SERIES REPRESENTATION OF THE SECOND ORDER HYPERGEOMETRIC ZETA FUNCTION

HUNDUMA LEGESSE GELETA, SEID MOHAMMED, ABDUL HASSEN

Abstract. A Hypergeometric zeta function is a generalization of the Riemann zeta function via integral representation. Hassen and Nguyen in ([2])introduced these families of functions and in subsequent papers, developed many properties analogous to those satisfied by the classical zeta function. (See [2], [3], [4], [5].) They showed that these functions have Dirichlet series type representations with coefficients $\mu(n,s)$, which depend on both n and s. In this paper, we will express $\mu(n,s)$ explicitly and use this formula to write the hypergeometric zeta function of order 2 as a power series in s with Dirichlet series as coefficients. We also use this series representation to demonstrate that the second order hypergeometric zeta function has a zero free region to the right half plane.

1. Introduction

The Riemann zeta function defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

where $s=\sigma+it,\ \ \sigma>1$, admits an integral representation given by

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx. \tag{1}$$

Hassen and Nguyen in [2] introduced and investigated a generalization of the integral representation of the Riemann zeta function by replacing e^x-1 in the denominator in (1) with arbitrary Taylor difference $e^x-T_{N-1}(x)$ where N is a positive integer and $T_{N-1}(x)$ is the Taylor polynomial of e^x at the origin having degree $N-1$. This defines a family of what Hassen and Nguyen called hypergeometric zeta functions denoted by $\zeta_N(s)$:

$$\zeta_N(s)=\frac{1}{\Gamma(s+N-1)}\int_0^{\infty}\frac{x^{s+N-2}}{e^x-T_{N-1}(x)}dx \tag{2}$$

Observe that $\zeta_1(s)=\zeta(s)$. In several papers, ([2], [3]) Hassen and Nguyen established many of the properties of the classical zeta function for the hypergeometric zeta functions. However, the hypergeometric zeta functions do not appear to have a product formula. The zero of the $e^x-T_{N-1}(x)$ can be approximated but cannot be found precisely. This makes it difficult to expect a functional equation. On a right half plane, the hypergeometric zeta functions can be represented in the form:

$$\zeta_N(s)=\sum_{n=1}^{\infty}\frac{\mu_N(n,s)}{n^{s+N-1}}, \tag{3}$$

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for $\sigma > 1$, where

$$\mu_N(n,s) = \sum_{k=0}^{(N-1)(n-1)} \frac{a_k(N,n)\Gamma(s+N+k-1)}{n^k\Gamma(s+N-1)} \tag{4}$$

and $a_k(N,n)$ is generated by

$$(T_{N-1}(x))^{n-1} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{N-1}}{(N-1)!}\right)^{n-1} = \sum_{k=0}^{(N-1)(n-1)} a_k(N,n)x^k.$$

Following Riemann, it is possible, as Hassen and Nguyen demonstrated, to extend the hypergeometric zeta function to a left half plane. In fact, it has been shown that the hypergeometric zeta functions $\zeta_N(s)$ can be extended analytically to the entire complex plane, except for N simple poles at $s = 1, 0, -1, ..., 2-N$.

The coefficients $\mu_N(n,s)$ in the series representation of $\zeta_N(s)$ depend on both n and s. It would be desirable to find an expression of $\mu_N(n,s)$ that allows us to write $\zeta_N(s)$ as a linear combination of ordinary Dirichlet series. It is the objective of this paper to investigate the properties of the coefficients $\mu_2(n,s)$ and write $\zeta_2(s)$ as a "power" series in s with Dirichlet Series as coefficients. To this end, we will find an explicit form of the coefficients of these polynomials and rewrite $\zeta_2(s)$.

We note that the expression $\frac{\Gamma(s+N+k-1)}{\Gamma(s+N-1)}$ which appears in the definition of the coefficient of the hypergeometric zeta function of order N as given in(4) is just $(s+N-1)_k$:

$$\frac{\Gamma(s+N+k-1)}{\Gamma(s+N-1)} = (s+N-1)_k$$

where $(s+a)_k$ is Pochhammer symbol:

$$(s+a)_k = (s+a)(s+a+1)\cdots(s+a+k-1),$$

with initial value $(s+a)_0 = 1$. Moreover the following recursive relation holds:

$$(s+a)_{k+1} = (s+a+k)(s+a)_k.$$

To express the polynomials $\mu_2(n,s)$ explicitly, we need to define a sequence of numbers recursively. For $m = 1, 2, \cdots$, and $k = m+1, m+2, m+3, \cdots$, we define

$$A_k^m = kA_{k-1}^m + A_{k-1}^{m-1}, \tag{5}$$

with initial values

$$A_k^0 = 0, \quad A_{k-1}^k = 1.$$

Note then that

$$A_k^1 = k! \quad \text{and} \quad A_k^k = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}. \tag{6}$$

