PLAYING WITH PARTITIONS ON THE COMPUTER

Mathematics and Computer Education , Vol. 35, No. 1, (2001), pp. 5-17.

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1. INTRODUCTION

One of the joys of mathematical study is the discovery of unexpected relations. In this paper we explore the strange interplay between partitions and pentagonal numbers.

An important function in number theory is $p(n)$, the number of unrestricted partitions of the positive integer *n,* that is, the number of ways of writing *n* as a sum of positive integers. For example, 4+2+2+1 is a partition of the number 9. The order of the summands is irrelevant here, so 4+2+2+1 is the same partition as 2+2+4+1. In Table 1 we show all the partitions of the numbers from 1 to 5 along with the values of $p(n)$.

Table 1: Partitions of a natural number *n*

n	Partitions of *n*	$p(n)$
1	1	1
2	2, 1+1	2
3	3, 2+1, 1+1+1	3
4	4, 3+1, 2+2, 2+1+1, 1+1+1+1	5
5	5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1	7

While it is simple to determine $p(n)$ for very small numbers n by actually counting all the partitions, this becomes difficult as the numbers grow. For example, $p(10) = 42$, and $p(20) = 627$, while $p(100) = 190{,}569{,}292$. It is the purpose of this paper to show how to write a simple program in BASIC to calculate $p(n)$. Along the way we will encounter several nifty mathematical relations.

The values of the partition function for large values of n can be obtained from the following remarkable recursive algorithm:

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) \\ &\quad + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \ldots, \end{aligned} \tag{1.1}$$

where we define $p(-1) = p(-2) = p(-3) = \ldots = 0$. We also define $p(0) = 1$.

This recursive formula was discovered by Euler. In Section 3, we will outline how (1.1) can be proved, but will leave the details to the references. The most mysterious feature in (1.1) is the appearance of the numbers 1, 2, 5, 7, 12, 15,... . These are related to the *pentagonal numbers* and will be discussed in the next section.

In Section 4, we will write a QUICK BASIC program that uses (1.1) to generate a table of the partition function. We have given one such table at the end of this paper. Students can use the table and the program to make and test conjectures concerning partitions.

The notions of pentagonal numbers and partitions are extremely simple and can be understood by students at the precalculus level. The ideas presented here should work well in a first course in programming for high school or college students. They could also be used in courses in discrete mathematics and in number theory. We hope that the opportunity to conjecture properties of partitions from the computer program as well as the intrinsic fascination of the relations like (1.1) will spark student interest.

2. THE PENTAGONAL NUMBERS

Since pentagonal numbers play a central role in this study, we take a brief moment to examine their origin.

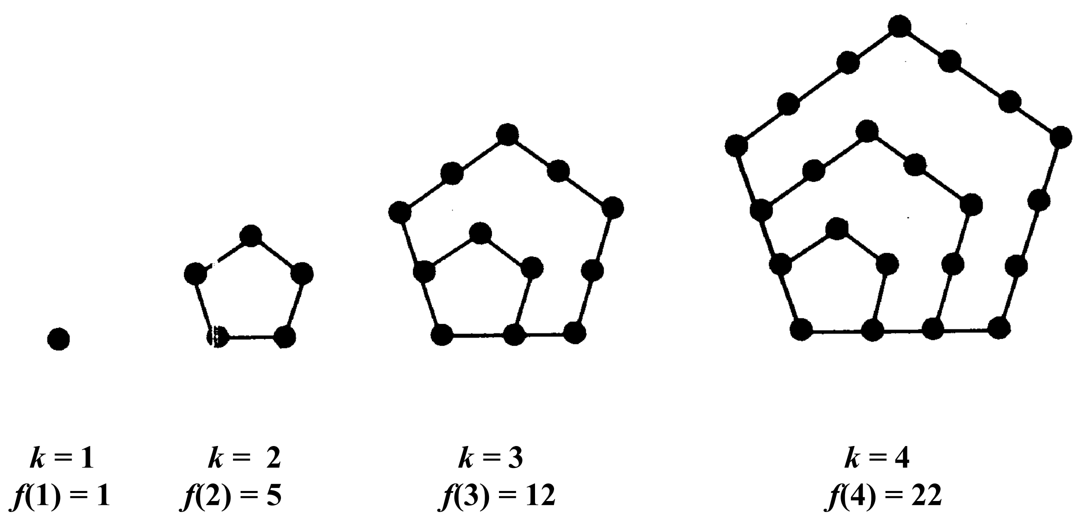

Figure 1: The First Four Pentagonal Numbers

We can easily verify that the sequence of pentagons defined by dots in Figure 1 have the property that when a pentagon has *k* dots on a side, it contains

(2.1) $$f(k) = k(3k-1)/2$$

dots within the pentagon. Thus the sequence of *pentagonal numbers* 1, 5, 12, 22, 35, 51, ... emerges from (2.1) by taking *k* = 1, 2, 3, 4, 5, 6, ...

