THE RIEMANN ZETA FUNCTION AND ITS APPLICATION TO NUMBER THEORY

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1. Introduction

This paper is based on lecture notes given by the second author at Temple University in the spring of 1994. It was in these lectures that the first author was introduced to the theory of the Riemann zeta function. We claim no originality in this exposition. All results and proofs are due to others and our contribution here is the selection of material and presentation. We hope that this paper will introduce young mathematicians to this beautiful theory and inspire them to go beyond these pages!

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for } \sigma > 1, \tag{1}$$

where $s = \sigma + it$. The notation s for a complex number is due to Riemann and is now standard in this context. In this article we discuss and prove some of the basic properties of $\zeta(s)$. Our main goal will be to show how to apply the zeta function in the proof of the Prime Number Theorem, henceforth abbreviated by PNT.

Legendre and Gauss independently conjectured the PNT as follows. Let $\pi(x)$ be the number of primes less than or equal to x, where x is a positive real number. Then

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1. \tag{2}$$

In his only paper in number theory, in 1859, Riemann uncovered a deep relationship between the zeros of the zeta function and $\pi(x)$. This eight-page paper in fact gave rise to what is now known as analytic number theory, a branch of number theory that uses complex analysis in tackling problems involving integers. Based on the ideas of Riemann, Hadamard and de la Valle Poussin independently proved PNT in 1896. Both mathematicians used methods from complex analysis, establishing as a main step of the proof that the Riemann zeta function $\zeta(s)$ is nonzero for all complex values of the variable s that have the form $s = 1 + it$ with $t > 0$.

To expose this point of view is one of the intentions of this paper. We hope that the reader will be curious and interested enough to explore this rich and vibrant field of mathematics. For this we recommend the introductory texts in this area, among which we mention Apostol [1], Chandrasekharan [4], Hardy and Wright [6], Ireland and Rosen [8], Niven, Zuckerman and Montgomery [16], Patterson [17] and Rademacher [18].

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Section 2 will review the necessary background material needed to develop the theory of the Riemann zeta function as it pertains to the proof of PNT. In section 3 we will develop the properties of the zeta function and prove the functional equation it satisfies. In section 4 we will first give some elementary theorems involving $\pi(x)$ and conclude the section with D. J. Newman's much simplified (and much admired) complex-variables proof of PNT.

2. Preliminaries

Clearly the series defining $\zeta(s)$ in (1) convergence absolutely for $\sigma > 1$. However the most interesting properties of zeta are observed in the region where $\sigma \le 1$. The series representation given by (1) is invalid in this region and therefore we have to find a way to extend it to this region. The most fruitful *analytic continuation* of zeta is by way of integration. Thus we will be defining functions using integrals and justify that such functions are analytic. In many instances we need to interchange the processes of integration, limit and summation.

In order to keep our exposition brief and to focus on the important technical aspects of the application of the zeta function to PNT, we shall assume that the reader is familiar with the theory of the functions of one complex variable and convergence theorems of real analysis. One of the theorems of real analysis we will be using often is the Weierstrass M-test for uniform convergence of series of functions. We will also be using consequences of uniform convergence. For readers with graduate level real analysis, we point out that the integrals we deal with can be considered as Lebesgue integrals and thus we can easily appeal to the Lebesgue Dominated Convergence Theorem. For proofs of theorems related to these topics, we refer the reader to any standard textbook of real and complex analysis but we mention Bak and Newman [2], Goldberg [5](Chapter 9), Hijab [7], Knopp [9](Chapter XII, Sections 56 to 58), and Titchmarsh [21].

One of the most important theorems of complex analysis that we will be using frequently is the Identity Theorem. Here is the statement of the theorem. For the proof we refer the reader to Marsden [13](Page 397).