We note that the recurrence relation above is the same as that of the unsigned Stirling numbers which are denoted by $\left[\begin{array}{c} k \\ m \end{array}\right]$ and is given by

$$\left[\begin{array}{c} k \\ m \end{array}\right] = k\left[\begin{array}{c} k-1 \\ m \end{array}\right] + \left[\begin{array}{c} k-1 \\ m-1 \end{array}\right].$$

Lemma 1.1. *$(s+1)_k$ is a polynomial of degree k given as follows:*

$$(s+1)_k = s^k + A_k^k s^{k-1} + A_k^{k-1} s^{k-2} + A_k^{k-2} s^{k-3} + \cdots + A_k^2 s + A_k^1.$$

Proof. We shall use induction on k.
For $k=0$ we have $(s+1)_0 = 1 = A_0^1 s^0$, and assume the assertion is true for k:

$$(s+1)_k = s^k + A_k^k s^{k-1} + A_k^{k-1} s^{k-2} + A_k^{k-2} s^{k-3} + \cdots + A_k^2 s + A_k^1,$$

we first observe that the leading coefficient in $(s+1)_{k+1}$ is 1.
From the Pochhammer symbol since we have,

$$(s+1)_{k+1} = (s+1)_k (s+k+1)$$

the coefficient of s^{k-j} in $(s+1)_{k+1}$ is the sum of the coefficient of s^{k-j-1} in $(s+1)_k$ and $k+1$ times the coefficient of s^{k-j} in $(s+1)_k$. From the induction hypothesis the coefficient of s^{k-j-1} in $(s+1)_k$ is A_k^{k-j} and the coefficient of s^{k-j} in $(s+1)_k$ is A_k^{k+1-j}.

Therefore, the coefficient of s^{k-j} in $(s+1)_{k+1}$ is $(k+1)(A_k^{k+1-j}) + A_k^{k-j} = A_{k+1}^{k+1-j}$, as desired. □

Returning to the series representation of the hypergeometric zeta function given in (3), we note that for $N=2$ we have,

$$\zeta_2(s) = \sum_{n=1}^{\infty} \frac{\mu_2(n,s)}{n^{s+1}},$$

where

$$\mu_2(n,s) = \sum_{k=0}^{n-1} \frac{a_k(2,n)\Gamma(s+k+1)}{n^k \Gamma(s+1)}$$

and the $a_k(2,n)$ is generated by,

$$(T_1(x))^{n-1} = (1+x)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k.$$

Thus,

$$\mu_2(n,s) = \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}\Gamma(s+k+1)}{n^k \Gamma(s+1)} = \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}(s+1)_k}{n^k}. \tag{7}$$

Now we want to write these coefficients of the hypergeometric zeta function as a polynomial whose coefficient is explicit as the following lemma shows:

Lemma 1.2. *$\mu_2(n,s)$ can be written as a polynomial of degree "$n-1$" with its explicit coefficients as given below:*

$$\mu_2(n,s) = \sum_{m=1}^{n-1} b_{nm} s^{m-1}$$

where

$$b_{nm} = \sum_{j=m-1}^{n-1} n^{-j} \binom{n-1}{j} A_j^m$$

Proof. To see this we begin from the very definition of $\mu_2(n,s)$

$$\mu_2(n,s) = \sum_{k=0}^{n-1} n^{-k} \binom{n-1}{k} (s+1)_k.$$

By Lemma 1.1, the coefficients of s^{m-1} in $(s+1)_k$ is

$$\sum_{j=m-1}^{n-1} A_j^m$$

for each $m = 1, 2, \cdots$ and hence the coefficient of s^{m-1} in

$$\mu_2(n,s) = \sum_{k=0}^{n-1} n^{-k} \binom{n-1}{k} (s+1)_k$$

becomes

$$\sum_{j=m-1}^{n-1} n^{-j} \binom{n-1}{j} A_j^m$$

which is equal to b_{nm}. □

As an example we can have the first few $\mu_2(n,s)$ as follows:

