TRIANGLES ON THE LATTICE OF INTEGERS

Andrew Roibal and Abdulkadir Hassen
Department of Mathematics
Rowan University
Glassboro, NJ 08028

I. Introduction

In this article we will be studying triangles whose vertices in the plane are integers with the property that their sides and area are integers. These triangles are called Heron triangles, named after Heron of Alexandria who is credited with the formula that relates perimeter and area of a triangle. (See Section II below.) For example, the triangle with vertices at $(0,0)$, $(3,0)$ and $(3,4)$ has side lengths 3, 4, 5 and area 6. This particular triangle is also a very well-known Pythagorean Triangle. The relationship between Heron and Pythagorean triangles will be explored in the following sections. Construction of Heron triangles will also be examined, along with divisibility of the sides and area. These triangles have been studied by many authors. See [1] and the references therein for further reading. We do not assume any more than high school algebra and geometry in our discussion.

II. Heron's Formula

Theorem 1. Let a, b, and c represent sides of a triangle and s represent the half perimeter, $s=\frac{a+b+c}{2}$. Then the area A of the triangle is

$$A=\sqrt{s(s-a)(s-b)(s-c)}.$$

This is known as Heron's Formula. It can be proven using the Pythagorean Theorem.

Proof: Let ABC be a triangle with sides of arbitrary lengths a, b, and c. Let altitude h be the height of this triangle such that it divides b into two segments m and n (see *Figure 1*).

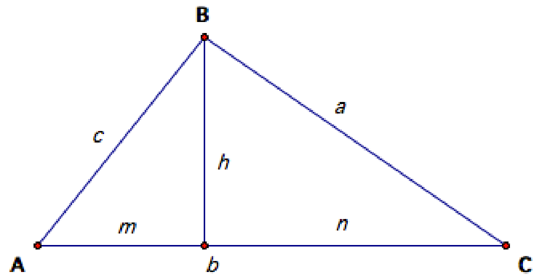

Figure 1*: Triangle ABC with Height *h

It follows from the Pythagorean Theorem that $a^2 = h^2 + n^2$ and $c^2 = h^2 + m^2$.

Subtracting these two equations, we obtain $a^2 - c^2 = n^2 - m^2$. Dividing by $b = n + m$ we have $\frac{a^2 - c^2}{b} = n - m$. Now adding $b = n + m$ gives us $\frac{a^2 - c^2 + b^2}{2b} = n$.

Now we can use $A = bh/2$ and $h = \sqrt{a^2 - n^2}$ to find the area of ABC. Substituting n and h into A, we have

$$A = \frac{1}{2}\sqrt{(ab)^2 - \left(\frac{a^2 - c^2 + b^2}{2}\right)}$$
$$4A = \sqrt{(2ab)^2 - \left(a^2 - c^2 + b^2\right)^2}.$$

Squaring both sides of the equation and factoring the right gives us

$$\begin{aligned} 16A^2 &= \left(2ab - a^2 + c^2 - b^2\right)\left(2ab + a^2 - c^2 + b^2\right) \\ &= \left[c^2 - (a-b)^2\right]\left[(a+b)^2 - c^2\right] \\ &= (c - a + b)(c + a - b)(a + b - c)(a + b + c). \end{aligned} \tag{2.1}$$

Substituting $2s = a + b + c$, we obtain

$$\begin{aligned} 16A^2 &= \left[2(s-a)\right]\left[2(s-b)\right]\left[2(s-c)\right][2s] \\ A^2 &= s(s-a)(s-b)(s-c) \\ A &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned} \tag{2.2}$$

This formula will be used throughout the paper.

III. Lattice Triangles

As previously stated, lattice triangles are those which can be drawn with vertices at integral coordinates. An example is the triangle with sides 5, 5, 6 (see *Figure 2*). This example happens to be a Heron triangle since it has integral length sides and an area of 12. (Notice that the vertical altitude splits the triangle into two 3, 4, 5 triangles.)

Not all lattice triangles are Heron. The triangle with sides $2\sqrt{2}$, $\sqrt{13}$, $\sqrt{17}$ can also be drawn on the lattice (see *Figure 2*). Although this triangle's area is 5, it is not Heronian since its sides do not have integer lengths.

