A REMARK ON NONDISCRETE HECKE GROUPS

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Let \( \lambda > 0 \). The Hecke group \( G(\lambda) \) is the group generated by

\[
S_{\lambda} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

\( G(\lambda) \) acts on \( \mathcal{H} \) by \( Mz = \frac{az + b}{cz + d} \), where \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda) \) and \( z \in \mathcal{H} \). In this case we identify \( M \) with its negative \( -M \) and consider the elements of \( G(\lambda) \) as fractional linear transformations. For this reason, we will use the convention that the order of a matrix is the order of the linear fractional transformation associated to it. Thus the order of \( T \) is 2.

It is well-known (see [1] and [2]) that the only discrete Hecke groups are those for which \( \lambda \geq 2 \) or \( \lambda = 2 \cos \left( \frac{\pi}{p} \right), p \in \mathbb{Z}, p \geq 3 \).

When \( \lambda = 2 \cos \left( \frac{r\pi}{p} \right) \), where \( r \) and \( p \) are relatively prime integers with \( 2 \leq 2r < p \), the group \( G(\lambda) \) has an elliptic element of order \( p > 2 \). In fact;

\[
(S_{\lambda}T)^p = (-1)^RI
\]

To see this let

\[
(S_{\lambda}T)^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}
\]

By induction we can show that

\[
a_n = \frac{\sin \left( \frac{(n+1)\pi}{p} \right)}{\sin \left( \frac{\pi}{p} \right)}, \quad b_n = -\frac{\sin \left( \frac{n\pi}{p} \right)}{\sin \left( \frac{\pi}{p} \right)}, \quad c_n = \frac{\sin \left( \frac{n\pi}{p} \right)}{\sin \left( \frac{2\pi}{p} \right)}, \quad d_n = -\frac{\sin \left( \frac{(n-1)\pi}{p} \right)}{\sin \left( \frac{\pi}{p} \right)}
\]

and the assertion follows from this.

The main objective of this article is to show that when \( \lambda = 2 \cos(\pi\theta) \) and \( 0 < \theta < 1/2 \) is irrational, the group \( G(\lambda) \) has no elliptic elements of finite order other than those conjugate to \( T \). For the remainder of our discussion, we shall assume that \( m_1, \ldots, m_n \) are nonzero integers, where \( n \) is a positive integer. \( A_n \) denotes the product

\[
A_n = S_{\lambda}^{m_1}T S_{\lambda}^{m_2}T \cdots TS_{\lambda}^{m_n} = \begin{pmatrix} a_n(\lambda) & b_n(\lambda) \\ c_n(\lambda) & d_n(\lambda) \end{pmatrix}.
\]

Note then that \( a_n(\lambda), b_n(\lambda), c_n(\lambda) \) and \( d_n(\lambda) \) are polynomials in \( \lambda \) with integer coefficients. We will denote by \( a_{n,0}, b_{n,0}, c_{n,0}, \) and \( d_{n,0} \), respectively, the constant terms of these polynomials.

**Lemma 1.** If \( n \geq 2 \), then (i) \( \deg (a_n(\lambda)) = n - 1 \), \( \deg (b_n(\lambda)) = n \), \( \deg (c_n(\lambda)) = n - 2 \), and \( \deg (d_n(\lambda)) = n - 1 \).

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(ii) \( b_n (\lambda) = (m_1 m_2 \cdots m_n) \lambda^n + \cdots \).

(The dots here stand for terms of degree less than \( n \).)