$$\begin{aligned}
\mu_2(1,s) &= 1 = b_{11} \\
\mu_2(2,s) &= 1 + 2^{-1} + 2^{-1}s = b_{21} + b_{22}s \\
\mu_2(3,s) &= 1 + \binom{2}{1}3^{-1} + \binom{2}{2}2!3^{-2} + [\binom{2}{1}3^{-1} + \binom{2}{2}3^{-2}A_2^2]s + \binom{2}{2}3^{-2}s^2 \\
&= b_{31} + b_{32}s + b_{33}s^2
\end{aligned}$$

$$\begin{aligned}
\mu_2(4,s) &= 1 + \binom{3}{1}4^{-1} + \binom{3}{2}2!4^{-2} + \binom{3}{3}3!4^{-3} + \left[\binom{3}{1}4^{-1} + \binom{3}{2}4^{-2}A_2^2 + \binom{3}{3}4^{-3}A_3^2\right]s \\
&\quad + \left[\binom{3}{2}4^{-2} + \binom{3}{3}4^{-3}A_3^3\right]s^2 + \binom{3}{3}4^{-3}s^3 \\
&= b_{41} + b_{42}s + b_{43}s^2 + b_{44}s^3.
\end{aligned}$$

Observe that

$$\mu_2(4,s) = \sum_{m=1}^{3} b_{4m}s^{m-1},$$

where

$$b_{4m} = \sum_{j=m-1}^{3} 4^{-j}\binom{3}{j}A_j^m.$$

2. Series Representation

We are now in a position to state and prove our main result:

Theorem 2.1. *The hypergeometric zeta function of order 2 can be rewritten as follows*

$$\zeta_2(s) = \sum_{m=1}^{\infty} D_m(s)s^{m-1},$$

where the coefficient $D_m(s)$ is a Dirichlet series for each $m = 1, 2, 3, \cdots$.

Proof. We can rewrite $\zeta_2(s)$ as follows:

$$\begin{aligned}
\zeta_2(s) &= \sum_{n=1}^{\infty} \frac{\mu_2(n,s)}{n^{s+1}} \\
&= \mu_2(1,s) + \frac{\mu_2(2,s)}{2^{s+1}} + \frac{\mu_2(3,s)}{3^{s+1}} + \frac{\mu_2(4,s)}{4^{s+1}} + \cdots \\
&= 1 + \frac{1+2^{-1}+2^{-1}s}{2^{s+1}} + \frac{1+\binom{2}{1}3^{-1}+\binom{2}{2}2!3^{-2}+[\binom{2}{1}3^{-1}+\binom{2}{2}3^{-2}A_2^2]s+\binom{2}{2}3^{-2}s^2}{3^{s+1}} + \cdots \\
&= \frac{b_{11}}{1^{s+1}} + \frac{b_{21}+b_{22}s}{2^{s+1}} + \frac{b_{31}+b_{32}s+b_{33}s^2}{3^{s+1}} + \cdots \\
&= \frac{b_{11}}{1^{s+1}} + \frac{b_{21}}{2^{s+1}} + \frac{b_{31}}{3^{s+1}} + \cdots \\
&\quad + \left[\frac{b_{22}}{2^{s+1}} + \frac{b_{32}}{3^{s+1}} + \frac{b_{42}}{4^{s+1}} + \cdots\right] s + \cdots + \left[\frac{b_{nj}}{n^{s+1}} + \frac{b_{(n+1)j}}{(n+1)^{s+1}} + \frac{b_{(n+2)j}}{(n+2)^{s+1}} + \cdots\right] s^{j-1} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{b_{n1}}{n^{s+1}} + s\sum_{n=2}^{\infty} \frac{b_{n2}}{n^{s+1}} + s^2\sum_{n=3}^{\infty} \frac{b_{n3}}{n^{s+1}} + \cdots + s^{j-1}\sum_{n=j}^{\infty} \frac{b_{nj}}{n^{s+1}} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{n^{-1}b_{n1}}{n^{s}} + s\sum_{n=2}^{\infty} \frac{n^{-1}b_{n2}}{n^{s}} + s^2\sum_{n=3}^{\infty} \frac{n^{-1}b_{n3}}{n^{s}} + \cdots + s^{j-1}\sum_{n=j}^{\infty} \frac{n^{-1}b_{nj}}{n^{s}} + \cdots .
\end{aligned}$$