Theorem 1. Identity Theorem or The Principle of Analytic Continuation *Let f and g be analytic in a region R. Suppose that there is a sequence $\{z_n\}$ of distinct points of R converging to a point $z_0 \in R$ such that $f(z_n) = g(z_n)$ for all $n = 1,\ 2,\ 3,\ \cdots$. Then $f = g$ on all of R.*

Example: Let $g(z) = \sum_{n=0}^{\infty} z^n$ and $f(z) = \frac{1}{1-z}$. If $|z| < 1$, $g(z) = f(z)$.(The series is a geometric series.) $f(z)$ is analytic everywhere except at $z = 1$. Thus f is an analytic continuation of g in the sense that we define $g(z)$ to be $f(z)$ for $z \neq 1$.

To obtain a different representation of the Riemann zeta function it is essential to use the gamma and theta functions. We shall define these two functions next and state the main properties that we shall need for our investigation of ζ.

Definition 1. *The gamma function, denoted by $\Gamma(s)$, is defined by*

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx, \qquad \sigma > 0.$$

Integration by parts ($u = e^{-x}, dv = x^{s-1}dx$) yields

$$\Gamma(s) = \frac{1}{s}\Gamma(s+1).$$

Note then that if $s = n$ is a positive integer, then $\Gamma(n) = (n-1)!$. More importantly, we note that the integral defining $\Gamma(s+1)$ is convergent for $Re(s) > -1$ and hence $\frac{1}{s}\Gamma(s+1)$ is the analytic continuation of $\Gamma(s)$ to the region $Re(s) > -1$. We repeat this process to extend $\Gamma(s)$ to the whole plane with simple poles at the nonpositive integers. One of the classical books on special functions, Lebedev [12](Chapter 1), treats many interesting properties and applications of the gamma function.

Next we introduce the theta function.

Definition 2. *The theta function* $\Theta(z)$ *is defined by*

$$\Theta(z) \;=\; \sum_{n=-\infty}^{\infty} e^{n^2\pi iz} \;=\; 1+2\sum_{n=1}^{\infty} e^{n^2\pi iz}, \qquad\qquad Im(z)=y>0.$$

The importance of the theta function lies in the property that we state in

Theorem 2. (The Transformation Law of Theta)

(1) $\Theta(z+2) \;=\; \Theta(z)$.

(2) $\Theta\left(\frac{-1}{z}\right) \;=\; e^{\frac{-\pi i}{4}} z^{\frac{1}{2}}\Theta(z)$.

As we shall see later, the transformation law stated above plays an important role in the analytic continuation of the zeta function. In fact, the functional equation of $\zeta(s)$ is a consequence of this transformation law. For the sake of simplicity we will proof a special case of Theorem 2 that we state in the following proposition. However, we note that the Identity Theorem can easily be used to deduce Theorem 2 from

Proposition 1. *If* $x>0$*, then*

$$\Theta\left(\frac{i}{x}\right) \;=\; x^{\frac{1}{2}}\Theta(ix).$$

To prove this form of the transformation law of the theta function, we first need the following theorem from analysis.

Theorem 3. (Poisson Summation Formula) *If* f *is continuous on the real line and* $\sum_{n=-\infty}^{\infty} f(n+t)$ *converges uniformly on* $0\le t\le 1$ *and if* $\sum_{n=-\infty}^{\infty} f(n)e^{2\pi int}$ *converges, then*

$$\sum_{n=-\infty}^{\infty} f(n+t) \;=\; \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int},$$

where

$$\hat{f}(n) \;=\; \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\,dx.$$

Proof: Define $\phi(t) \;=\; \sum_{m=-\infty}^{\infty} f(t+m)$. By our hypotheses, on the real line ϕ is continuous and clearly $\phi(t+1)=\phi(t)$. Thus $\phi(t)$ has a Fourier series expansion given by

$$\phi(t)=\sum_{n=-\infty}^{\infty} a_n e^{2\pi int},$$

where

$$a_n=\int_0^1 \phi(x)e^{-2\pi inx}\,dx.$$

That the Fourier series for $\phi(x)$ is equal to the function follows from the fact that $\phi(t)$ is uniformly continuous on $[0,1]$ and that the Fourier series is uniformly convergent in $[0,1]$. (For a detailed proof of this see [21], page 414. See also [10], page 39.) Let us find a_n by substituting the summation for $\phi(x)$ in the integral.(We leave it to the reader to justify the permissibility of interchanging summation and integration.)

$$\begin{aligned} a_n = \int_0^1 \phi(x)e^{-2\pi inx}\,dx \;&=\; \int_0^1 \sum_{m=-\infty}^{\infty} f(x+m)e^{-2\pi inx}\,dx \\ &= \sum_{m=-\infty}^{\infty} \int_0^1 f(x+m)e^{-2\pi inx}\,dx \;=\; \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(x)e^{-2\pi inx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\,dx = \hat{f}(n), \end{aligned}$$

as desired.

