A ZERO FREE REGION FOR HYPERGEOMETRIC ZETA FUNCTIONS

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Abstract. This paper investigates the location of 'trivial' zeros of some hypergeometric zeta functions. Analogous to Riemann's zeta function, we demonstrate that they possess a zero free region on a left-half complex plane, except for infinitely many zeros regularly spaced on the negative real axis.

1. INTRODUCTION

One of the outstanding problems in mathematics regards the location of the zeros the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

It is well known that $\zeta(s)$ is zero free outside of the *critical strip* $\{0 \le \Re(s) \le 1\}$, except for *trivial zeros* located at negative even integers. In particular, Euler's product formula for the zeta function,

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.2}$$

establishes that $\zeta(s) \ne 0$ on the right-half plane $\{\Re(s) > 1\}$. Its much celebrated functional equation,

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s)\zeta(1-s), \tag{1.3}$$

then reveals that $\zeta(s) \ne 0$ on the left-half plane $\{\Re(s) < 0\}$, except at the aforementioned trivial zeros. Regarding the zeros inside the critical strip, it is conjectured that these *nontrivial zeros* all must be located on the *critical line* at $\Re(s) = 1/2$. This conjecture is known as Riemann's Hypothesis.

In this paper we will investigate the nature of 'trivial' zeros of *hypergeometric zeta functions*, defined by the integral formula

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^{\infty} \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}\, dx. \tag{1.4}$$

Here N is a positive integer and $T_N(x) = 1+x+x^2/2!+\cdots+x^N/N!$. A study of these functions were initiated in [2] as a natural generalization of the Riemann zeta function. Notice that for $N = 1$, (1.4) reproduces the well known integral representation of $\zeta(s)$. In this same paper we established some analytic properties of $\zeta_N(s)$ where, among other things, we proved that for $\Re(s) < 0$, $\zeta_N(s)$ satisfies the following 'pre-functional' equation:

$$\zeta_N(s) = 2(-1)^{N-1}(N-1)!\Gamma(2-N-s) \sum_{k=1}^{\infty} r_k^{s-1} \cos\left[(s-1)(\pi - \theta_k)\right]. \tag{1.5}$$

Here $z_k = x_k + iy_k = r_k e^{i\theta_k}$ are the nonzero roots of $e^z - T_{N-1}(z) = 0$ in the upper-half complex plane arranged in increasing order of modulus. Observe that for $N = 1$, equation (1.5) simplifies to the classical functional equation given by (1.3) since in this case the zeros are elegantly located at $z_k = 2k\pi i$.

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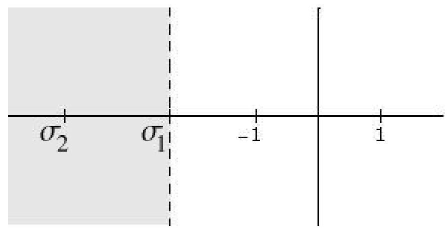

FIGURE 1. Zero-free left half-plane of $\zeta_2(s)$ (shaded)

In particular, we will focus on the zeros of the second-order hypergeometric zeta function $\zeta_2(s)$:

$$\zeta_2(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{x^s}{e^x - 1 - x}\, dx. \tag{1.6}$$

Unlike the situation with classical zeta, there is no product formula for $\zeta_2(s)$ to take advantage of here in establishing a zero free region to the left. Therefore, we must resort solely on its pre-functional equation, which in this case takes the form

$$\zeta_2(s) = -2\Gamma(-s) \sum_{k=1}^{\infty} r_k^{s-1} \cos\left[(s-1)(\pi - \theta_k)\right]. \tag{1.7}$$

Using (1.7) and appropriate bounds on the roots z_k, we intend to establish the following result for $\zeta_2(s)$, analogous to that for Riemann's zeta:

Theorem 1.1. *$\zeta_2(s)$ has no zeros in the left-half complex plane $\{s = \sigma + it | \sigma < \sigma_2\}$ (cf. Figure 1), except for infinitely many 'trivial' zeros on the negative real axis, one in each of the intervals*

$$S_m = [\sigma_{m+1}, \sigma_m], \tag{1.8}$$

where $m \geq 2$ is a positive integer and $\sigma_m = 1 - \frac{\pi}{\pi - \theta_1} m$.

