Hypergeometric Bernoulli Polynomials

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1. Introduction

The *Bernoulli* numbers are given by

$$\sum_{n=1}^{N} n^p = \sum_{n=0}^{p} (-1)^{\delta_{np}} \frac{p!}{n!(p+1-n)!} B_n N^{p+1-n} \tag{1.1}$$

They were studied by Jacob Bernoulli to establish formulas for sums of powers involving certain coefficients B_n that today bear his name. Euler observed that these numbers can also be obtained from the Taylor series expansion of

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{1.2}$$

The first few are $B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_3 = 0,\ B_4 = -1/30,\ B_5 = 0.$ There are many identities involving Bernoulli numbers and there are generalizations as well. One recursive formula is given by

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0\,.$$

It has also been shown that

$$B_n = \frac{(-1)^{n-1}}{(n+1)!} \begin{vmatrix} 1 & 2 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 3 & 0 & \cdots & 0 \\ 1 & 4 & 6 & 4 & \cdots & 0 \\ 1 & 5 & 10 & 10 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n+1}{0} & \binom{n+1}{1} & \binom{n+1}{2} & \binom{n+1}{3} & \cdots & \binom{n+1}{n-1} \end{vmatrix}_n \tag{1.3}$$

The Bernoulli polynomials $B_n(x)$, defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.4}$$

and whose value at $x = 0$ equals B_n, can be similarly expressed by the formula (see [1]):

$$B_n(x)=\frac{(-1)^{(n)}}{(n-1)!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ x & 1/2 & 1 & 0 & 0 & 0 & \dots & 0 \\ x^2 & 1/3 & 1 & 2 & 0 & 0 & \dots & 0 \\ x^3 & 1/4 & 1 & 3 & 3 & 0 & \dots & 0 \\ x^4 & 1/5 & 1 & 4 & 6 & 4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x^n & 1/(n+1) & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-2} \end{vmatrix}_{n+1} \tag{1.5}$$

Recently Booth and Nguyen [1] demonstrated how formula (1.5) can in fact be extended to a class of generalized Bernoulli polynomials first studied by F. Howard [3], who considered the following natural generalization of (1.4):

$$\frac{\frac{t^N}{N!}e^{xt}}{e^t-T_{N-1}(t)}=\sum_{n=0}^{\infty}B_n(N,x)\frac{t^n}{n!} \tag{1.6}$$

Here, N is any positive integer and

$$T_{N-1}(x)=\sum_{n=0}^{N-1}\frac{t^n}{n!} \tag{1.7}$$

is the Maclaurin polynomial of e^x having degree $N-1$. Howard was able to show that the polynomials $B_n(N,x)$ defined by (1.6) share many of the properties possessed by the classical Bernoulli numbers. Booth and Nguyen [1] found a determinant formula for $B_n(N,x)$ that is analogous to(1.5):

$$B_n(N,x)=\frac{(-1)^{(n)}(N!)^n 1!2!3!\dots(n-N-1)!}{1!2!3!\dots(n-1)!1!2!3!\dots N!}\left|b_{ij}\right| \tag{1.8}$$

where the matrix $\left(b_{ij}\right)$ has entries

$$b_{ij}=\begin{cases} 0 & i+1>j \\ x^{i-1} & i+1\geq j,\ j=1 \\ \dfrac{(i-1)!(N+2-j)!}{(i-j+N+1)!} & i+1\geq j, 2\leq j\leq N+2 \\ \dfrac{(i-1)!}{(i-j+N+1)!(j-N-2)!} & i+1\geq j,\ j\geq N+2 \end{cases} \tag{1.9}$$

Our aim here is to show that their result can be extended in a more general setting. We observe the connection between hypergeometric Bernoulli polynomials and hypergeometric functions given by the relation

$$\frac{\frac{t^N}{N!}e^{xt}}{e^t - T_{N-1}(t)} = \frac{e^{xt}}{{}_1F_1(1, N+1, t)} \tag{1.10}$$

where the confluent hypergeometric function ${}_1F_1(1, N+1, t)$ is defined by

$${}_1F_1(a,b,t) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{t^n}{n!} \tag{1.11}$$

Thus further generalize $B_n(N, x)$ given by (1.5) is possible:

$$\sum_{n=0}^{\infty} B_n(N, x) \frac{t^n}{n!} = \frac{e^{xt}}{{}_1F_1(1, N+1, t)}, \tag{1.12}$$

which is valid for all positive real values of *N*. We shall focus on the case when *N=1/2*.