Proof of Proposition 1: In what follows x is a fixed positive real number. We shall write $\exp(z)$ instead of e^z whenever it is convienient. Again we leave it to the reader to justify interchanging summations and integration.

Define $f(u) = e^{-\pi u^2 x}$. Then $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$ converges and by the Poisson Summation Formula (with $t = 0$), we have

$$\Theta(ix) = \sum_{n=-\infty}^{\infty} f(n) \;=\; \sum_{n=-\infty}^{\infty} \hat{f}(n) \;=\; \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)e^{-2\pi inu}\,du. \tag{3}$$

But $f(u) = \exp\left(-u^2 x\right)$ and hence by completing the square we have

$$\exp\left(-\pi u^2 x - 2\pi inu\right) = \exp\left(-\pi x\left(u^2 + 2inu/x\right)\right) = \exp\left(-\pi x\left(u + in/x\right)^2 - \pi n^2/x\right).$$

The change of variable $t = u + in/x$ then yields

$$\int_{-\infty}^{\infty} f(u)e^{-2\pi inu}\,du = \int_{-\infty}^{\infty} \exp\left(-\pi x\left(u+in/x\right)^2 - \pi n^2/x\right)\,du = e^{-\pi n^2/x} \int_{-\infty+in/x}^{\infty+in/x} \exp\left(-\pi x t^2\right)\,dt.$$

It can be shown that

$$\int_{-\infty+\frac{in}{x}}^{\infty+\frac{in}{x}} \exp\left(-\pi x t^2\right)\,dt = \int_{-\infty}^{\infty} \exp\left(-\pi x t^2\right)\,dt. \tag{4}$$

Thus we have

$$\int_{-\infty}^{\infty} f(u)e\exp\left(-2\pi inu\right)\,du = \exp\left(-\frac{\pi n^2}{x}\right)\int_{-\infty}^{\infty} \exp\left(-\pi x t^2\right)\,dt.$$

Finally to remove x form the integral, we let $t = y/\sqrt{\pi x}$. This yields

$$\int_{-\infty}^{\infty} f(u)\exp\left(-2\pi inu\right)\,du = \frac{\exp\left(-\frac{\pi n^2}{x}\right)}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \exp\left(-y^2\right)\,dy = \gamma\,\frac{\exp\left(-\frac{\pi n^2}{x}\right)}{\sqrt{x}}, \tag{5}$$

where

$$\gamma = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy.$$

Substituting (5) in (3) and noting that γ is a constant, we obtain

$$\Theta(ix) = \frac{\gamma}{\sqrt{x}}\Theta\left(\frac{i}{x}\right). \tag{6}$$

To complete the proof wee need to show $\gamma = 1$. Since (6) holds for all $x > 0$, putting $x = 1$ in the equation yields $\gamma = 1$, thereby completing the proof of the proposition.

As a consequence of Proposition 1, we have the following

Corollary 1. *For $t > 0$, let $\Psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Then*

$$\Psi\left(\frac{1}{t}\right) = -\frac{1}{2} + \frac{1}{2}t^{1/2} + t^{1/2}\Psi(t). \tag{7}$$

Proof: This follows from the fact that $\Psi(t) = \frac{\Theta(it)-1}{2}$.

3. The Riemann Zeta Function

Recall that the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for} \quad \sigma > 1,$$

where $s = \sigma + it$. Since $|n^{-s}| = n^{-\sigma}$, it follows from the integral test for convergence of infinite series that the series converges absolutely for $\sigma > 1$. Furthermore, if $a > 1$ and $\sigma \geq a$, then

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^a} < \infty.$$

Thus convergence is uniform for $\sigma > a$, and therefore $\zeta(s)$ is analytic in the region $\sigma > 1$. For an in-depth analysis and detailed proofs of properties of the Riemann zeta function, we recommend Titchmarsh [20].