A similar result will also be proven for $\zeta_3(s)$ (cf. Theorems 2.3 and 2.4). Tables 1 and 2 appearing in Appendix II list the approximate values of the first ten trivial zeros of $\zeta_2(s)$ and $\zeta_3(s)$, respectively.

Numerical evidence from the roots of $e^x - T_{N-1}(x) = 0$ suggests that statements similar to Theorem 1.1 should hold for $\zeta_N(s)$ in general. We will have more to say on this in our concluding remarks. The more difficult problem of finding a functional equation or product representation of these functions remains open.

2. A Zero Free Region on the Left

In [7], R. Spira established that the Hurwitz zeta function has a zero free region to the left of the complex plane by demonstrating that its functional equation is essentially dominated by the first term. We shall adopt his method to prove that $\zeta_2(s)$ has no zeros in the left half plane $\{\Re(s) < \sigma_2\}$, except for infinitely many zeros on the negative real axis, one in each of the intervals S_m for $m \geq 2$ (cf. (1.8)). To this end, we use (1.7) to rewrite $\zeta_2(s)$ as

$$\zeta_2(s) = f(s)(1 + g(s)), \tag{2.1}$$

where

$$f(s) = -2\Gamma(-s) r_1^{s-1} \cos[(s-1)(\pi - \theta_1)] \tag{2.2}$$

and

$$g(s) = \sum_{k=2}^{\infty} \left(\frac{r_k}{r_1}\right)^{s-1} \left(\frac{\cos[(s-1)(\pi - \theta_k)]}{\cos[(s-1)(\pi - \theta_1)]}\right). \tag{2.3}$$

The idea behind Spira's method is to demonstrate that $|g(s)| < 1$ inside a given domain (or respectively on its boundary). This essentially means that $\zeta_2(s)$ is dominated by its first term, $f(s)$. It follows that

$|\zeta_2(s)| \neq 0$ (or respectively by Rouche's Theorem both $\zeta_2(s)$ and $f(s)$ must have the same number of zeros). To determine a suitable zero free domain, we observe that for $\sigma = \Re(s) < 0$, the series

$$\sum_{k=2}^{\infty} \left(\frac{r_k}{r_1}\right)^{\sigma-1} \tag{2.4}$$

is increasing in σ. On the other hand, the modulus of the factor

$$\frac{\cos[(s-1)(\pi-\theta_k)]}{\cos[(s-1)(\pi-\theta_1)]} \tag{2.5}$$

is decreasing in $|t|$. This led us to consider a left half-plane as a zero-free region for $\zeta_2(s)$ (excluding the negative real axis), since if we can bound (2.5) and (2.4) by constants P and Q with $PQ < 1$, then it will follow from (2.3) that $|g(s)| < PQ < 1$.

In order to obtain bounds on the series (2.4), we first recall (see [2]) that the zeros $z_k = x_k + iy_k = r_k e^{i\theta_k}$ of $e^z - 1 - z = 0$ can be arranged in increasing order of modulus along with their arguments. Thus both sequences $\{\theta_k\}$ and $\{r_k\}$ are increasing with

$$0 < \theta_k < \pi/2 \quad \text{and} \quad r_k \to \infty \text{ as } k \to \infty. \tag{2.6}$$

The following lemma provides an estimate for (2.4) based on estimates for the roots $\{z_k\}$.

Lemma 2.1. *Suppose the even and odd roots $\{z_k\}$ of $e^x - T_{N-1}(x) = 0$ are bounded in modulus:*

$$\begin{aligned} r_{2m} &< Am, \quad m \geq 1 \\ r_{2m-1} &< Bm, \quad m \geq M \end{aligned} \tag{2.7}$$

where A and B are positive constants and $M \geq 2$ a positive integer. Then for $x > 1$,

$$\sum_{k=2}^{\infty} \left(\frac{r_1}{r_k}\right)^{x} < \phi_N(x), \tag{2.8}$$

where

$$\phi_N(x) = \left\{ \left(\frac{r_1}{A}\right)^x + \left(\frac{r_1}{B}\right)^x \right\} \zeta(x) - \left(\frac{r_1}{B}\right)^x + \sum_{m=3}^{M} \left[\left(\frac{r_1}{r_{2m-1}}\right)^x - \left(\frac{r_1}{Bm}\right)^x \right] \tag{2.9}$$

and $\zeta(x)$ is the Riemann zeta function.