We will also need to use $f(k)$ when k is a negative integer. It is easy to see that

(2.2) $$f(-k) = k(3k+1)/2\,.$$

Thus the sequence of numbers 2, 7, 15, 26, 40, 57, ... emerges by placing consecutive negative integers in (2.1). This same sequence is generated by (2.2) by using the sequence of positive integers for *k*. We do not know any geometric figure associated with the numbers generated by (2.2), but they could be referred to as *pentagonal numbers of negative index.*

The following is a short table of pentagonal numbers used in the calculation of partitions with the recursion relation (1.1):

Table 2: Pentagonal Numbers *f*(*k*) = *k*(3*k*-1)/ 2

K	*f*(*k*)	*f*(-*k*)	*k*	*f*(*k*)	*f*(-*k*)
1	1	2	11	176	187
2	5	7	12	210	222
3	12	15	13	247	260
4	22	26	14	287	301
5	35	40	15	330	345
6	51	57	16	376	392
7	70	77	17	425	442
8	92	100	18	477	495
9	117	126	19	532	551
10	145	155	20	590	610

3. SOME IMPORTANT RELATIONS INVOLVING PARTITIONS

We now examine three important relations involving the partition function $p(n)$. In some cases, we will give a heuristic explanation of the properties. In all cases we give references where systematic and rigorous treatments can be found.

3.1 The generating function

Euler [4], began the mathematical theory of partitions in 1748 by discovering the so called "generating function"

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n \ . \tag{3.1}$$

The infinite product on the left side of (3.1) "generates" the $p(n)$ as coefficients of the power series on the right side.

What follows is a brief glimpse at why (3.1) works. A full proof is found in Andrews' book [1] on pages 160 to 162. If we expand each of the factors $1/(1-x^n)$ using the geometric series we get the following:

$$\begin{aligned}
\frac{1}{1-x^1} &= 1 + x^{1\bullet 1} + x^{1\bullet 2} + x^{1\bullet 3} + x^{1\bullet 4} + x^{1\bullet 5} + \ldots \\
\frac{1}{1-x^2} &= 1 + x^{2\bullet 1} + x^{2\bullet 2} + x^{2\bullet 3} + x^{2\bullet 4} + x^{2\bullet 5} + \ldots \\
\frac{1}{1-x^3} &= 1 + x^{3\bullet 1} + x^{3\bullet 2} + x^{3\bullet 3} + x^{3\bullet 4} + x^{3\bullet 5} + \ldots \\
\frac{1}{1-x^4} &= 1 + x^{4\bullet 1} + x^{4\bullet 2} + x^{4\bullet 3} + x^{4\bullet 4} + x^{4\bullet 5} + \ldots \\
\frac{1}{1-x^5} &= 1 + x^{5\bullet 1} + x^{5\bullet 2} + x^{5\bullet 3} + x^{5\bullet 4} + x^{5\bullet 5} + \ldots \\
&\ldots
\end{aligned} \tag{3.2}$$

When we multiply the series on the right side of (3.2) and carefully observe what is taking place, we see that the partition function is being generated. To see a particular case, look at the terms that generate x^5. They are

$$x^{1\bullet 5}+x^{1\bullet 3}x^{2\bullet 1}+x^{1\bullet 2}x^{3\bullet 1}+x^{1\bullet 1}x^{4\bullet 1}+x^{1\bullet 1}x^{2\bullet 2}+x^{2\bullet 1}x^{3\bullet 1}+x^{5\bullet 1}=7x^5$$

(Here we interpret the power of $x^{a\bullet b}$ to mean $a+a+...+a$ with b terms). Notice that each of the exponents is a particular partition of the number 5. These are, respectively, 1+1+1+1+1, 1+1+1+2, 1+1+3, 1+4, 1+2+2, 2+3 and 5. Thus there are 7 partitions of the number 5. This illustrates how the generating function (3.1) works.