Theorem 4. *$\zeta(s)$ can be extended meromorphically to the right half-plane $\sigma > 0$ and, in fact, $\zeta(s) - \frac{1}{s-1}$ is analytic in $\sigma > 0$.*

Poof: For $\sigma > 0$, define

$$\phi_n(s) \;=\; \frac{1}{n^s} - \int_n^{n+1} \frac{1}{u^s}\,du \;=\; \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{u^s}\right) du.$$

Then

$$|\phi_n(s)| \;=\; \left|\int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{u^s}\right) du\right| \leq \max_{u\in[n,n+1]} \left(n^{-s} - u^{-s}\right).$$

But $n^{-s} - u^{-s} = \int_n^u sx^{-s-1}\,dx$. Thus we have

$$\left|n^{-s} - u^{-s}\right| \leq |s| \int_n^{n+1} x^{-\sigma-1}\,dx \;\leq\; \frac{|s|}{\sigma}\left(n^{-\sigma} - (n+1)^{-\sigma}\right).$$

Adding over n we get

$$\sum_{n=1}^{\infty} |\phi_n(s)| \;\leq\; \frac{|s|}{\sigma}\sum_{n=1}^{\infty}\left(n^{-\sigma} - (n+1)^{-\sigma}\right) \;=\; \frac{|s|}{\sigma}.$$

Thus $\sum_{n=1}^{\infty} \phi_n(s)$ converges absolutely and uniformly in $\sigma > a$, $a > 0$. Since $\phi_n(s)$ is entire, it follows that the function $F(s)$ defined by

$$F(s) \;=\; \sum_{n=1}^{\infty} \phi_n(s)$$

is analytic function in $\sigma > 0$.
On the other hand, we have, for $\sigma > 1$,

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} \phi(s) = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{u^s}\, du \right) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{u^s}\, du \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \left(\frac{(n+1)^{-s+1} - n^{-s+1}}{-s+1} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{s-1} \\ &= \zeta(s) - \frac{1}{s-1}. \end{aligned}$$

Thus $\zeta(s) = F(s) + \frac{1}{s-1}$ for $\sigma > 1$. Since $F(s)$ and $\frac{1}{s-1}$ are analytic for $\sigma > 0$, we see that $F(s) + \frac{1}{s-1}$ is the analytic continuation of $\zeta(s)$ to the region $\sigma > 0$ with a simple pole at $s = 1$ and residue 1 there.

Next we extend this theorem by proving the functional equation of the Riemann zeta function.

Theorem 5. *Let*

$$\Phi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Then $\Phi(s)$ can be continued analytically to the whole plane and it satisfies the functional equation

$$\Phi(s) = \Phi(1-s).$$

Proof: From the definitions of $\zeta(s)$ and $\Gamma(s)$, we have, for $\sigma > 1$,

$$\Phi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-s/2} \left(\int_0^{\infty} x^{s/2-1} e^{-x}\, dx \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right) = \sum_{n=1}^{\infty} \int_0^{\infty} x^{-1} \left(\frac{x}{n^2 \pi} \right)^{s/2} e^{-x}\, dx.$$

Let $y = x/\left(n^2\pi\right)$. Then

$$\Phi(s) = \sum_{n=1}^{\infty} \int_0^{\infty} y^{s/2-1} \exp\left(-n^2 \pi y\right)\, dy = \int_0^{\infty} y^{s/2-1} \sum_{n=1}^{\infty} \exp\left(-n^2\pi y\right)\, dy = \int_0^{\infty} y^{s/2-1} \Psi(y)\, dy.$$

Now split the integral into two pieces to get

$$\Phi(s) = \int_0^1 y^{s/2-1} \Psi(y)\, dy + \int_1^{\infty} y^{s/2-1} \Psi(y)\, dy.$$