Proof. The inequality above follows from rearrangement of the series (2.4) and the assumed bounds. □

We are now ready to state and prove our first result. First define for each positive integer m,

$$\sigma_m := \sigma_m(N) = 1 - \frac{\pi}{\pi - \theta_1} m, \tag{2.10}$$

where θ_1 is the argument of z_1 (the smallest nonzero root of $e^x - T_{N-1}(x) = 0$ in modulus). For example, if $N = 2$, then $\theta_1 \approx 1.2978341024$ and $\theta_2 \approx 1.3811541551$ (cf. Table 1); hence $\sigma_1 \approx -0.703907$ and $\sigma_2 \approx -2.40781$ from (2.10).

Theorem 2.1. *($N = 2$) Let $s = \sigma + it$. If $\sigma < \sigma_2$ and $|t| > 1$, then $\zeta_2(s) \neq 0$. Here, $\sigma_2 \approx -2.40781$ is defined by (2.10).*

Proof. For $N = 2$, the roots of $e^z - 1 - z = 0$ are known to be bounded as follows (cf. [2]):

$$\begin{aligned} r_{2m} &< 4\pi m, \quad m \geq 1 \\ r_{2m-1} &< 3\pi m, \quad m \geq 2 \end{aligned} \tag{2.11}$$

If $|t| > 1$, then it follows from (4.3) and Lemma 2.1 (with $A = 4\pi$, $B = 3\pi$, and $M = 2$) that

$$\begin{aligned} |g(s)| &\leq \sum_{k=2}^{\infty} \left(\frac{r_k}{r_1}\right)^{\sigma-1} \left|\frac{\cos((s-1)(\pi-\theta_k))}{\cos((s-1)(\pi-\theta_1))}\right| \\ &\leq \frac{\cosh(\pi-\theta_2)}{\sinh(\pi-\theta_1)} \sum_{k=2}^{\infty} \left(\frac{r_1}{r_k}\right)^{1-\sigma} \qquad (2.12) \\ &\leq \frac{\cosh(\pi-\theta_2)}{\sinh(\pi-\theta_1)} \phi_2(1-\sigma). \qquad (2.13) \end{aligned}$$

Since $\phi_2(1-\sigma)$ is increasing on $(-\infty, \sigma_2)$, it follows that $\phi_2(1-\sigma) \leq \phi_2(1-\sigma_2) < 0.29$ where we have used the fact that $r_1 \approx 7.748360311$ (cf. Table 1). Thus (2.13) implies that for $\sigma < \sigma_2$,

$$\begin{aligned} |g(s)| &\leq \frac{\cosh(\pi-\theta_2)}{\sinh(\pi-\theta_1)} \phi_2(1-\sigma_2) \qquad (2.14) \\ &< (0.9716)(0.29) \qquad (2.15) \\ &< 1. \qquad (2.16) \end{aligned}$$

By the reverse triangle inequality it also follows that

$$|\zeta_2(s)| = |f(s)||1+g(s)| \geq |f(s)|\,(1-|g(s)|) > 0,$$

since $|f(s)| > 0$ in the region of the hypothesis. This proves the theorem. □

Theorem 2.2. *($N = 2$) Let $s = \sigma + it$. If $\sigma \leq \sigma_2$ and $|t| \leq 1$, then $\zeta_2(s)$ has exactly one zero in the interval $[\sigma_{m+1}, \sigma_m]$, where $\sigma_m = 1 - \frac{\pi}{\pi-\theta_1} m$, for each $m = 2, 3, 4, \cdots$.*

Proof. Let γ_m be the rectangle with vertices $\sigma_{m+1} \pm i$ and $\sigma_m \pm i$. Let $f(s)$ and $g(s)$ be as in (2.2) and (2.3), respectively. We shall show that

$$|g(s)| < 1 \qquad (2.17)$$

on γ_m. Since

$$|\zeta_2(s) - f(s)| = |f(s)g(s)| < |f(s)|, \qquad (2.18)$$

it follows from Rouche's Thereom that $f(s)$ and $\zeta_2(s)$ have the number of roots inside γ_m. Clearly the roots of $f(s) = -2\Gamma(-s)r_1^{s-1}\cos[(s-1)(\pi-\theta_1)]$ are the roots of $\cos[(s-1)(\pi-\theta_1)]$ and the latter has exactly one root in the interval $[\sigma_{m+1}, \sigma_m]$. On the other hand, from (1.7), we observe that $\zeta_2(\bar{s}) = \bar{\zeta}_2(s)$. Thus, if s_m is a root of $\zeta_2(s)$ inside γ_m, then $\bar{s}_m$ is also a root in the same region. Hence $s_m = \bar{s}_m$ and the theorem follows.