2. Hypergeometric Bernoulli Polynomials and Matrix Determinants

Starting with the definition for hypergeometric Bernoulli polynomials and using the power series representation of the functions involved, we get:

$$\sum_{n=0}^{\infty} B_n(N, x) \frac{t^n}{n!} = \frac{e^{xt}}{{}_1F_1(1, N+1, t)} = \frac{\sum_{n=0}^{\infty} \frac{x^n t^n}{n!}}{\sum_{n=0}^{\infty} \frac{(1)_n t^n}{(N+1)_n n!}}$$

If we let

$$c_n = \frac{x^n}{n!} \text{ and } a_n = \frac{1}{(N+1)_n} = \frac{\Gamma(N+1)}{\Gamma(N+1+n)}$$

and observe that $(1)_n = n!$, we can rewrite the above as

$$\left(\sum_{n=0}^{\infty} \frac{\Gamma(N+1)}{\Gamma(N+n)} t^n \right) \cdot \left(\sum_{n=0}^{\infty} \frac{B_n(N, x)}{n!} t^n \right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n .$$

Multiplying the two power series on the left, collecting like terms and comparing coefficients will give a system of linear equations with variables $B_n(N, x)$. We then use Cramer's Rule, to obtain

$$B_n(N,x) = n!(-1)^n \begin{vmatrix} 1 & 1 & 0 & \ldots & 0 \\ x & \frac{\Gamma(N+1)}{\Gamma(N+2)} & 1 & \ldots & 0 \\ \frac{x^2}{2!} & \frac{\Gamma(N+1)}{\Gamma(N+3)} & \frac{\Gamma(N+1)}{\Gamma(N+2)} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ \frac{x^n}{n!} & \frac{\Gamma(N+1)}{\Gamma(N+1+n)} & \frac{\Gamma(N+1)}{\Gamma(N+n)} & \ldots & \frac{\Gamma(N+1)}{\Gamma(N+2)} \end{vmatrix}_{n+1} = n!(-1)^n \left|b_{ij}^1\right|$$

where

$$b^1_{ij} = \begin{cases} 0 & i+1<j \\ \frac{x^{i-1}}{\Gamma(i)} & j=1 \\ \frac{\Gamma(N+1)}{\Gamma(i-j+N+2)} & else \end{cases}$$

(For details of these steps, see [1].) We note that it is a simple matter to rearrange this determinant in a completely analogous way to Booth and Nguyen to attain Booth and Nguyen’s formula for the limiting case of binomials.

For $N=\frac{1}{2}$, here are the first few polynomials

$$B_1(1/2,x) = x-\frac{2}{3}$$

$$B_2(1/2,x) = x^2-\frac{4x}{3}+\frac{16}{45} = x^2-\frac{8}{15}-\frac{4}{3}B_1(1/2,x)$$

$$B_3(1/2,x) = x^3-2x^2+\frac{16x}{15}-\frac{32}{315} = x^3-\frac{112}{315}-\frac{8}{5}B_1(1/2,x)-2B_2(1/2,x)$$

$$B_4(1/2,x) = x^4-\frac{8x^3}{3}+\frac{32x^2}{15}-\frac{128x}{315}-\frac{246}{4725} = x^4+\frac{99728}{51975}-\frac{74}{105}B_1(1/2,x)+\frac{48}{15}B_2(1/2,x)-\frac{8}{3}B_3(1/2,x).$$

Note that the second equalities in the above example can be generalized to yield the formula:

$$B_n(1/2,x) = n!\left(\pm\frac{2^{n-1}}{(2(n)-1)!!}B_1(1/2,x) \pm \frac{2^{n-2}}{2!(2(n-1)-1)!!}B_2(1/2,x) \pm \ldots \pm \frac{2}{3(n-1)!}B_{n-1}(1/2,x) \pm \begin{vmatrix} 1 & 1 \\ \frac{x^n}{n!} & \frac{\Gamma(3/2)}{\Gamma(n+3/2)} \end{vmatrix}\right),$$

or

$$B_n(1/2,x)=n!\left(\pm\frac{2^{n-1}}{(2(n)-1)!!}B_1(1/2,x)\pm\frac{2^{n-2}}{2!(2(n-1)-1)!!}B_2(1/2,x)\pm\ldots\pm\frac{2}{3(n-1)!}B_{n-1}(1/2,x)\pm\left(\frac{\Gamma(3/2)}{\Gamma(n+3/2)}-\frac{x^n}{n!}\right)\right).$$

