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ON NUMBERS EQUAL TO THE SUM OF TWO SQUARES IN MORE THAN ONE WAY

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1. Introduction

Recently at Rowan University we were examining Euler's paper [1] where we found the following statement: "Since r must be the sum of two squares in two different ways, r must be the product of two factors of this same form. In other words, $r = (aa + bb)(cc + dd)$ … ". Euler gives no proof of this statement, so he probably assumed that it was familiar to his readers. Since none of us were aware of this idea, we tried to prove it, and after about two weeks found an advanced proof using the unique factorization property of Gaussian integers. This proof had the disappointing feature that it did not show how to find the numbers a, b, c, and d, but simply demonstrated their existence. Later we found a constructive proof and were surprised to see how elementary it was using only precalculus mathematics. It is the purpose of this paper to show this derivation which we hope will be of interest to serious students of mathematics at all levels.

The smallest number that can be written as the sum of two squares in two ways is $50 = 1^2 + 7^2 = 5^2 + 5^2$. So, according to Euler, we should be able to write it as the product of two factors each of which is the sum of two squares. After a little thought we see that $50 = 5 \times 10 = (1^2 + 2^2)(1^2 + 3^2)$. As a bonus, it can also be written a second way as

$50 = 2 \times 25 = (1^2 + 1^2)(3^2 + 4^2)$. As an exercise the reader should find similar

factorizations for $65 = 1^2 + 8^2 = 4^2 + 7^2$, and $85 = 2^2 + 9^2 = 6^2 + 7^2$. When the numbers

are small it is not too difficult to guess the product. However, when the number is larger,

such as $10004 = 2^2 + 100^2 = 20^2 + 98^2$, it is much harder. We will demonstrate a

systematic method for finding the factors of any appropriate number in section 3.

      While we have derived all the material in this paper ourselves, we make no claim

that it is new, as the literature on sums of two squares is extensive. Rather, it is our hope

that teachers and students will find this subject of unusual interest. Since the only

mathematics used is the Pythagorean theorem, similar triangles, and the greatest common

divisor, this material could be used as a special exploration for interested students in

precalculus and beyond.

2. The general theorem

      Throughout this paper, all letters are positive integers. We now explore the

following theorem:

Theorem: Suppose the number N can be expressed as the sum of two squares in two

ways

(1) $N = a^2 + b^2 = c^2 + d^2$.

Then N can be expressed as the product of two factors, each of which is the sum of two

squares

(2) $N = (\alpha^2 + \beta^2)(s^2 + t^2)$.

Proof: Notice the following that follow from (1) $\bullet$ (To see that these are true, consider

mod 4 and the values of squares in mod 4).

If both a and b are even, then c and d are even.

If both a and b are odd, then c and d are odd.

If a is odd and b is even, then c and d also have opposite parity. In this case, let c be odd.

In Figure 1 we see a circle of radius $\sqrt{N}$. From (1) we see that the points $A(a,b)$ and $B(c,d)$ are on this circle. Let M be the midpoint of the line AB and call its coordinates (e,f). It follows that the line OM is perpendicular to the line AB. We see that

(3) $e = \dfrac{a+c}{2}$ and $f = \dfrac{b+d}{2}$.

These show that the point M is the average of the points A and B. Notice that both e and f are integers, which follows from the three conditions on parity given above.
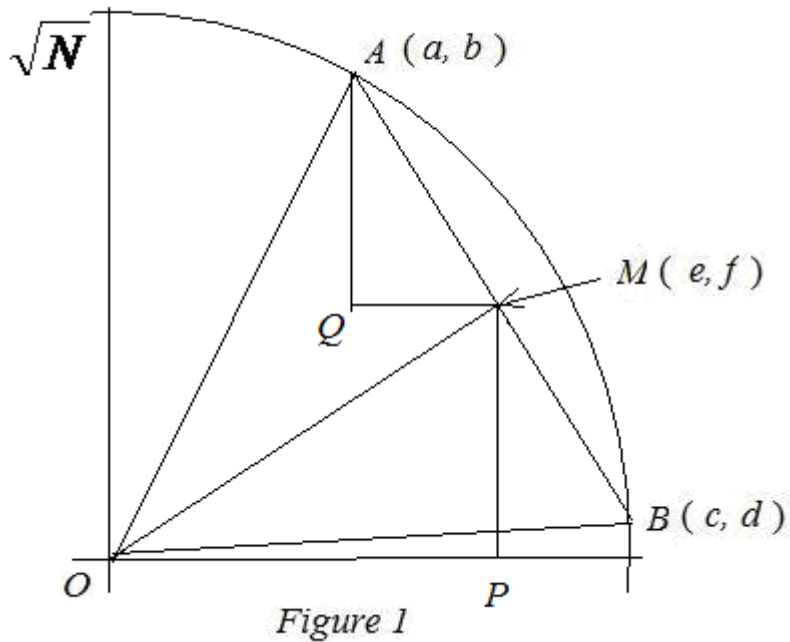
Figure 1

Call

(4) $\alpha = GCD(e, f)$,

the greatest common divisor of e and f. Then natural numbers s and t exist such that

(5) $OP = e = \alpha s$ and $PM = f = \alpha t$.