To prove (2.17), we first consider the right vertical side of γ_m where $\sigma = \sigma_m$ and $|t| \leq 1$. Using (4.4) we have

$$\begin{aligned} \left|\frac{\cos((s-1)(\pi-\theta_k))}{\cos((s-1)(\pi-\theta_1))}\right| &= \left[\frac{\cos^2\left(-\frac{\pi-\theta_k}{\pi-\theta_1} m\right) + \sinh^2(t(\pi-\theta_k))}{\cos^2(-m\pi) + \sinh^2(t(\pi-\theta_1))}\right]^{1/2} \\ &\leq \left[\frac{1+\sinh^2(t(\pi-\theta_k))}{1+\sinh^2(t(\pi-\theta_1))}\right]^{1/2} = \frac{\cosh(t(\pi-\theta_k))}{\cosh(t(\pi-\theta_1))} \leq 1. \qquad (2.19) \end{aligned}$$

It then follows from (2.19), as in the proof of Thereom 2.1, that $|g(s)| \leq \phi_2(1-\sigma)$, where $\phi_2(1-\sigma)$ is again given by (2.9). Since $\phi_2(1-\sigma) < 1$ for $\sigma = \sigma_m \leq \sigma_2$, we conclude that $|g(s)| < 1$. A similar argument can be applied on the left vertical side of γ_m where $\sigma = \sigma_{m+1}$ and $|t| \leq 1$.

Next we consider the top side of γ_m where $t = 1$ and $\sigma_{m+1} \leq \sigma \leq \sigma_m$. On this part we have

$$\begin{aligned}\left|\frac{\cos\left((s-1)(\pi-\theta_k)\right)}{\cos\left((s-1)(\pi-\theta_1)\right)}\right| &= \left[\frac{\cos^2\left((\sigma-1)(\pi-\theta_k)\right)+\sinh^2\left(\pi-\theta_k\right)}{\cos^2\left((\sigma-1)(\pi-\theta_1)\right)+\sinh^2\left(t(\pi-\theta_1)\right)}\right]^{1/2} \\ &\leq \left[\frac{1+\sinh^2\left(\pi-\theta_k\right)}{\sinh^2\left(\pi-\theta_1\right))}\right]^{1/2} = \frac{\cosh\left(\pi-\theta_k\right)}{\sinh\left(\pi-\theta_1\right)} \\ &\leq \frac{\cosh\left(\pi-\theta_2\right)}{\sinh\left(\pi-\theta_1\right)}.\end{aligned}$$

Since since $\sigma_m \leq \sigma_2$, we once again have

$$|g(s)| < \frac{\cosh\left(\pi-\theta_2\right)}{\sinh\left(\pi-\theta_1\right)}\phi_2(1-\sigma_2) < 1. \tag{2.20}$$

The argument for the bottom side of γ_m is exactly the same. This proves our theorem. □

Remark 2.1. Observe that our results do not account for the first zero $s_1 \approx -1.605486847$ of $\zeta_2(s)$ on the negative real axis since Theorem 2.2 holds only for $m \geq 2$. Also, the zeros s_m converge asymptotically to the zeros of $\cos[(s-1)(\pi-\theta_1)]$, which are located at $(\sigma_m + \sigma_{m+1})/2$, i.e.

$$s_m \sim \frac{\sigma_m + \sigma_{m+1}}{2} = 1 - \frac{\pi}{\pi - \theta_1}\left(\frac{2m+1}{2}\right). \tag{2.21}$$

Table 3 lists the first ten values of s_m and $(\sigma_m + \sigma_{m+1})/2$.

Having now established a zero free region for $\zeta_2(s)$, it becomes clear that this method applies to $\zeta_N(s)$ in general as long as one can establish suitable bounds on the corresponding roots z_k. We demonstrate this for $\zeta_3(s)$.