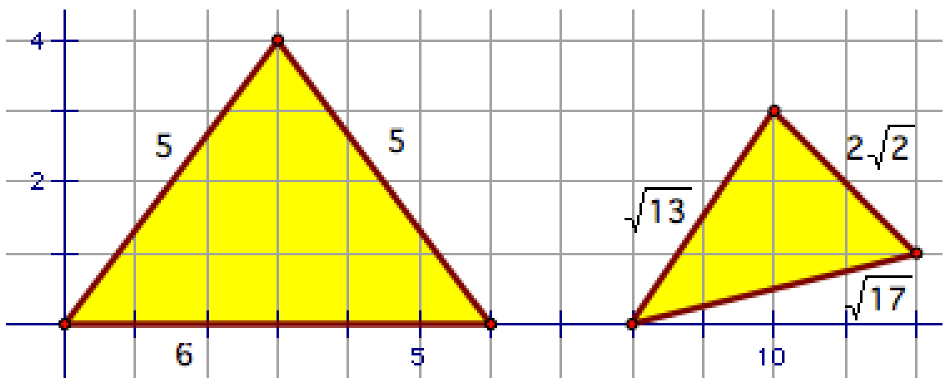

***Figure 2*: Triangles on the Integral Lattice**

An example of a triangle that cannot be drawn on the lattice is the triangle with sides 12, 13, 20 and an area of $\frac{15\sqrt{399}}{4}$ (see *Figure 3*). Another triangle that cannot be drawn on the lattice is the triangle with sides 15, 17, 29 and an area of $\frac{9\sqrt{1891}}{4}$.

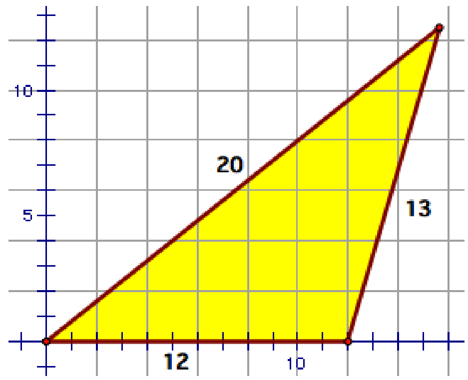

***Figure 3*: Triangle not Represented on the Integral Lattice**

An easy way to see that these triangles cannot be represented on the lattice is by using the Pythagorean Theorem. Let us examine the triangle with sides of length 12, 13, 20. We can see that 12^2 cannot be represented as a sum of two squares. This means that a side of length 12 must be horizontal or vertical to be drawn on the lattice. Hence, there is no arrangement of the other two sides such that the 12, 13, 20 triangle can be drawn on the lattice.

This method can also be used to find Heron triangles on the integral lattice. Let us observe the three cases of Heron triangles on the lattice of integers. The first is a triangle with one slanted (non-vertical and non-horizontal) side. (Note that this is a right angle triangle.) The second is a triangle with two slanted sides and the third is a triangle with three slanted sides (see *Figures 4 – 6*). It can be observed in the diagrams that every case contains a Pythagorean triangle in some form.

The following are examples of Heron triangles with one slanted side:

3, 4, 5 (A = 6)

5, 12, 13 (A = 30)

6, 8, 10 (A = 24)

8, 15, 17 (A = 60)

9, 12, 15 (A = 54)

12, 16, 20 (A = 96)

7, 24, 25 (A = 84)

The following are examples of Heron triangles with two slanted sides:

6, 5, 5 (A = 12)

8, 5, 5 (A = 12)

14, 13, 15 (A = 84)

21, 10, 17 (A = 84)

9, 10, 17 (A = 36)

11, 13, 20 (A = 66)

7, 15, 20 (A = 42)

The following are examples of Heron triangles with three slanted sides:

15, 20, 25 (A = 150)

5, 29, 30 (A = 72)

10, 35, 39 (A = 168)

13, 40, 45 (A = 252)

15, 34, 35 (A = 252)

17, 40, 41 (A = 336)

25, 34, 39 (A = 420)