A computer algebra system, like *Mathematica*, can use this idea to calculate $p(n)$. However it would not be a good way to find the partitions of a large number. One of the important implications of (3.1) is that the function defined by the infinite product can be studied analytically to get asymptotic expressions for *p(n),* which we will describe next.

3.2 The asymptotic formula

A glance at a table of the partition function shows that p(n) grows "very fast". How fast is "very fast"? Hardy and Ramanujan have given us an asymptotic formula for p(n). Before we present this formula, we mention one of the most common asymptotic expression known as Stirling’s formula:

(3.3) $$n!\approx\sqrt{2\pi n}\,n^n/e^n,$$

which can be used to estimate large values of the factorial. In a similar spirit we have the asymptotic formula for the partition function

(3.4) $$p(n)\approx\frac{\exp(\pi\sqrt{2n/3})}{4\sqrt{3}\,n}.$$

Hardy and Ramanujan [11] published (3.4) in 1917 and again in 1918 using advanced methods from the theory of functions of a complex variable. (See Kanigel's book [6] for a readable description of the collaboration of Hardy and Ramanujan on (3.4).) These asymptotic formulas contain a marvelous mystery. The left hand sides of both (3.3) and (3.4) are integers. But the right hand sides contain π , *e*, and square roots. What does π have to do with factorials or partitions? When we leave this world, this is the first question we would like to ask God!

3.3 The recursion relation

As we mentioned in Section 1, the values of the partition function can be obtained from the following remarkable recursive algorithm (1.1). We reproduce this formula here for an easy reference.

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) \\ &\quad + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \ldots \end{aligned} \tag{3.5}$$

where we define $p(-1) = p(-2) = p(-3) = \ldots = 0$. We also define $p(0) = 1$. We can also write (3.5) in the following form

$$p(n) = \sum_{k=1}^{\infty} (-1)^{k+1} \{ p(n - f(k)) + p(n - f(-k)) \}, \tag{3.6}$$

where $f(k) = k(3k-1)/2$ generates the sequence of pentagonal numbers For example (3.5) tells us that

$$\begin{aligned} p(11) &= p(10)+p(9)-p(6)-p(4)+p(-1)+p(-4)-\ldots \\ &= p(10)+p(9)-p(6)-p(4)+0+0+\ldots \end{aligned}.$$

The remaining terms all have negative arguments and are thus zero. In this way we can calculate the number of partitions of 11 if we know the partitions of 10, 9, 6 and 4. Using Table 3 we have

$p(11) = 42 + 30 - 11 - 5 = 56$

The full proof of the recursion relation (3.6) is beyond the scope of this paper. This proof can be found in Hardy and Wright [5] and in Andrews [1]. However, since the proof is itself very interesting, we give here a brief outline of the main steps.

The proof of (3.6) begins with Euler's remarkable discovery known as "Euler's pentagonal number theorem":

$$\begin{aligned} \prod_{n=1}^{\infty}(1-x^n) &= \sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n-1)/2} \\ &= 1+\sum_{n=1}^{\infty}(-1)^n\left\{x^{n(3n-1)/2}+x^{n(3n+1)/2}\right\} \end{aligned} \tag{3.7}$$

Writing out the terms in (3.7) explicitly we get

$$\begin{aligned} &(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots= \\ &\quad 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots \end{aligned} \tag{3.8}$$

The reader can multiply out a few of the factors on the left side of (3.8) to see that the terms involving pentagonal numbers as exponents appear on the right side.

Notice that the left side of (3.1) is the reciprocal of the left side of (3.8). From this it follows that

$$\sum_{n=0}^{\infty} p(n)x^n = \left(1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots\right)^{-1},$$

and therefore

$$\left(\sum_{n=0}^{\infty} p(n)x^n\right)\left(1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots\right)=1.$$

Multiplying out the product of series above we get

$$\begin{aligned}1 = 1 &+ (p(1)-p(0))x + (p(2)-p(1)-p(0))x^2 + \\ &(p(3)-p(2)-p(1))x^3 + (p(4)-p(3)-p(2))x^4 + . \\ &(p(5)-p(4)-p(3)+p(0))x^5 + \ldots\end{aligned} \tag{3.9}$$

Since the left side of (3.9) is 1, all the coefficients of the powers of x on the right side are zero. Thus we get

$$\begin{aligned}&p(1) = p(0), \\ &p(2) = p(1) + p(0), \\ &p(3) = p(2) + p(1), \\ &p(4) = p(3) + p(2), \\ &p(5) = p(4) + p(3) - p(0), \\ &\ldots\end{aligned}$$

This last list of relations is the first five values of our recursion relation (3.6). This completes our brief look at how this important recursion relation emerges.