In Figure 2 we see the right triangle AMQ which is similar to the triangle OMP because the angle MOP equals the angle QAM. (Recall that AM is perpendicular to OM.) Since the sides of this triangle AMQ are integers, a natural number β exists such that

(6) $QM = e - a = \beta t$ and $AQ = b - f = \beta s$.
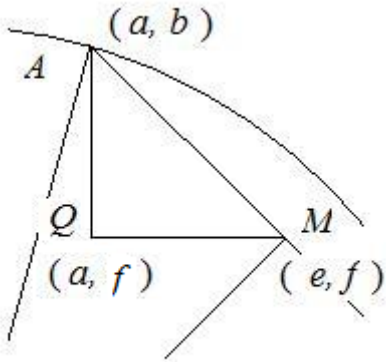
Figure 2

Notice that

(7) $\beta = GCD(e - a, b - f)$.

Finally we examine the right triangle OMA from Figure 1. The Pythagorean theorem tells us that

(8) $N = OA^2 = OM^2 + MA^2$.

But from (5) we have $OM^2 = OP^2 + PM^2 = \alpha^2 s^2 + \alpha^2 t^2$, and from (6) we have

$MA^2 = QM^2 + AQ^2 = \beta^2 t^2 + \beta^2 s^2$. Using these last two relations with (8) we get

$$N = \left(\alpha^2 s^2 + \alpha^2 t^2\right) + \left(\beta^2 s^2 + \beta^2 t^2\right) = \alpha^2\left(s^2 + t^2\right) + \beta^2\left(s^2 + t^2\right)$$

which simplifies to

(9) $N = \left(\alpha^2 + \beta^2\right)\left(s^2 + t^2\right)$.

Thus our theorem is proved.

3. Applications of the general theorem

We now show how to find the factors $N = \left(\alpha^2 + \beta^2\right)\left(s^2 + t^2\right)$ when we are given

the sums of squares $N = a^2 + b^2 = c^2 + d^2$. We do this in a streamlined fashion based on

the reasoning in the above theorem.

1. If one of the given squares is odd and one even, let a and c denote odd numbers with $a < c$.

2. Calculate the natural numbers $e = \dfrac{a+c}{2}$ and $f = \dfrac{b+d}{2}$.

3. Find $\alpha = GCD(e, f)$ and $s = \dfrac{e}{\alpha}$ and $t = \dfrac{f}{\alpha}$.

4. Find $\beta = GCD(e-a, b-f)$.

We now have $N = \left(\alpha^2 + \beta^2\right)\left(s^2 + t^2\right)$.

Example 1: Given $N = 10004 = 2^2 + 100^2 = 20^2 + 98^2$, use the above four steps to express 10004 as the product of two factors which are each the sum of two squares.

Solution: We can ignore the first part of step 1 since both a and b are even. We take $a = 2$, $b = 100$, $c = 20$ and $d = 98$. See Figure 3 where we are using the points A (2, 100) and B (20, 98) with midpoint M. From step 2 we get the midpoint M with coordinates $e = 11$ and $f = 99$. In step 3 we find $\alpha = GCD(11,99) = 11$, $s = 1$ and $t = 9$. From step 4 we get $\beta = GCD(9,1) = 1$. So we finally have

$10004 = \left(11^2 + 1^2\right)\left(1^2 + 9^2\right) = 122 \times 82$.

A (2, 100)