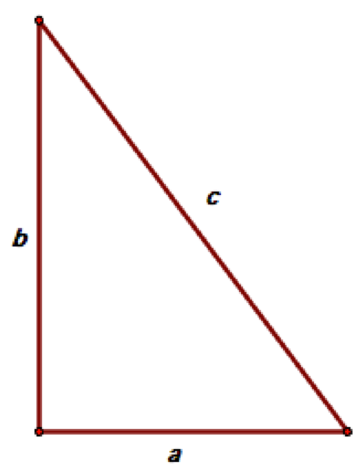

***Figure 4*: Lattice Triangle with One Slanted Side**

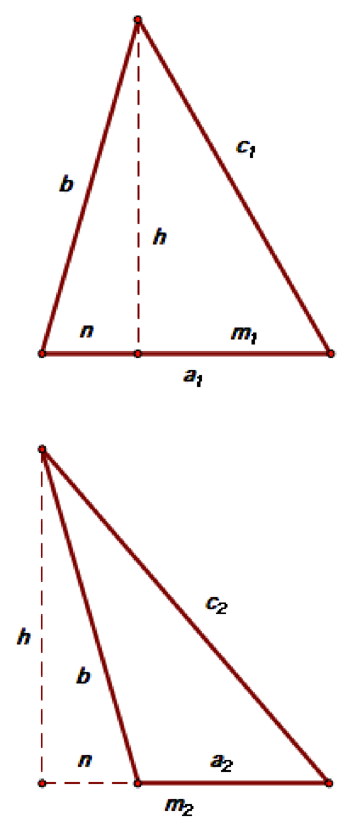

***Figure 5*: Lattice Triangles with Two Slanted Sides**

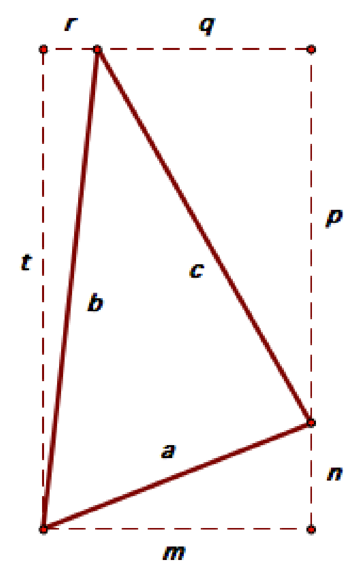

***Figure 6*: Lattice Triangle with Three Slanted Sides**

The triangle represented in *Figure 4* is a Pythagorean triangle. In fact, every Heron triangle that can be drawn on the lattice with one slanted side is a Pythagorean triangle. We will prove this in the next section. The triangles represented in *Figure 5* are constructed using two Pythagorean triangles that share at least one congruent side. In the first case, we can see that the two Pythagorean triangles have sides of length b, n, h and c_1, m_1, h (the shared side being h). In the second case, we can see that the construction is similar except the triangles are oriented so that they create an obtuse triangle rather than an acute. This time, the two Pythagorean triangles have sides of length b, n, h and c_2, m_2, h (the shared side being h). The triangle represented in *Figure 6* is constructed using three Pythagorean triangles. We can observe that the sum of the heights of two Pythagorean triangles has to be equivalent to the height of the third. The case is similar for the bases. In *Figure 6*, we can see that we have $m = r + q$ and $t = p + n$.

IV. Properties of Heron Triangles

Theorem 2. All Pythagorean triangles are Heronian.

Proof: This is easily realized since the definition of a Pythagorean triangle guarantees the sides to be of integral length. All that is left to be shown is that the area of a Pythagorean triangle is an integer value. We will prove the following lemma.

Lemma 1. If triangle ABC is Pythagorean, then sides a, b, and c can be represented as $m^2 - n^2$, $2mn$, and $m^2 + n^2$, respectively. Furthermore, all Pythagorean triangles have integral area.

Proof: Let Pythagorean triangle ABC be as in *Figure 4*. We start with the fact that $a^2 + b^2 = c^2$. By rearranging this equivalency and factoring we obtain $\frac{b}{(c-a)} = \frac{(c+a)}{b}$. Realizing that the numerator and denominator of the left and right sides are integers since a, b, and c are all integers, we can let $\frac{b}{(c-a)} = \frac{m}{n}$ such that $\frac{m}{n}$ is a reduced rational number. Note that also

$\frac{(c+a)}{b} = \frac{m}{n}$. We can now solve these equations for $\frac{c}{b}$ and $\frac{a}{b}$.

$$\frac{c}{b} = \frac{m^2+n^2}{2mn} \text{ and } \frac{a}{b} = \frac{m^2-n^2}{2mn}.$$

Because $\frac{m}{n}$ is a reduced fraction, we can equate numerators and denominators and obtain

$$a = m^2 - n^2,\ b = 2mn,\ c = m^2 + n^2.$$

Now taking the area of ABC to be $\frac{ab}{2} = \frac{(m^2-n^2)(2mn)}{2} = (m^2-n^2)(mn)$, we see that the area of a Pythagorean triangle is always an integer. This completes the proof of *Lemma* 1. *Theorem 2* follows from this result. Note this theorem will allow us to construct Heron Triangles through the use of Pythagorean Triangles.

Theorem 3. The length of at least one side of a Heron triangle is an even integer.