Theorem 2.3. *($N = 3$) Let $s = \sigma + it$. If $\sigma < \sigma_2$ and $|t| > 1$, then $\zeta_3(s) \neq 0$. Here, $\sigma_2 \approx -2.14012$ is defined by (2.10).*

Proof. As in Theorem 2.1 we shall again use (1.5) but with $N = 3$. Let $f(s)$ and $g(s)$ be as in (2.2) and (2.3) respectively, where $z_k = r_k e^{i\theta_k}$ are roots of $e^z - 1 - z - z^2/2 = 0$. In [2], these roots were shown to satisfy (2.6). Moreover, z_k is bounded in modulus: $r_k > (2k + 1/2)\pi$. Thus

$$r_{2m} > 4m\pi, \quad (m = 1,\ 2,\ 3,\ \cdots) \quad \text{and} \quad r_{2m-1} > 3.24m\pi, \quad (m = 2,\ 3,\ 4,\ \cdots).$$

Using the approximate values $r_1 \approx 9.2053499$, $\theta_1 \approx 1.1406576364$, and $\theta_2 \approx 1.2568294158$ (cf. Table 2), it follows from (4.3) that for $|t| > 1$,

$$\left|\frac{\cos\left(s-1\right)(\pi-\theta_k))}{\cos\left(s-1\right)(\pi-\theta_1))}\right| \leq \frac{\cosh(\pi-\theta_2)}{\sinh(\pi-\theta_1)} < 0.927818.$$

On the other hand, for $\sigma < \sigma_2$, we have from Lemma 2.1 (with $A = 4\pi$, $B = 3\pi$, and $M = 2$) that

$$\sum_{k=2}^{\infty}\left(\frac{r_1}{r_k}\right)^{1-\sigma} < \phi_3(1-\sigma) \leq \phi_3(1-\sigma_2) < 1.$$

As before, it follows that $|g(s)| < 1$ and hence $\zeta_3(s) \neq 0$. This completes the proof of the theorem. □

Combining Theorem 2.3 and same arguments used in Theorem 2.2 (with the natural modifications), we obtain an analogous result for the case $N = 3$:

Theorem 2.4. *($N = 3$) Let $s = \sigma + it$. If $\sigma \leq \sigma_2$ and $|t| \leq 1$, then $\zeta_3(s)$ has exactly one zero in the interval $[\sigma_{m+1}, \sigma_m]$, where $\sigma_m = 1 - \frac{\pi}{\pi-\theta_1}m$, for each integer $m \geq 2$.*

See Table 3 for a listing of the first ten trivial zeros of $\zeta_3(s)$ compared to those of $\cos[(s-1)(\pi-\theta_1)]$.

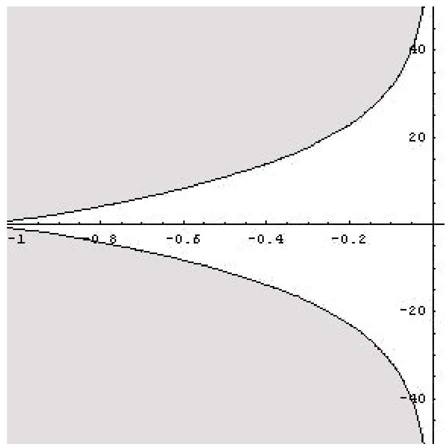

FIGURE 2. Extended zero-free region of $\zeta_2(s)$ to the left (shaded)

3. Concluding Remarks

It is possible to enlarge the zero-free region $\{\Re(s) < \sigma_2\}$ as illustrated in Figure 2. Let us assume for $N = 2$ that $|t| > t_0 > 0$ and $\sigma < \sigma_0 < 0$. It follows from Lemmas 2.1 and 4.1 that

$$|g(s)| \leq \frac{\cosh[t_0(\pi - \theta_2)]}{\sinh[t_0(\pi - \theta_1)]}\phi_2(1 - \sigma_0). \tag{3.1}$$

Since the region $\{s : |\Re(s)| > t_0, \Im(s) < \sigma_0\}$ will be zero free when $|g(s)| < 1$, the boundary of the largest possible zero-free region in terms of t_0 and σ_0 is then defined by the constraining equation

$$\frac{\cosh[t_0(\pi - \theta_2)]}{\sinh[t_0(\pi - \theta_1)]}\phi_2(1 - \sigma_0) = 1. \tag{3.2}$$

This is the boundary drawn in Figure 2.

