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A TABLE OF THE PARTITION FUNCTION

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1. INTRODUCTION

A ***partition*** of a positive number n is a representation of this number as a sum of natural numbers, called parts or summands. The order of the summands is irrelevant. For example, 4+2+1 is a partition of the number 7. Since order is irrelevant, 4+2+1 is the same partition as 2+4+1. The number of unrestricted partitions of the positive integer n is denoted by $p(n)$. For example, the partitions of 5 are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1. Thus, $p(5) = 7$. The reader can easily verify that $p(1)=1$, $p(2)=2$, $p(3)=3$, $p(4)=5$, $p(6)=11$, and $p(7)=15$.

While it is simple to determine $p(n)$ for small numbers by actually counting all the partitions, this becomes difficult as the numbers grow. As George Andrews put it: *"Actual enumeration of the 3, 972, 999, 029, 388 partitions of 200 would certainly take more than a lifetime."* ([1], page 150).

The theory of partitions is a very active area of modern mathematical research. The authors searched ten on line electronic journals available in JSTOR (Journal STORage on the world wide web) and found 2213 papers which discussed partitions! However, most textbooks on number theory do not discuss partitions. Two of the exceptions are the excellent book of Andrews [1] and the bible of number theory by Hardy and Wright [5].

In [7], the first two authors used the following remarkable recursive algorithm to show how one can create a table of $p(n)$ with a BASIC program for large values of n. We also gave a heuristic explanation behind the proof of this formula.

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) \\ &\quad + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \ldots \end{aligned} \tag{1.1}$$

(The progression of numbers 1, 2, 5, 7, 12, 15, 22, 26, … is related to the pentagonal numbers. Space does not permit us to give details here, but they can be found in [7].)

Most relations in the theory of partitions are not directly accessible. Euler initiated a beautiful theory of these in 1748 [3] using generating functions. The interplay between the discrete and the continuous branches of mathematics contains a wonderland of amazing relations. For the reader unfamiliar with Euler's Pentagonal Number Theorem or Jacobi's Triple Product, [1] has many delightful gems to sample.

It is the purpose of this article to show how $p(n)$ and certain related functions can be calculated. This is a story about building a table of numbers in the spirit of Pascal's triangle. We will use a recursion relation that is easy to understand and easy to implement, by hand or by computer. Dickson, in his extensive history of number theory [2], seems to indicate that this method was known to Euler [3].

In Section 2, we will look at some restricted partitions and some relations between them. In Section 3, we will show how to create a table of partitions by hand. Finally, we will give a BASIC program that generates a table of partitions. Since we have given such a table in [7], we will not do so here.

2. RESTRICTED PARTITIONS AND A RECURSIVE RELATION

The partition function $p(n)$ is sometimes called the *unrestricted partition function* of n. Besides the partition function $p(n)$, we also consider partitions formed from numbers from some restricted set. For example, let $q(n)$ denotes the number of partitions of the number n where all parts are distinct. The partitions of the number 7 into distinct parts are 7, 6+1, 5+2, 4+3 and 4+2+1. Thus $q(7) = 5$. Other examples of restricted partitions are the following:

- *p(m, n)= the number of partitions of n in which no part is larger than m*
- $q(m,n)$ = *the number of partitions of n in which at most m parts appear*
- *d(m, n) = the number of partitions of n into m distinct parts*
- *D(m, n) = the number of partitions of n into distinct parts in which no part is greater than m*
- $h(m,n)$ = *the number of partitions of n whose least part is m*
- *o(n)=the number of partitions of in which all parts are odd*
- *e(n)= the number of partitions of n in which all parts are even*
- *p(S, n) = the number of partitions of n using summands in a set S.*

Note that the above notation is not standard. For example, many authors use the notation $p_m(n)$ for our function $p(m,n)$. We challenge the reader to find the values of each of these functions for *n*=1, 2, 3, …,6 and for different values of *m.* For relationships between these partition functions and similar ones, we advise the interested reader to read G. Andrews [1] and Hardy and Wright [5].

The relation (1.1) is very good for calculating $p(n)$, but not easy to derive. We will use the restricted partition function *p(m, n)* to compute *p(n).* Let us consider an example. The partitions of the number 7 using parts not exceeding 2 are 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, and 1+1+1+1+1+1+1. Thus *p*(2,7) = 4.
The partition function *p(m, n)* has a simple relation given by:

Theorem

(1) $$p(m,n) = p(m-1,n) + p(m,n-m).$$

Proof:

Imagine counting the partitions of *n* whose parts do not exceed *m*, namely $p(m,n)$. First we count all the partitions of *n* whose parts do not exceed *m*-1. This is

$p(m-1,n)$, the first term on the right side of (1). It remains to count all the partitions of n using the number m at least once. All of these partitions are of the form

(2) $m+x=n$,

where x is a sum of parts that do not exceed m. Looking only at x, we see that the number of partitions of the form (2) is $p(m,x)$. But (2) tells us that $x=n-m$, so $p(m,x)=p(m,n-m)$, which is the second term on the right hand side of (1). This completes our proof.