To verify this we consider the determinant generated by the following formula, noting that we can add in the factor n! back in as long as we divide each B_r by r! so as to not double count the factored factorial:

$$b^1_{ij}=\begin{cases} 0 & i+1<j \\ \dfrac{x^{i-1}}{\Gamma(i)} & j=1 \\ \dfrac{\Gamma(3/2)}{\Gamma(i-j+5/2)} & else \end{cases}$$

Clearly, the *nth* column will consist of entirely 0, except for the last two entries which will be 1 and 2/3. We can expand this determinant along the last column. The formula is now

$$(-1)^{n+1}\frac{2}{3}B_{n-1}(1/2,x)+(-1)^n A_1(x),$$

where $A_1(x)$ is the new determinant resulting from the expansion along n^{th} column and $(n-1)st$ row. We can now expand $A_1(x)$ in a similar manner and attain

$$(-1)^{n+1}\frac{2}{3}B_{n-1}(1/2,x)+(-1)^n(-1)^n\frac{4}{15}B_{n-2}(1/2,x)+(-1)^n(-1)^{n-1}A_2(x).$$

We can continue this process to obtain

$$B_n(1/2,x)=n!\left(\pm\frac{2^{n-1}}{(2(n)-1)!!}B_1(1/2,x)\pm\frac{2^{n-2}}{2!(2(n-1)-1)!!}B_2(1/2,x)\pm\ldots\pm\frac{2}{3(n-1)!}B_{n-1}(1/2,x)\pm\begin{vmatrix} 1 & 1 \\ \dfrac{x^n}{n!} & \dfrac{\Gamma(3/2)}{\Gamma(n+3/2)}\end{vmatrix}\right)$$

All that remains to be proved is that

$$\frac{\Gamma(3/2)}{\Gamma(k-(k-r)+5/2)}=\frac{2^{r+1}}{(2r+3)(2r-2+3)\ldots(3)},k,n\in\mathbb{Z}^+\text{(including 0)},\ r=0,1,\ldots k.$$

But this follows from the fact that

$$\frac{\Gamma(3/2)}{\Gamma(k-(k-r)+5/2)}=\frac{\Gamma(3/2)}{\Gamma(r+5/2)}=\frac{1}{(r+3/2)(r-1+3/2)\ldots(3/2)}=\frac{2^{r+1}}{(2r+3)(2r-2+3)\ldots(3)}$$

Concluding Remarks

The associated generalized Bernoulli are worth investigate. For instance it is not clear if there is any sign pattern when $N=\frac{1}{2}$ while the signs alternate for $N=-\frac{1}{2}$.

In [2], Hassen and Nguyen have proved that the hypergeometric Bernoulli polynomials satisfy the following three condition

$$
\begin{aligned}
&B_0(N,x)=1,\\
&B_n^{'}(N,x)=nB_{n-1}(N,x),\\
&\int_0^1 (1-x)^{N-1}B_n(N,x)dx=\begin{cases}\dfrac{1}{N}, \text{ if } n=0,\\ 0, \text{ if } n>0.\end{cases}
\end{aligned}
$$

In fact in the paper mentioned above, Hassen and Nguyen have shown that the only polynomials that satisfy these three properties are the hypergeometric polynomials. The first two equations can be used to show that

$$B_n(N,x)=\sum_{k=0}^{n} B_k(N,0)\binom{n}{k}x^k$$

The third condition then gives the relationship between the associated Bernoulli numbers. It is worth exploring this approach for fractional values of *N*.

References

1. R. Booth and H. D. Nguyen *Bernoulli Polynomials and Pascal's Square*, to appear in The Fibonacci Quarterly.
2. A. Hassen and H. D. Nguyen, *Hypergeometric Bernoulli Polynomials and Appell Sequences*, to appear in Intern. J. Number Theory
3. F. T. Howard, *Numbers Generated by the Reciprocal of* e^*x-1-x*, Math. Comput. 31 (1977) No. 138, 581-598.