(iii) \( a_{n,0} = d_{n,0} = \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad b_{n,0} = -c_{n,0} = \begin{cases} (\frac{1}{2}), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases} \)

**Proof:** Induction on \( n \). If \( n = 2 \), then

\[
A_2 = S^m \lambda^2 T S^m \lambda^m = \begin{pmatrix} a_2 (\lambda) & b_2 (\lambda) \\ c_2 (\lambda) & d_2 (\lambda) \end{pmatrix} = \begin{pmatrix} m_1 \lambda & m_1 m_2 \lambda^2 - 1 \\ 1 & m_2 \lambda \end{pmatrix},
\]

and so all statements of the lemma are true. Suppose the assertions are true for some \( n \geq 2 \). Now,

\[
A_{n+1} = A_n T S^m \lambda^{n+1} = \begin{pmatrix} a_n (\lambda) & b_n (\lambda) \\ c_n (\lambda) & d_n (\lambda) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & m_{n+1} \lambda \end{pmatrix} = \begin{pmatrix} b_n (\lambda) & m_{n+1} \lambda b_n (\lambda) - a_n (\lambda) \\ d_n (\lambda) & m_{n+1} \lambda d_n (\lambda) - c_n (\lambda) \end{pmatrix}.
\]

Thus

\[
a_{n+1} (\lambda) = b_n (\lambda), \quad b_{n+1} (\lambda) = m_{n+1} \lambda b_n (\lambda) - a_n (\lambda), \quad c_{n+1} (\lambda) = d_n (\lambda), \quad d_{n+1} (\lambda) = m_{n+1} \lambda d_n (\lambda) - c_n (\lambda).
\]

From these equations and the induction assumptions we see that the assertions in (i) and (ii) are valid for \( n + 1 \) as well, thereby completing the induction step. To complete the proof of (iii), we observe that a repeated application of the above procedure gives

\[
a_{n+1} (\lambda) = m_n \lambda b_{n-1} (\lambda) - a_{n-1} (\lambda), \quad b_{n+1} (\lambda) = (m_n m_{n+1} \lambda^2 - 1) b_{n-1} (\lambda) - m_{n+1} \lambda a_{n-1} (\lambda), \\
c_{n+1} (\lambda) = m_n \lambda c_{n-1} (\lambda) - d_{n-1} (\lambda), \quad d_{n+1} (\lambda) = (m_n m_{n+1} \lambda^2 - 1) d_{n-1} (\lambda) - m_{n+1} \lambda c_{n-1} (\lambda).
\]

Upon comparing coefficients, we see from these equations that (iii) holds for \( n + 1 \) whenever it holds for \( n \). This completes the proof of the lemma.

In what follows we shall assume that \( \lambda = 2 \cos (\pi \theta) \) is transcendental. Note that this is the case, for example, if \( \theta \) is an irrational algebraic number. For in that case, by the Hilbert-Gelfond-Schneider Theorem (See [3], Chapter III), \( \rho = e^{\pi i g} \) is transcendental and from the relation \( \lambda = \rho + \frac{1}{\rho} \), we have \( \rho^2 - \lambda \rho + 1 = 0 \).

**Lemma 2.** Suppose \( B \in G (\lambda) \) is an elliptic element of finite order. Then its order is either 2 or 3 and \( \text{tr} (B) = -1, 0, 1 \), where \( \text{tr} (B) \) is the trace of \( B \).

**Proof:** Let \( B = \begin{pmatrix} a (\lambda) & b (\lambda) \\ c (\lambda) & d (\lambda) \end{pmatrix} \). Note then that \( a (\lambda), \ b (\lambda), \ c (\lambda), \) and \( d (\lambda) \) are polynomials in \( \lambda \) with coefficients in \( \mathbb{Z} \). We let \( a_0 \) and \( d_0 \) denote the constant terms of \( a (\lambda) \) and \( d (\lambda) \), respectively. Clearly \( B \) is diagonalizable. Let \( C \) be a matrix such that

\[
A = CBC^{-1} = \begin{pmatrix} f (\lambda) & 0 \\ 0 & g (\lambda) \end{pmatrix}.
\]