4. A BASIC PROGRAM TO GENERATE PARTITIONS

In this section we examine a simple program written in QUICK BASIC(also QBASIC) to calculate a list of the values of the partition function $p(n)$ for $n = 1,2,3,\ldots$. The program can be easily modified to work in any version of BASIC or any computer language.

The lines that begin with an "apostrophe" are merely remarks and can be omitted.

Line 100 sets all variables to double precision mode. This allows 16 digits for integers (but only 15 digits of certain accuracy) in the computations. The BASIC interpreter used by the author gave accurate exact values of $p(n)$ for n from 1 to 293.

Line 100 dimensions the array P, and line 120 defines the value of $p(0)$.

Each time the FOR - NEXT loop from lines 200 to 500 is executed, we calculate another value of the partition function $p(n)$. Each time the FOR - NEXT loop in lines 220 to 300 is performed we find another value of the term

$$(-1)^{k+1}\{p(n-f(k))+p(n-f(-k))\} \tag{4.1}$$

from the recursion relation (3.4). The variable SIGN in lines 210, 250, 280 and 290 contains the value of $(-1)^{k+1}$ from (4.1). We exit this loop in line 240 or 270 where we check to see if $n-f(k)$ or $n-f(-k)$ is negative. (Recall from the previous section that $p(m)=0$ when m is a negative integer.)

In line 230 we calculate the pentagonal number $f(k)=k(3k-1)/2$. In line 250 we add the term $(-1)^{k+1}p(n-f(k))$ to the present value of the sum for $p(n)$. Again in line 260 we calculate the value of $f(-k)=k(3k+1)/2$ needed in (4.1), and in line 280 we add the term $(-1)^{k+1}p(n-f(-k))$ to the sum for $p(n)$.

In line 400 we print the value just calculated for n and for $p(n)$ on the screen. Line 450 causes the screen calculations to pause after 20 lines are printed so that they can be examined before they scroll out of view.

This completes our explanation of the program that calculates the partition function.

Program 1: Calculate Partitions

```
        'Calculate partitions of N,  P(N)
        'exactly up to P(301).
        'Set double precision, dimension array P, initialize P
90     CLS
100    DEFDBL A-Z
110    DIM P(400)
120    P(0) = 1

        'Main loop, for each N find P(N)
200    FOR N = 1 TO 293
210      SIGN = 1
215      P(N) = 0

220      FOR K = 1 TO 100
          'Calculate two terms in recursion relation for P(N)
230        F = K * (3 * K - 1) / 2
          'Exit loop if argument negative
240        IF N - F < 0 THEN GOTO 400
250        P(N) = P(N) + SIGN * P(N - F)
260        F = K * (3 * K + 1) / 2
          'Exit loop if argument negative
270        IF N - F < 0 THEN GOTO 400
280        P(N) = P(N) + SIGN * P(N - F)
290        SIGN = -SIGN
300      NEXT K
        'Print results
400      PRINT N, P(N)
        'Pause after printing 20 lines on the screen
450      IF 20 * INT(N / 20) = N THEN INPUT A$: CLS
500    NEXT N
```

5. USING THE PROGRAM TO CHECK CONJECTURES

Now that we can easily generate many values of the partition function, we examine the results to see if any observable patterns are emerging.

Ramanujan examined a table of the first 200 values of $p(n)$ calculated by Major Mac Mahon and conjectured and proved the following in 1921, (see [11] on pages 233 to 238).

(5.1) $$p(5m+4) \equiv 0 \pmod{5},$$

(5.2) $p(7m+5) \equiv 0 \pmod{7}$,

(5.3) $p(11m+6) \equiv 0 \pmod{11}$.

Evidence of the validity of (5.1) is easily seen in Table 3. We look at the values of n that end in the digit 4 or 9. These are the numbers of the form $n = 5m+4$ with $m = 0,1,2,\ldots$. Notice that the values of $p(5m+4)$ all end in the digit 0 or 5, thereby supporting (5.1). (See Kanigel's book [6], page 250, for a brief description of Major Mac Mahon and his work with Ramanujan.)