As for zero-free regions of $\zeta_N(s)$ where $N \geq 4$, one needs to obtain bounds on the roots of $e^z - T_{N-1}(z) = 0$ similar to those required by Lemma 2.1. While numerical evidence for the first several values of N suggest such bounds exists, we do not have a proof of this beyond $N \geq 4$.

Lastly, we do not yet know if $\zeta_2(s)$ has any nontrivial zeros on a right half-plane. We recall from [2] that $\zeta_N(s)$ for $\Re(s) > 1$ has 'Dirichlet series' type representation of the form

$$\zeta_N(s) = \sum_{n=1}^{\infty} \frac{\mu_N(n, s)}{n^{s+1}},$$

where

$$\mu_N(n, s) = \sum_{k=0}^{(N-1)(n-1)} \frac{a_k(N, n)}{n^k}(s + N - 1)_k$$

and $a_k(N, n)$ is generated by

$$(T_{N-1}(x))^{n-1} = \left(\sum_{k=0}^{N-1} \frac{x^k}{k!}\right)^{n-1} = \sum_{k=0}^{(N-1)(n-1)} a_k(N, n)x^k.$$

In the absence of a functional equation or Euler product for hypergeometric zeta, one needs a good understanding of the properties of the "coefficients" $\mu_N(n, s)$ in order to investigate the existence of zeros to the right of the complex plane.

4. Appendix I

Lemma 4.1. *Let $\{\theta_k\}$ be a strictly increasing sequence of arguments satisfying (2.6). Then the following bounds hold true:*

(a) If $s = \sigma + it$ and $|t| \geq t_0 > 0$, then

$$\left|\frac{\cos\left[(s-1)(\pi-\theta_k)\right]}{\cos\left[(s-1)(\pi-\theta_1)\right]}\right| \leq \frac{\cosh[t_0(\pi-\theta_2)]}{\sinh[t_0(\pi-\theta_1)]} \quad \text{for all } \; k > 1. \tag{4.3}$$

(b) Let t be any real value. Then

$$\left|\frac{\cosh\left(t(\pi-\theta_k)\right)}{\cosh\left(t(\pi-\theta_1)\right)}\right| \leq 1. \tag{4.4}$$

Proof. To prove (a), we observe that $|\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y$ and hence $|\sinh y| \leq |\cos(x+iy)| \leq \cosh y$. Since $\cosh[t(\pi-\theta_k)]/\sinh[t(\pi-\theta_1)]$ is monotonically decreasing in $|t|$ and monotonically increasing in k, it follows for $|t| \geq t_0$ that

$$\begin{aligned}\left|\frac{\cos\left[(s-1)(\pi-\theta_k)\right]}{\cos\left[(s-1)(\pi-\theta_1)\right]}\right| &\leq \frac{\cosh\left[t(\pi-\theta_k)\right]}{|\sinh\left[t(\pi-\theta_1)\right]|} \\ &\leq \frac{\cosh\left[t_0(\pi-\theta_k)\right]}{|\sinh\left[t_0(\pi-\theta_1)\right]|} \\ &\leq \frac{\cosh\left[t_0(\pi-\theta_2)\right]}{|\sinh\left[t_0(\pi-\theta_1)\right]|}.\end{aligned}$$

This proves (4.3).

To prove (b), we merely observe that $\cosh x$ is increasing in $|x|$ and if $0 < \alpha \leq \beta$, then $\left|\frac{\cosh(\alpha t)}{\cosh(\beta t)}\right| \leq 1$. As $\{\theta_k\}$ is increasing in k towards π, we have $\pi - \theta_k < \pi - \theta_1$ and thus (4.4) immediately follows. This completes the proof of the lemma. □

5. Appendix II

Tables 1 and 2 list the first ten non-trivial zeros of $e^z - 1 - z = 0$ and $e^z - 1 - z - z^2/2 = 0$ corresponding to $N = 2$ and $N = 3$, respectively. Tables 3 and 4 list the first ten trivial zeros of $\zeta_2(s)$ and $\zeta_3(s)$, respectively. These values were computed using the pre-functional equation (1.5) and the software program Mathematica.