Now we put,

$$D_1(s) = \sum_{n=1}^{\infty} \frac{n^{-1}b_{n1}}{n^s},$$

$$D_2(s) = \sum_{n=2}^{\infty} \frac{n^{-1}b_{n2}}{n^s},$$

$$D_3(s) = \sum_{n=3}^{\infty} \frac{n^{-1}b_{n3}}{n^s}.$$

So in general let $D_m(s)$ be the coefficient of s^{m-1} for each $m = 1, 2, 3, \cdots$, then,

$$D_m(s) = \sum_{n=m}^{\infty} \frac{n^{-1}b_{nm}}{n^s}$$

Therefore,

$$\zeta_2(s) = \sum_{m=1}^{\infty} D_m(s)s^{m-1}.$$

This completes the proof. □

For notational convenience, let us define

$$a_{nm} = \sum_{k=m-1}^{n-1} \binom{n-1}{k} A_k^m n^{-(k+1)} = \frac{b_{nm}}{n} \tag{8}$$

for each $m = 1, 2, 3, \cdots$, so that the Dirichlet series $D_m(s)$ can be expressed as

$$D_m(s) \quad = \quad \sum_{n=m}^{\infty} \frac{\sum_{k=m-1}^{n-1} \binom{n-1}{k} A_k^m n^{-(k+1)}}{n^s} = \sum_{n=m}^{\infty} \frac{n^{-1} b_{nm}}{n^s} = \sum_{n=m}^{\infty} \frac{a_{nm}}{n^s}, \tag{9}$$

where the A_k^m are given by (5).

We observe that the first few Dirichlet series given in (9) are

$$D_1(s) = \sum_{n=1}^{\infty} \frac{a_{n1}}{n^s}$$

$$D_2(s) = \sum_{n=2}^{\infty} \frac{a_{n2}}{n^s}$$

$$D_3(s) = \sum_{n=3}^{\infty} \frac{a_{n3}}{n^s}$$

It is also interesting to list some few coefficients a_{nm} and look at what they represent as a remark.

Remark 2.1. The first few values of a_{nm} are given by

$$a_{11} = 1, \quad a_{21} = \tfrac{3}{4}, \quad a_{31} = \tfrac{17}{27}, \quad a_{41} = \tfrac{142}{256}$$

$$a_{22} = \tfrac{1}{4}, \quad a_{32} = \tfrac{1}{3}, \quad a_{42} = \tfrac{95}{256}, \quad a_{52} = \tfrac{1220}{3125}$$

$$a_{33} = \tfrac{1}{27}, \quad a_{43} = \tfrac{18}{256}, \quad a_{53} = \tfrac{305}{3125}$$

We note that

$$a_{n1} = \sum_{k=0}^{n-1} \binom{n-1}{k} A_k^1 n^{-(k+1)} = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k!}{n^{k+1}}, \tag{10}$$

where we have used (6) in the last equality. It is interesting to note that a_{n1} has a closed form given by

$$a_{n1} = \frac{e^n \Gamma[n, n]}{n^n},$$

where $\Gamma[a, z]$ is the upper incomplete gamma functions. This can proven by mathematical induction. The number a_{n1} is the probability of selecting a ball from an urn containing n different balls, with replacement until exactly one ball has been selected twice and that ball was also the first ball selected once. Further more the sequence $\{n^n a_{n1}\}$ begins as

$$1, 3, 17, 142, 1569, 21576, 355081, 6805296, 148869153, 3660215680, \ 99920609601, \cdots$$

and is listed as $A001865$ in the On-Line Encyclopedia of Integer Sequences (OEIS) [10]. This sequence represents the number of connected functions on n-labeled nodes as indicated in OEIS. We also mention here that the sequence $\{n^n a_{n2}\}$ appears as $A065456$ on OEIS([11]), and it is the number of functions on n-labeled nodes whose representation as a digraph has two components. However, we have not seen a list that corresponds to other sequences $\{a_{nm}\}$ for $m \geq 3$ we have here. We will explore this in a future work. Finally, if we define $a_{nm} = 0$ for $n < m$, then the Dirichlet series given in (9) is an ordinary Dirichlet series:

$$D_m(s) = \sum_{n=1}^{\infty} \frac{a_{nm}}{n^s}.$$

We do not know if these Dirichlet series have functional equations. We also note that each of the coefficients has the following relations on the real line:

$$\cdots, D_3(\sigma) < D_2(\sigma) < D_1(\sigma) < \zeta(\sigma) < \zeta_2(\sigma).$$

Moreover,

$$D_2(\sigma) < \frac{1}{4}\zeta(\sigma)$$
$$D_3(\sigma) < \frac{1}{27}\zeta(\sigma)$$
$$D_4(\sigma) < \frac{1}{256}\zeta(\sigma)$$