We can also check these relations with the computer. If we add the following lines to our program:

```
1000   M = 5 :  R = 4
1010    FOR N = R TO 293 STEP M
1020    IF P(N) = M *  INT( P(N)/M) THEN PRINT N; "TRUE",
        ELSE PRINT N; "FALSE",
1030    NEXT N
```

This FOR - NEXT loop runs through the values N = M, M+R, M+2R, M+3R, ..., where M (modulus) and R (residue) are defined in line 1000. Line 1020 checks to see if P(N) is divisible by the modulus M. It then prints N and the word TRUE if the division was successful, otherwise it prints FALSE. By changing line 1000 to M = 7: R = 5, we can check (5.2). We can check (5.3) by changing line 1000 to M = 11 : R = 6.

These "arithmetic properties" of the partition function have been the subject of recent research. Ken Ono [7], [8] and [9] proved new results regarding these congruences. In particular he showed that if $m \geq 5$ is prime, then there are positive integers a and b for which $p(an+b) \equiv 0 \pmod{m}$, for every non-negative integer n. When is $p(n)$ even or odd? This question remains unanswered. You can use the above program to check for even $p(n)$ by changing line 1000 to M = 2 : R = 0. Few results are known for modulus M = 3. Perhaps the reader can find the answer.

A proof of (5.1) is given in Hardy and Wright [5] on pages 287 to 290, along with a few more arithmetical results.

We can also use the program to verify the asymptotic relation (3.4) for some values of n. Replace line 400 with the lines

```
400 A = EXP(3.14159*(2 * N/3)^.5)/(4 * (3)^.5 * N)
410  E = A - P(N) : PCT = 100*E/P(N)
420  U$ = " ###  #################  #################  ##.## "
430  PRINT USING U$; N,P(N), A, PCT
```

In line 400 we use (3.10) to find A which is the asymptotic estimate of P(N). In line 410 we find the error E and the percentage error PCT. Lines 420 and 430 print out the results in four columns. We see that there is almost a 10 percent error for small N. Gradually this error diminishes to about 2 percent when N = 300.

6. FINAL REMARKS

In addition to pentagonal numbers discussed in Section 2, there are triangular numbers, square numbers, hexagonal numbers, etc. The initial study of these numbers is attributed to the Pythagoreans, as early as 500 BC. They are called *figurative numbers* and many interesting relations exist among them. The Pythagoreans believed that "everything is number", and therefore took great interest in this study. For a lively discussion of figurative numbers and the Pythagoreans see Burton [3].

Two major branches of the theory of numbers are the *multiplicative theory* and the *additive theory.* In the multiplicative theory we decompose a natural number n into prime factors $n = p_1 p_2 p_3 \dots p_k$ and consider the consequences. In the additive theory we decompose our natural number into a sum of elements from some set. For example we could try to express n as a sum of squares. Our study of partitions is part of this additive theory. Most textbooks on number theory ignore partitions. Exceptions are the excellent text by Andrews [1] and the bible of number theory Hardy and Wright [5].

In the multiplicative theory we examine many functions, one of which is the *sum of the divisors of n,* $\sigma(n)$. For example the divisors of 6 are 1, 2, 3, and 6. Thus the sum of the divisors of 6 is $\sigma(6) = 1+2+3+6 = 12$. Now divisors of numbers are related to primes, and primes seem unrelated to partitions. We are not surprised that partitions satisfy a recursion relation, although the appearance of pentagonal numbers in the relation is a wonder. We do not expect $\sigma(n)$ to satisfy a recursion relation. What do the divisors of n have to do with the divisors of $n-1, n-2, \ldots$? Yet Euler showed that $\sigma(n)$ satisfies the same recursion relation (3.4) as does $p(n)$. Only $\sigma(0)$ is different from $p(0)$. Euler was astonished at this result, and you can read a translation of his own words in Polya [10] and in Young [14]. (Every lover of mathematical analysis should own Young's book [14]). There are even relations "marrying" the two functions such as (Schroeder [12])

$$n\,p(n) = \sum_{k=1}^{n} \sigma(k)\,p(n-k) .$$

We plan to explore these items in a sequel to this paper called *The unlikely marriage of partitions and divisors.*

For additional programs in number theory in the spirit of this paper see the fun book by Spencer [13].