References

[1] H. M. Edwards, *Riemann's Zeta Function*, Pure and Applied Mathematics Series, Academic Press, 1974.

[2] A. Hassen and H. D. Nguyen, *Hypergeometric zeta functions*, Preprint (2005).

[3] F. T. Howard, *A sequence of numbers related to the exponential function*, Duke Math. J. 34 (1967), 599-616.

[4] F. T. Howard,*Some sequences of rational numbers related to the exponential function*, Duke Math. J. **34** (1967), 701-716.

[5] F. T. Howard, *Numbers Generated by the Reciprocal of $e^x - 1 - x$*, Mathematics of Computation **31** (1977) No. 138, 581-598.

[6] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse (On the Number of Prime Numbers less than a Given Quantity)*, 1859, Translated by D. R. Wilkins (1998).

[7] R. Spira, *Zeros of Hurwitz Zeta Functions*, Mathematics of Computation **30** (1976), No. 136, 863-866.

[8] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University Press, 1967.

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k	z_k	r_k	θ_k
1	2.088843016+7.461489286i	7.748360311	1.2978341024
2	2.664068142+13.87905600i	14.13242564	1.3811541551
3	3.026296956+20.22383500i	20.44900915	1.4222583654
4	3.291678332+26.54323851i	26.74656346	1.4474143156
5	3.501269010+32.85054823i	33.03660703	1.4646154233
6	3.674505305+39.15107412i	39.32313052	1.4772159363
7	3.822152869+45.44738491i	45.60782441	1.4868931567
8	3.950805215+51.74088462i	51.89150222	1.4945866979
9	4.064795694+58.03240938i	58.17459155	1.5008669923
10	4.167125550+64.32248998i	64.45733203	1.5061018433

TABLE 1. First ten nonzero roots of $e^z - 1 - z = 0$ in the upper-half complex plane.

k	z_k	r_k	θ_k
1	3.838602048+8.366815507i	9.205349934	1.1406576364
2	4.857263960+14.95891141i	15.72774757	1.2568294158
3	5.520626554+21.39846201i	22.09912880	1.3183102795
4	6.016178416+27.77895961i	28.42296607	1.3575169538
5	6.412519686+34.12944500i	34.72663855	1.3850733959
6	6.743013428+40.46233161i	41.02034263	1.4056646865
7	7.026523305+46.78391852i	47.30863623	1.4217195916
8	7.274789053+53.09777556i	53.59380865	1.4346366398
9	7.495625078+59.40609018i	59.87710703	1.4452835555
10	7.694499832+65.71028350i	66.15925246	1.4542298245

TABLE 2. First ten nonzero roots of $e^z - 1 - z - z^2/2 = 0$ in the upper-half complex plane.

m	s_m	$\frac{\sigma_m+\sigma_{m+1}}{2} = 1 - \frac{\pi}{\pi-\theta_1}\left(\frac{2m+1}{2}\right)$
1	-1.605486847	-1.555860135
2	-3.279946232	-3.259766892
3	-4.972103587	-4.963673649
4	-6.671076580	-6.667580406
5	-8.372911061	-8.371487163
6	-10.07596151	-10.07539392
7	-11.77952201	-11.77930068
8	-13.48329192	-13.48320743
9	-15.18714578	-15.18711419
10	-16.89103253	-16.89102095

TABLE 3. First ten trivial zeros $\{s_m\}$ of $\zeta_2(s)$ on the negative real axis and corresponding zeros of $\cos[(s-1)(\pi-\theta_1)]$.

m	s_m	$\frac{\sigma_m+\sigma_{m+1}}{2} = 1 - \frac{\pi}{\pi-\theta_1}\left(\frac{2m+1}{2}\right)$
1	-1.441343934	-1.355093464
2	-2.965619536	-2.925155773
3	-4.514693033	-4.495218082
4	-6.074606688	-6.065280391
5	-7.639732900	-7.635342700
6	-9.207425391	-9.205405009
7	-10.77637373	-10.77546732
8	-12.34592528	-12.34552963
9	-13.91575965	-13.91559194
10	-15.48572309	-15.48565425

TABLE 4. First ten trivial zeros $\{s_m\}$ of $\zeta_3(s)$ on the negative real axis and corresponding zeros of $\cos[(s-1)(\pi-\theta_1)]$.