A REMARK ON NONDISCRETE HECKE GROUPS

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Let $\lambda > 0$. *The Hecke group* $G(\lambda)$ is the group generated by

$$S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$G(\lambda)$ acts on $\mathcal{H}$ by $Mz = \frac{az+b}{cz+d}$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda)$ and $z \in \mathcal{H}$. In this case we identify M with its negative $-M$ and consider the elements of $G(\lambda)$ as fractional linear transformations. For this reason, we will use the convention that the order of a matrix is the order of the linear fractional transformation associated to it. Thus the order of T is 2.

It is well-known (see [1] and [2]) that the only discrete Hecke groups are those for which

$$\lambda \geq 2 \quad \text{or} \quad \lambda = 2\cos(\pi/p), \ p \in \mathbf{Z}, \ p \geq 3.$$

When $\lambda = 2\cos(r\pi/p)$, where r and p are relatively prime integers with $2 \leq 2r < p$, the group $G(\lambda)$ has an elliptic element of order $p > 2$. In fact;

$$(S_\lambda T)^p = (-1)^R I$$

To see this let

$$(S_\lambda T)^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

By induction we can show that

$$a_n = \frac{\sin\left(\frac{(n+1)r\pi}{p}\right)}{\sin\left(\frac{r\pi}{p}\right)}, \quad b_n = -\frac{\sin\left(\frac{nr\pi}{p}\right)}{\sin\left(\frac{r\pi}{p}\right)}, \quad c_n = \frac{\sin\left(\frac{nr\pi}{p}\right)}{\sin\left(\frac{r\pi}{p}\right)}, \quad d_n = -\frac{\sin\left(\frac{(n-1)r\pi}{p}\right)}{\sin\left(\frac{r\pi}{p}\right)}$$

and the assertion follows from this.

The main obejective of this article is to show that when $\lambda = 2\cos(\pi\theta)$ and $0 < \theta < 1/2$ is irrational , the group $G(\lambda)$ has no elliptic elements of finite order other than those conjugate to T. For the remainder of our discussion, we shall assume that $m_1, \ldots, m_n$ are nonzero integers, where n is a positive integer. A_n denotes the product

$$A_n = S_\lambda^{m_1} T S_\lambda^{m_2} T \cdots T S_\lambda^{m_n} = \begin{pmatrix} a_n(\lambda) & b_n(\lambda) \\ c_n(\lambda) & d_n(\lambda) \end{pmatrix}.$$

Note then that $a_n(\lambda), b_n(\lambda), c_n(\lambda)$ and $d_n(\lambda)$ are polynomials in λ with integer coefficients. We will denote by $a_{n,0}, b_{n,0}, c_{n,0}$, and $d_{n,0}$, respectively, the constant terms of these polynomials.

Lemma 1. *If* $n \geq 2$, *then* (*i*) $\quad deg(a_n(\lambda)) = n-1, \ deg(b_n(\lambda)) = n, \ deg(c_n(\lambda)) = n-2,$ *and* $\quad deg(d_n(\lambda)) = n-1.$

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(ii) $\qquad b_n(\lambda) = (m_1 m_2 \cdots m_n)\lambda^n + \cdots$.

(The dots here stand for terms of degree less than n.)

(iii) $\qquad a_{n,0} = d_{n,0} = \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \end{cases} \qquad$ *and* $\qquad b_{n,0} = -c_{n,0} = \begin{cases} (-1)^{\frac{n}{2}}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$

Proof : Induction on n. If $n = 2$, then

$$A_2 = S_\lambda^{m_1} T S_\lambda^{m_2} = \begin{pmatrix} a_2(\lambda) & b_2(\lambda) \\ c_2(\lambda) & d_2(\lambda) \end{pmatrix} = \begin{pmatrix} m_1\lambda & m_1 m_2 \lambda^2 - 1 \\ 1 & m_2\lambda \end{pmatrix},$$

and so all statements of the lemma are true. Suppose the assertions are true for some $n \geq 2$. Now,

$$\begin{aligned} A_{n+1} &= A_n T S_\lambda^{m_{n+1}} \\ &= \begin{pmatrix} a_n(\lambda) & b_n(\lambda) \\ c_n(\lambda) & d_n(\lambda) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & m_{n+1}\lambda \end{pmatrix} \\ &= \begin{pmatrix} b_n(\lambda) & m_{n+1}\lambda b_n(\lambda) - a_n(\lambda) \\ d_n(\lambda) & m_{n+1}\lambda d_n(\lambda) - c_n(\lambda) \end{pmatrix}. \end{aligned}$$

Thus

$$a_{n+1}(\lambda) = b_n(\lambda), \quad b_{n+1}(\lambda) = m_{n+1}\lambda b_n(\lambda) - a_n(\lambda), \quad c_{n+1}(\lambda) = d_n(\lambda), \quad d_{n+1}(\lambda) = m_{n+1}\lambda d_n(\lambda) - c_n(\lambda).$$

From these equations and the induction assumptions we see that the assertions in (i) and (ii) are valid for $n+1$ as well, thereby completing the induction step. To complete the proof of (iii), we observe that a repeated application of the above procedure gives

$$a_{n+1}(\lambda) = m_n\lambda b_{n-1}(\lambda) - a_{n-1}(\lambda), \qquad b_{n+1}(\lambda) = \left(m_n m_{n+1}\lambda^2 - 1\right) b_{n-1}(\lambda) - m_{n+1}\lambda a_{n-1}(\lambda),$$

$$c_{n+1}(\lambda) = m_n\lambda c_{n-1}(\lambda) - d_{n-1}(\lambda), \qquad d_{n+1}(\lambda) = \left(m_n m_{n+1}\lambda^2 - 1\right) d_{n-1}(\lambda) - m_{n+1}\lambda c_{n-1}(\lambda).$$

Upon comparing coefficients, we see from these equations that (iii) holds for $n+1$ whenever it holds for n. This completes the proof of the lemma.