3. Zero Free Region on the Right Half Plane

Zero free regions of the hypergeometric zeta functions of order 2 and order 3 on the left half plane was established by Hassen and Nguyen in [5]. In this section we will establish a zero free region for $\zeta_2(s)$ in the right half plane. In the case of the classical Riemann Zeta function, the Euler product formula

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

can be used to conclude that it is zero free for $\sigma > 1$(see [1],[7], [8] and [9]). The hypergeometric zeta functions are not known to have such a product formula. Furthermore, due to lack of knowledge of the precise locations of the zeros of $e^z - 1 - z = 0$, we do not have a functional equation. However one can use the Cauchy theory to express $\zeta_2(s)$ in terms of a series that involves the roots of $e^z - 1 - z = 0$ and establish zero free region on the left half plane. (See [5] for details.)

It follows from Theorem 2.1 that the second order hypergeometric zeta function has no real zeros if $\sigma > 1$ where $s = \sigma + it$. The following result extends this domain to $\sigma > 0$:

Theorem 3.1. *$\zeta_2(s) \neq 0$ for $s = \sigma > 0$*

Proof. As remarked above, we need only to show that $\zeta_2(\sigma) \neq 0$ for $0 < \sigma < 1$. For this we note that when $N = 2$, the integral representation in (2) becomes

$$\zeta_2(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{x^s}{e^x - 1 - x} dx \tag{11}$$

and this can be rewritten as

$$\Gamma(s+1)\zeta_2(s) = \int_0^1 \frac{x^s}{e^x - 1 - x}\, dx + \int_1^\infty \frac{x^s}{e^x - 1 - x}\, dx \tag{12}$$

$$= \int_0^1 \left(\frac{1}{e^x - 1 - x} - \frac{2}{x^2}\right) x^s\, dx + \frac{2}{s-1} + \int_1^\infty \frac{x^s}{e^x - 1 - x}\, dx. \tag{13}$$

The last formula in (13) is analytic in the strip $0 < \sigma \leq 1$, except for the pole at $s = 1$, since both integrals on the right hand side are convergent on this domain. Moreover, for $0 < \sigma < 1$,

$$\frac{1}{s-1} = -\int_1^\infty \frac{x^s}{x^2} dx.$$

Therefore, we can rewrite

$$\zeta_2(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \left(\frac{1}{e^x - 1 - x} - \frac{2}{x^2}\right) x^s\, dx.$$

Since $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$, we see that

$$\left(\frac{1}{e^x - 1 - x} - \frac{2}{x^2}\right) x^{\sigma} \ < 0$$

for $\sigma > 0$ and all $x > 0$.
So the result follows. ∎

Now we show it has zero free regions to the right half plane by showing that the limit of $\zeta_2(s)$ as the real part tends to infinity converges to 1 uniformly with some restrictions on the imaginary part. This is the content of the following theorem:

Theorem 3.2. *If $F(s) = \zeta_2(s) - D_1(s)$, where $D_1(s)$ is given by (9), then*

$$\lim_{\sigma\to\infty} |F(s)| = 0$$

uniformly in t, where $s = \sigma + it$ satisfies the inequality $|s| < C\sigma$, for some constant C.

Proof. With

$$a_{nm} = \sum_{k=m-1}^{n-1} \binom{n-1}{k} A_k^m n^{-(k+1)}$$

we have,

$$D_m(s) = \sum_{n=m}^{\infty} \frac{a_{nm}}{n^s}.$$

We now use triangle inequality to obtain

$$\begin{aligned}
|F(s)| &= |\zeta_2(s) - D_1(s)| \\
&= \left|D_2(s)s + D_3(s)s^2 + D_4(s)s^3 + \cdots\right| \\
&\leq |D_2(s)s| + \left|D_3(s)s^2\right| + \left|D_4(s)s^3\right| + \cdots \\
&\leq \left|\frac{a_{22}}{2^s} + \frac{a_{32}}{3^s} + \frac{a_{42}}{4^s} + \cdots\right| |s| + \left|\frac{a_{33}}{3^s} + \frac{a_{43}}{4^s} + \frac{a_{53}}{5^s} + \cdots\right| |s|^2 + \cdots \\
&\leq \left(\frac{a_{22}}{2^\sigma} + \frac{a_{32}}{3^\sigma} + \frac{a_{42}}{4^\sigma} + \cdots\right) |s| + \left(\frac{a_{33}}{3^\sigma} + \frac{a_{43}}{4^\sigma} + \frac{a_{53}}{5^\sigma} + \cdots\right) |s|^2 + \cdots \\
&< \frac{C\sigma}{2^\sigma}\left(a_{22} + \frac{2^\sigma}{3^\sigma} a_{32} + \cdots\right) + \frac{(C\sigma)^2}{3^\sigma}\left(a_{33} + \frac{3^\sigma}{4^\sigma} a_{43} + \cdots\right) + \cdots .
\end{aligned}$$

From this we get that,

$$\lim_{\sigma\to\infty} |F(s)| \leq 0.$$

Hence,

$$\lim_{\sigma\to\infty} F(s) = 0.$$

∎

We also note that the constant C is larger than 1.