M
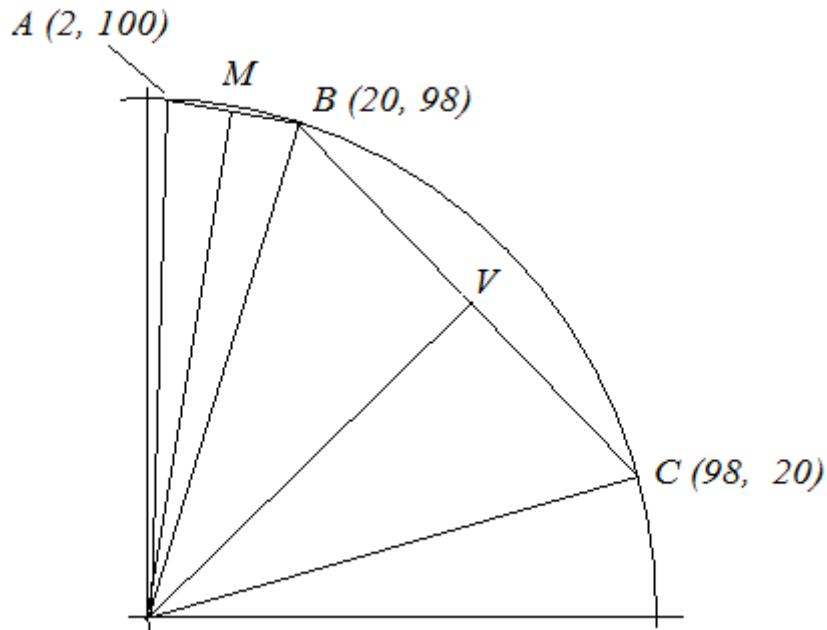
B (20, 98)

V

C (98, 20)

Figure 3

Example 2: Given $N = 10004 = 2^2 + 100^2 = 20^2 + 98^2$, the same number as in Example

1, find another factoring into sums of two squares.

Solution: Examine Figure 3 and observe that the solution obtained in Example 1 was

made by placing points *A* and *B* on the circle of radius $\sqrt{10004}$ and working with the

midpoint *M*. In this solution we use the point *C* (98, 20) by interchanging the coordinates

of point *B*. We now work with points *B* (20, 98) and *C* (98, 20). In step 1 we have

$a = 20$, $b = 98$, $c = 98$ and $d = 20$. In step 2 we calculate the midpoint of *B* and *C* at *V*

with $e = f = 59$. In step 3 we find $\alpha = GCF(59,59) = 59$ and $s = t = 1$. In step 4 we

calculate $\beta = GCF(39,39) = 39$. Thus we calculate a new product

$$10004 = \left(59^2 + 39^2\right)\left(1^2 + 1^2\right) = 5002 \times 2.$$

Example 3: Given again $N = 10004 = 2^2 + 100^2 = 20^2 + 98^2$, the same number as in

Examples 1 and 2, find yet a third factoring into sums of two squares.

Solution: Examine Figure 4 where now we use points A and C with the midpoint R. In

step 1 we take $a = 2$, $b = 100$, $c = 98$ and $d = 20$. In step 2 we find $e = 50$ and $f = 60$

as the coordinates of the midpoint R. In step 3 we find $\alpha = GCF(50.60) = 10$, and

calculate $s = 5$ and $t = 6$. In step 4 we find $\beta = GCF(48,40) = 8$. We now have a third

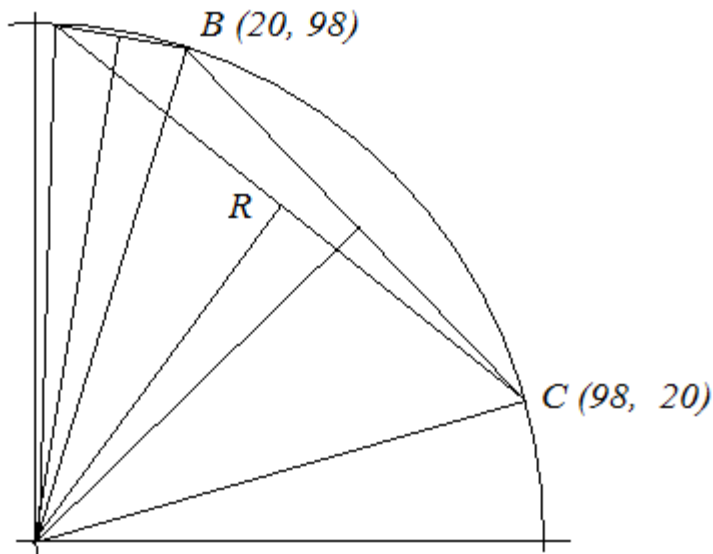factoring $10004 = (10^2 + 8^2)(5^2 + 6^2) = 164 \times 61$.

A (2, 100)

B (20, 98)

R

C (98, 20)

Figure 4

4. Data for future computations

Using a computer program, we found the numbers under 400 that can be written

as the sum of two squares in more than one way. We list them below.