Now if \( B^n = I \), then \( A^n = I \) and hence \( f (\lambda)^n = 1 \) and \( g (\lambda)^n = 1 \). Since \( \det (A) = 1 \), we see that \( f (\lambda) = e^{2 \pi i m/n} \) and \( g (\lambda) = e^{-2 \pi i m/n} \) for some integers \( m \) and \( n \). On the other hand, \( \text{tr} (B) = \text{tr} (A) \) implies that \( a (\lambda) + d (\lambda) = e^{2 \pi i m/n} + e^{-2 \pi i m/n} = 2 \cos (2 \pi m/n) \). Since \( 2 \cos (2 \pi m/n) \) is algebraic and by assumption \( \lambda \) is transcendental, we see that \( a (\lambda) + d (\lambda) = a_0 + d_0 \) and hence \( 2 \cos (2 \pi m/n) \) is an integer. Since \( B \) is
elliptic, we conclude that $2 \cos (2\pi m/n) = -1, 0, \text{ or } 1$. Consequently, $n = 2$ or $3$.

**Corollary 1.** If $A_n$ is elliptic of finite order, then $a_{n,0} = 0$ and so $n$ is even and $tr(A_n) = 0$.

**Proof:** By Lemma 2 $tr(A_n) = -1, 0, \text{ or } 1$. Thus $a_n(\lambda) + d_n(\lambda) = -1, 0, \text{ or } 1$ and since $\lambda$ is transcendental, we have $a_0 + d_0 = -1, 0, \text{ or } 1$. But by Lemma 1(iii), $a_0 = d_0$. Since these are integers, we must have $a_0 = 0$. The remaining assertions of the corollary now follow from Lemma 1(iii).

**Lemma 3.** If $B \in G(\lambda)$ is elliptic of finite order, then $tr(B) = 0$ and hence $B$ is of order 2.

**Proof:** There are four possibilities:

Case I: $B = A_n$. In this case, the first statement follows from Corollary 1 and the second follows from the proof of Lemma 1.

Case II: $B = TA_nT$. In this case $tr(B) = -tr(A_n) = 0$, where we have used Corollary 4.1.

Case III: $B = T A_n$. Then

$$B = \begin{pmatrix} -c_n(\lambda) & -d_n(\lambda) \\ a_n(\lambda) & b_n(\lambda) \end{pmatrix},$$

and so $tr(B) = b_n(\lambda) - c_n(\lambda)$. But then by Lemma 2(i), $tr(B)$ is not an integer and we have a contradiction to Lemma 1.

Case IV: $B = A_nT$. This is similar to Case III.

**Remark 1:** It follows from these lemmas that if $B \in G(\lambda)$ is elliptic of finite order, then it is conjugate to $A_n$. We are now in a position to prove that an elliptic element of finite order is conjugate to $T$.

**Theorem 1.** If $B \in G(\lambda)$ is elliptic of finite order, then it is conjugate to $T$.

**Proof:** By Remark 1 it suffices to show that if $A_n$ is elliptic of finite order, then it is conjugate to $T$. First observe that $n$ is even. We write

$$A_n^2 = S_{\lambda}^{m_1}T S_{\lambda}^{m_2}T \cdots S_{\lambda}^{m_n+m_1}T S_{\lambda}^{m_2}T \cdots S_{\lambda}^{m_n} = \begin{pmatrix} c_n(\lambda) & f_n(\lambda) \\ g_n(\lambda) & h_n(\lambda) \end{pmatrix}.$$ 

Then by Lemma 4.2(ii), we have $f_n(\lambda) = (m_1m_2 \cdots (m_n + m_1)m_2 \cdots m_n)\lambda^{2n-1} + \cdots$ (where the dots are terms of degree less than $2n - 1$). Since $\lambda$ is transcendental, it must be that the leading term of $f_n$ is zero. Since each $m_j$ is nonzero, we must have $m_1 + m_n = 0$, that is, $m_n = -m_1$. Hence $A_n$ is conjugate to $S_{\lambda}^{m_2}T S_{\lambda}^{m_2}T \cdots S_{\lambda}^{m_n-1}$ and the stage for induction has been set.

**References**


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