In what follows we shall assume that $\lambda = 2\cos(\pi\theta)$ is transcendental. Note that this is the case, for example, if θ is an irrational algebraic number. For in that case, by the Hilbert-Gelfond-Schneider Theorem (See [3], Chapter III), $\rho = e^{\pi i\theta}$ is transcendental and from the relation $\lambda = \rho + \frac{1}{\rho}$, we have $\rho^2 - \lambda\rho + 1 = 0$.

Lemma 2. *Suppose $B \in G(\lambda)$ is an elliptic element of finite order. Then its order is either 2 or 3 and $tr(B) = -1, 0, 1$, where $tr(B)$ is the trace of B.*

Proof : Let $B = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$. Note then that $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d(\lambda)$ are polynomials in λ with coefficients in $\mathbf{Z}$. We let a_0 and d_0 denote the constant terms of $a(\lambda)$ and $d(\lambda)$, respectively. Clearly B is diagonalizable. Let C be a matrix such that

$$A = CBC^{-1} = \begin{pmatrix} f(\lambda) & 0 \\ 0 & g(\lambda) \end{pmatrix}.$$

Now if $B^n = I$, then $A^n = I$ and hence $f(\lambda)^n = 1$ and $g(\lambda)^n = 1$. Since $det(A) = 1$, we see that $f(\lambda) = e^{2\pi i m/n}$ and $g(\lambda) = e^{-2\pi i m/n}$ for some integers m and n. On the other hand, $tr(B) = tr(A)$ implies that $a(\lambda) + d(\lambda) = e^{2\pi i m/n} + e^{-2\pi i m/n} = 2\cos(2\pi m/n)$. Since $2\cos(2\pi m/n)$ is algebraic and by assumption λ is transcendental, we see that $a(\lambda) + d(\lambda) = a_0 + d_0$ and hence $2\cos(2\pi m/n)$ is an integer. Since B is elliptic, we conclude that $2\cos(2\pi m/n) = -1, 0,$ or 1. Consequently, $n = 2$ or 3.

Corollary 1. *If A_n is elliptic of finite order, then $a_{n,0} = 0$ and so n is even and $tr(A_n) = 0$.*

Proof : By Lemma 2 $tr(A_n) = -1, 0,$ *or* 1. Thus $a_n(\lambda) + d_n(\lambda) = -1, 0,$ *or* 1 and since λ is transcendental, we have $a_0 + d_0 = -1, 0,$ *or* 1. But by Lemma $1(iii)$, $a_0 = d_0$. Since these are integers, we must have $a_0 = 0$. The remaining assertions of the corollary now follow from Lemma $1(iii)$.

Lemma 3. *If $B \in G(\lambda)$ is elliptic of finite order, then $tr(B) = 0$ and hence B is of order 2.*

Proof : There are four possibilities:
Case I: $B = A_n$. In this case, the first statement follows from Corollary 1 and the second follows from the proof of Lemma 1.
Case II: $B = TA_nT$. In this case $tr(B) = -tr(A_n) = 0$, where we have used Corollary 4.1
Case III: $B = TA_n$. Then

$$B = \begin{pmatrix} -c_n(\lambda) & -d_n(\lambda) \\ a_n(\lambda) & b_n(\lambda) \end{pmatrix},$$

and so $tr(B) = b_n(\lambda) - c_n(\lambda)$. But then by Lemma $2(i)$, $tr(B)$ is not an integer and we have a contradiction to Lemma 1.
Case IV: $B = A_nT$. This is similar to Case III.

Remark 1: It follows from these lemmas that if $B \in G(\lambda)$ is elliptic of finite order, then it is conjugate to A_n. We are now in a position to prove that an elliptic element of finite order is conjugate of T.

Theorem 1. *If $B \in G(\lambda)$ is elliptic of finite order, then it is conjugate to T.*

Proof : By Remark 1 it suffices to show that if A_n is elliptic of finite order, then it is conjugate to T. First observe that n is even. We write

$$A_n^2 = S_\lambda^{m_1} T S_\lambda^{m_2} T \cdots S_\lambda^{m_n+m_1} T S_\lambda^{m_2} T \cdots S_\lambda^{m_n} = \begin{pmatrix} e_n(\lambda) & f_n(\lambda) \\ g_n(\lambda) & h_n(\lambda) \end{pmatrix}.$$

Then by Lemma $4.2(ii)$, we have $f_n(\lambda) = (m_1 m_2 \cdots (m_n + m_1) m_2 \cdots m_n)\lambda^{2n-1} + \cdots$ (where the dots are terms of degree less than $2n-1$). Since λ is transcendental, it must be that the leading term of f_n is zero. Since each m_j is nonzero, we must have $m_1 + m_n = 0$, that is, $m_n = -m_1$. Hence A_n is conjugate to $S_\lambda^{m_2} T S_\lambda^{m_3} T \cdots S_\lambda^{m_{n-1}}$ and the stage for induction has been set.

References

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