Corollary 3.1. *Within the restriction given in Theorem 3.2,*

$$\lim_{\sigma\to\infty} |\zeta_2(s)| = 1.$$

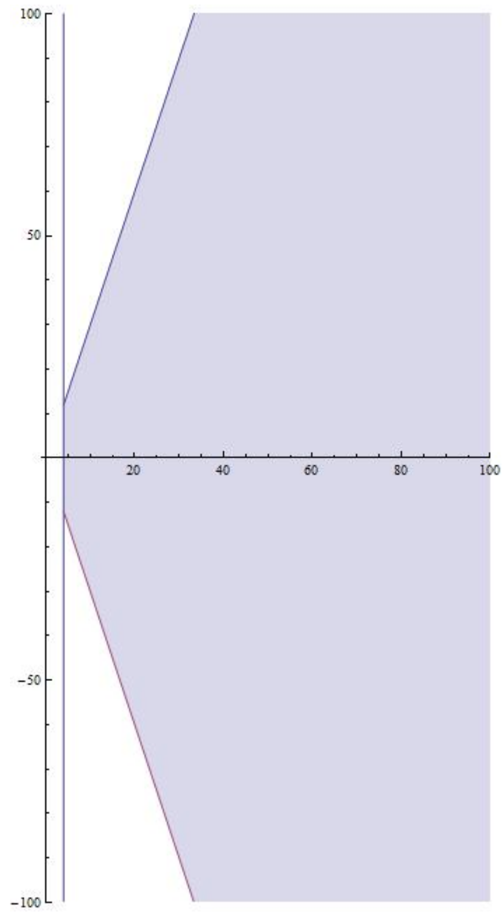

FIGURE 1. A Zero Free Region in the Right Half Plane

Proof. Since $\lim_{\sigma\to\infty}|F(s)|=0$ and $\lim_{\sigma\to\infty}|D(s)|=1$, we have

$$\begin{aligned}|D_1(s)| &= |D_1(s)-\zeta_2(s)+\zeta_2(s)|\\ &\le |D_1(s)-\zeta_2(s)|+|\zeta_2(s)|\end{aligned}$$

Hence

$$\begin{aligned}1=\lim_{\sigma\to\infty}|D_1(s)| &\le \lim_{\sigma\to\infty}|D_1(s)-\zeta_2(s)|+\lim_{\sigma\to\infty}|\zeta_2(s)|\\ &\le \lim_{\sigma\to\infty}|\zeta_2(s)|\end{aligned}$$

On the other hand,

$$\begin{aligned} |\zeta_2(s)| &= |D_1(s) - D_1(s) + \zeta_2(s)| \\ &\leq |D_1(s) - \zeta_2(s)| + |D_1(s)|. \end{aligned}$$

Thus we have,

$$\begin{aligned} \lim_{\sigma\to\infty} |\zeta_2(s)| &\leq \lim_{\sigma\to\infty} |D_1(s) - \zeta_2(s)| + \lim_{\sigma\to\infty} |D_1(s)| \\ &\leq \lim_{\sigma\to\infty} |\zeta_2(s)| \leq 1. \end{aligned}$$

Therefore, we have,

$$\lim_{\sigma\to\infty} |\zeta_2(s)| = 1.$$

□

We note that the condition $|s| < C\sigma$ in the the above theorem can be strengthened to $|s| < \sigma^\gamma$ for any $\gamma > 1$. This zero free region is shown roughly as in the figure. We conjecture, based on numerical evidence, that there is a $\sigma_0 > 1$ such that $\zeta_2(s) \neq 0$ for all s for which $\sigma > \sigma_0$. We will return to this conjecture in the future work. We also expect similar result for other hypergeometric zeta function of order N.

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Addis Ababa University Department of Mathematics, Rowan University
E-mail address: hunduma2006@ymail.com, seid-mt@yahoo.com, hassen@rowan.edu