tends to infinity we have the required result. □

**Corollary 4.2.** *Let $z$ be a complex number such that its real part is positive, then*

$$\lim_{|z|\to\infty}\gamma(a,z) = \Gamma(a)$$

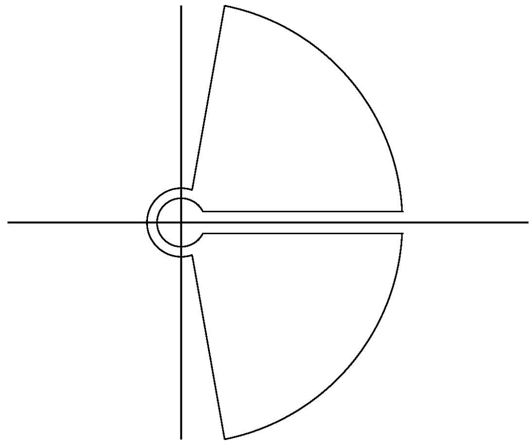

FIGURE 2. Contour $C_{R,\delta}$

*Proof.*

$$|\Gamma(a) - \gamma(a, z)| = |\Gamma(a) - \gamma(a, |z|) + \gamma(a, |z|) - \gamma(a, z)|$$

Thus we have,

$$|\Gamma(a) - \gamma(a, z)| \leq |\Gamma(a) - \gamma(a, |z|)| + |\gamma(a, |z|) - \gamma(a, z)|$$

Hence using the theorem and letting the modulus of $u$ tends to infinity we have the required result. □

From this theorem and its corollary it follows that $\gamma(a, z)$ has only a finite number of simple zeros to the right half of the complex plane. Moreover these zeros are symmetric with respect to the Real-axis. Thus all these zeros can be arranged in a sequence of increasing modulus. Let for each $n = 1, 2, 3, \cdots$ , $z_1^n, z_2^n, \cdots, z_{k_n}^n$ be the non-zero roots in the first quadrant of the lower incomplete gamma function having the same distance $r_n$ from the origin with different arguments $\theta_1^n, \theta_2^n, \cdots, \theta_{k_n}^n$, respectively. Since the lower incomplete gamma function is symmetric about the real axis, the conjugates of these roots are also its roots in the fourth quadrant.

Let $C_{R,\delta}$ be the annulus-shaped contour consisting of two concentric circles centered at the origin. The outer wedge $C_R$ having radius $R$, the outer circle centered at the origin having radius $\frac{\delta}{R}$, and the inner circle $C_\delta$ having radius $\frac{\delta}{2R}$. Here $\delta > 0$ is chosen so that no other zeros of the lower incomplete gamma function included besides the root zero inside the circle. This is possible since zero is an isolated zero of the lower incomplete gamma function. $R$ is chosen so that all the zeros to the right half of the complex plane are included in the wedge. This is possible , since there are only finite number of zeros of the lower incomplete gamma function occurred in the right half of the complex plane. The outer circle and the wedge traversed clockwise, the inner circle counterclockwise and the radial segment along the positive real axis is traversed in both directions. We define

$$I_{C_{R,\delta}}(s) = \frac{1}{2\pi i} \int_{C_{R,\delta}} \left( \frac{2(-w)^{2(s+N-1)} e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2}, w^2)} \right) \frac{dw}{w}$$

We claim that $I_{C_{R,\delta}}(s)$ converges to $I(s)$ as $R$ tends to infinity, for $\Re(s) < 1 - N$. Where

$$I(s) = \frac{1}{2\pi i} \int_C \frac{2(-w)^{2(s+N-1)} e^{-w^2}}{(2N+1)\gamma(\frac{3}{2}, w^2)} ) \frac{dw}{w}$$

where the contour $C$ is given as follows:

$$C = C_- + C_\delta + C_+$$

$C_-$ is taken along the real axis from $\infty$ to $\delta > 0$. $C_\delta$ is taken counterclockwise around a circle of radius $\delta > 0$. $C_+$ is taken along the real axis from $\delta > 0$ to $\infty$. Moreover, we let $-w$ have argument $-\pi$ backwards along $\infty$ to $\delta > 0$, and argument $\pi$ when going from $\delta > 0$ to $\infty$. as defined in the previous section.

Since out side the wedge with radius $R$, $\gamma(\frac{2N+1}{2}, w^2) \neq 0$ we have $|\gamma(\frac{2N+1}{2}, w^2)| > M > 0$.

Thus if $|w| = R$, we have

$$\left|\frac{w^{2(s+N-1)}e^{-w^2}}{\gamma(\frac{2N+1}{2}, w^2)}\right| \leq \left|\frac{R^{2(s+N-1)}}{M}\right| = \frac{R^{2\Re(s+N-1)}}{M} \to 0$$

as $R$ tends to infinity, since $\Re(s) < 1 - N$.
Therefore,

$$I(s) = \lim_{R\to\infty} I_{C_{R,\delta}}(s)$$

On the other hand, we have by residue theory that,

$$I_{C_{R,\delta}}(s) = \sum_{j=1}^{k_n} Res\left[\frac{2(-z)^{2(s+N-1)}e^{-z^2}}{(2N+1)\gamma(\frac{2N+1}{2}, z^2)}, z = z_j^n, z = \overline{z_j^n}\right]$$

But $Res\left[\frac{2(-z)^{2(s+N-1)}e^{-z^2}}{(2N+1)\gamma(\frac{2N+1}{2},z^2)}, z = w\right] = \frac{w^{2s-2}}{2N+1}$
Hence,

$$I_{C_{R,\delta}}(s) = \frac{2}{2N+1}\sum_{j=1}^{n} k_j r_j^{2s-2}\sum_{i=1}^{k_j}\cos(2(s-1)\theta_i^j)$$

Now taking limit as $n$ tends to infinity we have the required result. Thus,

$$I(s) = \frac{2}{2N+1}\sum_{j=1}^{\infty} k_j r_j^{2s-2}\sum_{i=1}^{k_j}\cos(2(s-1)\theta_i^j)$$

Since

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\frac{\Gamma\left(\frac{1}{2}-(s+N-1)\right)}{\cos(s+N-\frac{1}{2})}I(s)$$

Then we have,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\frac{\Gamma\left(\frac{1}{2}-(s+N-1)\right)}{\cos(s+N-\frac{1}{2})}\frac{2}{2N+1}\sum_{j=1}^{\infty} k_j r_j^{2s-2}\sum_{i=1}^{k_j}\cos(2(s-1)\theta_i^j)$$

In particular, if $k_j = 1$ for all $j = 1, 2, 3, \cdots$ we put $\theta_i^j = \theta_j$ so that the above relation becomes,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\frac{\Gamma\left(\frac{1}{2}-(s+N-1)\right)}{\cos(s+N-\frac{1}{2})}\frac{2}{2N+1}\sum_{j=1}^{\infty} r_j^{2s-2}\cos(2(s-1)\theta_j)$$

We will explore some other properties of the fractional hypergeometric zeta functions in the future work. One trivial problem with non-trivial answer is whether or not the pre-functional equation can be made functional equation.

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