Proof: This can be shown using Heron's Formula and modular arithmetic. We rearrange equation (2.1), which we restate as

$$16A^2 = (-a+b+c)(a-b+c)(a+b-c)(a+b+c) \tag{4.1}$$

for easier reference. This equation implies that the right hand side is even. Now consider the case of no sides being even. Odd integers can be written as 1 (mod 2) as they all have a remainder of 1 when divided by 2. So we have $a \equiv 1 \pmod 2$, $b \equiv 1 \pmod 2$, and $c \equiv 1 \pmod 2$. Hence, rewriting the left hand side of equation (4.1), we have

$$16A^2 \equiv (-1+1+1)(1-1+1)(1+1-1)(1+1+1) \equiv (1)(1)(1)(1) \equiv 1 \pmod 2$$

which is a contradiction. Hence, Heron triangles must have at least one even length side.

Theorem 4. The area of a Heron triangle is divisible by 6.

Proof: Here, we can use techniques similar to those in the proof of *Theorem 3*. It suffices to show that the areas of primitive Heron triangles are divisible by 2 and 3. Primitive Heron triangles have sides a, b, c such that g.c.d. $(a, b, c) = 1$. Hence, by *Theorem 3*, we know one side must be even and the other two odd. Let $a = 2k$, $b = 2m + 1$, $c = 2n + 1$. A rearrangement of equation (4.1) yields

$$16A^2 = -\left(a^4 + b^4 + c^4\right) + 2\left(a^2b^2 + a^2c^2 + b^2c^2\right) \tag{4.2}$$

Now simplification modulo 2 and modulo 3 yields the desired result.

Remark: Not every Heron triangle has an integral altitude.

Proof: The easiest way to show this is with a counterexample. Let us examine the Heron triangle with sides 5, 29, 30. First, we recall that an altitude is a line extended from any vertex of a triangle to the opposite side at a perpendicular angle. Using some basic arithmetic and geometry, we can see that the altitudes of this triangle are 4.8, 28.8, and 4.97. These are clearly not integral values.

It is useful to note that many Heron triangles do, in fact, have integral altitudes. All Heron triangles with one or two slanted sides possess this quality. Referring back to *Figure 4* and *Figure 5*, we can see that the heights of the triangles are integral altitudes. This does, however, provoke the following question: *Do any Heron triangles with three slanted sides have integral altitudes?* The answer is yes.

The triangle with sides 15, 20, 25 has an integral altitude of length 12. This implies that the triangle can also be drawn with only two slanted sides. Here, the height would be the altitude, guaranteeing an integer coordinate. Interestingly enough, this triangle can also be drawn with one slanted side. 15, 20, 25 is a Pythagorean triple, hence represents a Pythagorean triangle with corresponding lengths.

Theorem 5. Every Heron triangle can be drawn on the integer lattice.

Proof: We only need to concern ourselves with primitive Heron triangles. Let the sides of the triangle be represented by *a*, *b*, *c* such that g.c.d. (*a*, *b*, *c*) = 1. As in the proof of *Theorem 4*, we will let *c* be odd and *a*, *b* of different parity. Expanding equation (4.2) yields

$$(4A)^2 = -a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 . \tag{4.3}$$

This is a basic quartic equation, which can be solved by making some substitutions. Let $x = a^2$, $y = b^2$, $z = c^2$ and $w = 2A$. Thus, equation (4.3) becomes

$$(2w)^2 = -x^2 - y^2 - z^2 + 2xy + 2xz + 2yz$$

which can be reorganized into a more familiar quadratic equation for *z*

$$z^2 - 2(x+y)z + (x-y)^2 + (2w)^2 = 0 . \tag{4.4}$$

Using the quadratic formula, we can see that equation (4.4) has integer solutions if and only if the discriminant is the square of an even integer 2*d*. In other words,

$$(x+y)^2 - (x-y)^2 - (2w)^2 = (2d)^2 . \tag{4.5}$$

As Paul Yiu explains in [1], the solutions to (4.5) are $x = m^2 + n^2$, $y = r^2 + t^2$, $d = mr + nt$, and $w = mt - nr$ for integers *m*, *n*, *r*, *t*. This implies that (4.4) has integer roots

$$z = (x+y) \pm 2d = (m \pm r)^2 (n \pm t)^2 .$$

Hence,

$$a^2 = m^2 + n^2 \quad b^2 = r^2 + t^2 \qquad c^2 = (m \pm r)^2 (n \pm t)^2 .$$

It follows that the Heron triangle *a*, *b*, *c* can be represented with vertices at the origin and at points (*m*, *n*) and (*r*, *t*). *Figure 6* illustrates this.

References

[1] Yiu, Paul. "Heronian Triangles Are Lattice Triangles." *The American Mathematical Monthly* 108.3 (2001): 261-63.