To see how we obtain $p(n)$ from (1), we need only notice that

$$p(n)=p(n,n)=p(n+1,n)=p(n+2,n)=\ldots\ .$$

For example, the number of partitions of 5 using parts not exceeding 7, *p(7,5),* must be the same as the partitions of 5 using parts not exceeding 5, which is $p(5,5)=p(5)$.

3. CREATING A TABLE OF THE PARTITION FUNCTION

We now show how to use (1) for hand calculation of the partition function. Here are the steps:

Step 1. Create a table labeling the rows n = 0, 1, 2, 3, ... and the columns m = 1, 2, 3, ...(See the Table below. Notice that we use the *non-standard* notation in which m counts the columns and n counts the rows.)

Step 2. Fill the first row with one's. (We need to define $p(m,0) = 1$.) You can also complete a few more rows easily using the definition of $p(m, n)$.

Step 3. Next we fill the in the remaining rows one at a time using relation (1).

To see how this works, suppose we have completed all the rows down to $n = 9$ and are now working on row $n = 10$. We fill in the cells in this row from left to right. Relation (1) tells us that the number needed in the cell (m,n) is the number in the cell to the left plus the number directly above on the shaded diagonal. For example, the number in the cell where $n = 10$ and $m = 6$, $p(6,10)=35$, is the sum of the number to its left, $p(6-1,10) = 30$, plus the number above it on the shaded diagonal $p(6,10-6) = 5$.

Table: Calculation of *p*(*m,n*) using relation (2.1)

n \ *m*	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	*1*	1
1	1	1	1	1	1	1	1	1	*1*	1	1
2	1	2	2	2	2	2	2	*2*	2	2	2
3	1	2	3	3	3	3	*3*	3	3	3	3
4	1	3	4	5	5	**5**	5	5	5	5	5
5	1	3	5	6	*7*	7	7	7	7	7	7
6	1	4	7	*9*	10	11	11	11	11	11	11
7	1	4	*8*	11	13	14	15	15	15	15	15
8	1	*5*	10	15	18	20	21	22	22	22	22
9	*1*	5	12	18	23	26	28	29	30	30	30
10	1	6	14	23	**30**	**35**	38	40	41	42	42
11	1	6	16	27	37	44	49	52	54	55	56

Notice that, on each row, from $m = n$ all the values of $p(m,n)$ are constant and equal $p(n)$. Thus the values in the last column, where $n = 11$, give the correct values of $p(n)$ for $n = 1, 2, \ldots 11$. Having shown how to calculate a partition function by hand, we give a simple BASIC program to do the same thing in the next section.

4. A COMPUTER PROGRAM TO COUNT PARTITIONS

The following computer program calculates the partition function using (1). The reader familiar with BASIC will have no difficulty seeing how it works.

In line 120 the variable MAXN is the number of rows and columns in our table. In line 200 we fill the first row of the table with the number one as we did in the hand-calculation in the previous section. Each time the loop from lines 300 to 380 is executed, it calculates one row of the table and prints out the partition function calculated on that row in line 370. In lines 310 to 330 we calculate the entries along row N for columns M = 1, 2, 3, ..., N, using relation (1) in line 320. In lines 340 to 360 we fill in the rest of the row with the constant value P(N,N).

Program: Counting Partitions the Easy Way

```
100     CLS
110     DIM P(50, 50)
120     MAXN = 20
200     FOR M = 0 TO MAXN: P( M,0) = 1: P( M,1) = 1: NEXT M
300     FOR N = 2 TO MAXN
 310       FOR M = 1 TO N
 320       P(M,N) = P(M - 1,N) + P(M,N - M)
 330       NEXT M
 340       FOR M = N + 1 TO MAXN
 350       P(M,N) = P(N, N)
 360       NEXT M
 370     PRINT N, P(N, N)
 380     NEXT N
```

5. CONCLUDING REMARKS

The method shown above works for other restricted partition functions as well. The reader might want to study the partition function $D(m,n)$, which counts the number of partitions of the number *n* into distinct summands that do not exceed the number *m*. Corresponding to (1) we have the relation

(3) $$D(m,n) = D(m-1,n) + D(m-1,n-m) .$$

A table for the hand calculation of $D(m,n)$, similar to our table for $p(m,n)$, can then be constructed using (3).

Another example which the reader might wish to examine is the function $h(m,n)$, which counts the number of partitions of *n* whose least part is *m*. For example, the partitions of 5 whose least part is 1 are 1+1+1+1+1, 1+1+1+2, 1+2+2, 1+1+3, and 1+4. Thus $h(1,5) = 5$. This is a particularly interesting partition with many nice properties. It satisfies the relation

$$h(m,n) = h(m-1,n-1) - h(m-1,n-m) .$$

Finally, one has the deep results of so the called Rogers-Ramanujan Identities and the Ramanujan Congruences. Andrews' book [1] has an excellent treatment of these topics that should be accessible to most readers.

6.REFERENCES

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