$50 = 1^2 + 7^2 = 5^2 + 5^2$

$$65 = 1^2 + 8^2 = 4^2 + 7^2$$
$$85 = 2^2 + 9^2 = 6^2 + 7^2$$
$$125 = 2^2 + 11^2 = 5^2 + 10^2$$
$$130 = 3^2 + 11^2 = 7^2 + 9^2$$
$$145 = 1^2 + 12^2 = 8^2 + 9^2$$
$$170 = 1^2 + 13^2 = 7^2 + 11^2$$
$$185 = 4^2 + 13^2 = 8^2 + 11^2$$
$$200 = 2^2 + 14^2 = 10^2 + 10^2$$
$$205 = 3^2 + 14^2 = 6^2 + 13^2$$
$$221 = 5^2 + 14^2 = 10^2 + 11^2$$
$$250 = 5^2 + 15^2 = 9^2 + 13^2$$
$$260 = 2^2 + 16^2 = 8^2 + 14^2$$
$$265 = 3^2 + 16^2 = 11^2 + 12^2$$
$$290 = 1^2 + 17^2 = 11^2 + 13^2$$
$$305 = 4^2 + 17^2 = 7^2 + 16^2$$
$$325 = 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2$$
$$338 = 7^2 + 17^2 = 13^2 + 13^2$$
$$340 = 4^2 + 18^2 = 12^2 + 14^2$$
$$370 = 3^2 + 19^2 = 9^2 + 17^2$$
$$377 = 4^2 + 19^2 = 11^2 + 16^2$$

Notice that only 325 can be written three different ways in this list. By the way 425 is the second smallest number that is the sum of two squares in three ways:

$$425 = 5^2 + 20^2 = 8^2 + 19^2 = 13^2 + 16^2$$

Two additional numbers with this property are

$$10400 = 20^2 + 100^2 = 44^2 + 92^2 = 68^2 + 76^2,$$
$$11849 = 20^2 + 107^2 = 43^2 + 100^2 = 68^2 + 85^2.$$

A few numbers that can be expressed as the sum of two squares in four ways are

$$5125 = 15^2 + 70^2 = 30^2 + 65^2 = 34^2 + 63^2 = 47^2 + 54^2,$$
$$6409 = 3^2 + 80^2 = 28^2 + 75^2 = 35^2 + 72^2 = 53^2 + 60^2,$$
$$10585 = 28^2 + 99^2 = 37^2 + 96^2 = 44^2 + 93^2 = 48^2 + 91^2,$$
$$12505 = 29^2 + 108^2 = 48^2 + 101^2 = 52^2 + 99^2 = 69^2 + 88^2.$$

A number that can be expressed in six ways is

$$5525 = 7^2 + 74^2 = 14^2 + 73^2 = 22^2 + 71^2 = 25^2 + 70^2 = 41^2 + 62^2 = 50^2 + 55^2.$$

An applet to help finding the decomposition of a number into the sum of two squares is found at [3]. The reader can use the above data to examine many special cases of our theorem.

5. Final Remarks

There are many questions we could ask that the reader might enjoy exploring:

1. Many natural numbers cannot be written as the sum of two squares. These include 1, 3, 4, 6, 7, 9, 11, 12, … . Now $4 = (1^2 + 1^2)(1^2 + 1^2)$, but the other numbers we listed cannot be expressed as a product of factors each of which is the sum of two squares. When is this true for this set of numbers?

2. Many natural numbers can be written as the sum of two squares in only one way. The first few numbers in this set are 2, 5, 8, 10, 13, … Now $10 = (1^2 + 1^2)(1^2 + 2^2)$, but the other numbers we listed cannot be expressed as a product of factors each of which is the sum of two squares. When is this true for this set of numbers?

3. In this paper we considered numbers that can be expressed as the sum of two squares in two or more ways. We proved that such numbers can always be expressed, in at least one way, as the product of factors each of which is the sum of two squares. When is such a factoring possible in only one way?

Another interesting way of factoring numbers like those considered in this paper is called "Euler's Factorization Method." There is a very interesting discussion of this in [2] which includes an informative history. The webpage [4] also discusses this factorization.

References

[1] Euler, L., *Recherches ulterueures et tres curieuses sur le probléme de quatre nombres positives et en proportion arithmétique tels, que la somme de deux quelconques soit toujours un nombre carré.*

 (E797, Research into the problem of four positive numbers and such an arithmetic proportion of them, that the sum of any two of them is always a square number), on the web at the Euler Archive **http://www.math.dartmouth.edu/~euler/**.

[2] Ore, Oystein, *Number Theory and It's History*, Dover Publications, New York, 1976, pp.59-64. ISBN: 0-486-65620-9

[3] Webpage Applet for computing the sum of two squares

http://www.math.mtu.edu/mathlab/COURSES/holt/dnt/represent3.html

[4] Weisstein, Eric W. "Euler's Factorization Method." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/EulersFactorizationMethod.html