7. REFERENCES

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Table 3: Values of the partition function

n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$
1	1	41	44583	81	18004327	121	2056148051
2	2	42	53174	82	20506255	122	2291320912
3	3	43	63261	83	23338469	123	2552338241
4	5	44	75175	84	26543660	124	2841940500
5	7	45	89134	85	30167357	125	3163127352
6	11	46	105558	86	34262962	126	3519222692
7	15	47	124754	87	38887673	127	3913864295
8	22	48	147273	88	44108109	128	4351078600
9	30	49	173525	89	49995925	129	4835271870
10	42	50	204226	90	56634173	130	5371315400
11	56	51	239943	91	64112359	131	5964539504
12	77	52	281589	92	72533807	132	6620830889
13	101	53	329931	93	82010177	133	7346629512
14	135	54	386155	94	92669720	134	8149040695
15	176	55	451276	95	104651419	135	9035836076
16	231	56	526823	96	118114304	136	10015581680
17	297	57	614154	97	133230930	137	11097645016
18	385	58	715220	98	150198136	138	12292341831
19	490	59	831820	99	169229875	139	13610949895
20	627	60	966467	100	190569292	140	15065878135
21	792	61	1121505	101	214481126	141	16670689208
22	1002	62	1300156	102	241265379	142	18440293320
23	1255	63	1505499	103	271248950	143	20390982757
24	1575	64	1741630	104	304801365	144	22540654445
25	1958	65	2012558	105	342325709	145	24908858009
26	2436	66	2323520	106	384276336	146	27517052599
27	3010	67	2679689	107	431149389	147	30388671978
28	3718	68	3087735	108	483502844	148	33549419497
29	4565	69	3554345	109	541946240	149	37027355200
30	5604	70	4087968	110	607163746	150	40853235313
31	6842	71	4697205	111	679903203	151	45060624582
32	8349	72	5392783	112	761002156	152	49686288421
33	10143	73	6185689	113	851376628	153	54770336324
34	12310	74	7089500	114	952050665	154	60356673280
35	14883	75	8118264	115	1064144451	155	66493182097
36	17977	76	9289091	116	1188908248	156	73232243759
37	21637	77	10619863	117	1327710076	157	80630964769
38	26015	78	12132164	118	1482074143	158	88751778802
39	31185	79	13848650	119	1653668665	159	97662728555
40	37338	80	15796476	120	1844349560	160	107438159466

Table 3: Values of the partition function (continued)

n	*p(n)*	*n*	*p(n)*	*n*	*p(n)*
161	118159068427	201	4328363658647	241	114540884553038
162	129913904637	202	4714566886083	242	123888443077259
163	142798995930	203	5134205287973	243	133978259344888
164	156919475295	204	5590088317495	244	144867692496445
165	172389800255	205	6085253859260	245	156618412527946
166	189334822579	206	6622987708040	246	169296722391554
167	207890420102	207	7206841706490	247	182973889854026
168	228204732751	208	7840656226137	248	197726516681672
169	250438925115	209	8528581302375	249	213636919820625
170	274768617130	210	9275102575355	250	230793554364681
171	301384802048	211	10085065885767	251	249291451168559
172	330495499613	212	10963707205259	252	269232701252579
173	362326859895	213	11916681236278	253	290726957916112
174	397125074750	214	12950095925895	254	313891991306665
175	435157697830	215	14070545699287	255	338854264248680
176	476715857290	216	15285151248481	256	365749566870782
177	522115831195	217	16601598107914	257	394723676655357
178	571701605655	218	18028182516671	258	425933084409356
179	625846753120	219	19573856161145	259	459545750448675
180	684957390936	220	21248279009367	260	495741934760846
181	749474411781	221	23061871173849	261	534715062908609
182	819876908323	222	25025873760111	262	576672674947168
183	896684817527	223	27152408925615	263	621837416509615
184	980462880430	224	29454549941750	264	670448123060170
185	1071823774337	225	31946390696157	265	722760953690372
186	1171432692373	226	34643126322519	266	779050629562167
187	1280011042268	227	37561133582570	267	839611730366814
188	1398341745571	228	40718063627362	268	904760108316360
189	1527273599625	229	44132934884255	269	974834369944625
190	1667727404093	230	47826239745920	270	1050197489931117
191	1820701100652	231	51820051838712	271	1131238503938606
192	1987276856363	232	56138148670947	272	1218374349844333
193	2168627105469	233	60806135438329	273	1312051800816215
194	2366022741845	234	65851585970275	274	1412749565173450
195	2580840212973	235	71304185514919	275	1520980492851175
196	2814570987591	236	77195892663512	276	1637293969337171
197	3068829878530	237	83561103925871	277	1762278433057269
198	3345365983698	238	90436839668817	278	1896564103591584
199	3646072432125	239	97862933703585	279	2040825852575075
200	3972999029388	240	105882246722733	280	2195786311682516