In the first integral, let us change the variable by letting $y = 1/u$. Then

$$\Phi(s) = \int_1^{\infty} u^{-s/2-1} \Psi\left(\frac{1}{u}\right) du + \int_1^{\infty} y^{s/2-1} \Psi(y)\, dy. \tag{8}$$

Using (7), we can rewrite (8) as

$$\begin{aligned} \Phi(s) &= \int_1^{\infty} u^{-s/2-1} \left(-\frac{1}{2} + \frac{1}{2} u^{1/2} + u^{1/2} \Psi(u) \right) du + \int_1^{\infty} y^{s/2-1} \Psi(y)\, dy \\ &= -\frac{1}{2} \int_1^{\infty} u^{-s/2-1}\, du + \frac{1}{2} \int_1^{\infty} u^{-s/2-1/2}\, du + \int_1^{\infty} u^{-s/2-1} \Psi(u)\, du + \int_1^{\infty} y^{s/2-1} \Psi(y)\, dy \\ &= -\frac{1}{s} + \frac{1}{s-1} + \int_1^{\infty} u^{-s/2-1} \Psi(u)\, du + \int_1^{\infty} y^{s/2-1} \Psi(y)\, dy. \end{aligned}$$

Replacing u by y in the first integral on the last line of the above equation, we get

$$\Phi(s) = -\frac{1}{s} + \frac{1}{s-1} + G(s), \tag{9}$$

where

$$G(s) = \int_1^\infty y^{-s/2-1}\Psi(y)\,du + \int_1^\infty y^{s/2-1}\Psi(y)\,dy \tag{10}$$

Note then that $G(s)$ is an entire function, since clearly $\Psi(y) \le Ae^{-\alpha y}$, as $y \to \infty$ for some constants A and α. Also $\frac{1}{s}$ and $\frac{1}{s-1}$ are analytic except at 0 and 1, respectively. Therefore $-\frac{1}{s} + \frac{1}{s-1} + G(s)$ is the analytic continuation of $\Phi(s)$ to the whole plane.

To see that $\Phi(s)$ satisfies the functional equation $\Phi(s) = \Phi(1-s)$, we need only observe that under the transformation $s \to 1-s$ the first integral in $G(s)$ goes to the second and vice versa. Clearly $\frac{1}{s} - \frac{1}{s-1}$ goes back to itself when s is replaced by $1-s$. This completes the proof of the theorem.

Remark 1. (1) *From the fact that* $\Gamma(s) = \frac{1}{s}\Gamma(s+1)$, *we deduce that* $\lim_{s\to 0} s\Gamma(s) = \lim_{s\to 0}\Gamma(s+1) = 1$. *It follows that* $\lim_{s\to 0} s\Gamma\left(\frac{s}{2}\right) = 2$. *Since* $G(s)$ *is entire,* $G(0)$ *is finite and hence* $\lim_{s\to 0} sG(s) = 0$. *But then*

$$\lim_{s\to 0}(s\Phi(s)) = \lim_{s\to 0}\left(-1 + \frac{s}{s-1} + sG(s)\right) = -1.$$

From these facts and the definition of $\Phi(s)$, *we conclude that* $\zeta(0) = -\frac{1}{2}$.

(2) *The function* $G(s)$ *defined by* (10) *is entire. Thus* $\Phi(s)$ *is analytic everywhere except at* $s = 0$ *and* $= 1$, *where it has simple poles. Solving for* $\zeta(s)$ *from* $\Phi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, *we get*

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\Phi(s).$$

Since the only poles of $\Gamma(s)$ *are simple ones at* $0, -1, -2, -3, \cdot$, *it follows that* $1/\Gamma(s/2)$ *is entire with simple zeros at* $s = 0, -2, -4, -6, \cdots$. *The functional equation* $\Phi(s) = \Phi(1-s)$ *and the fact that* $\zeta(n)$ *and* $\Gamma(n)$ *are nonzero for positive integers* n, *implies that* $s = -2, -4, -6, \cdots$ *are zeroes of* $\zeta(s)$. *These zeroes are called the trivial zeroes.*

(3) *We will show shortly that if* $Re(s) = \sigma > 1$, *then* $\zeta(s) \neq 0$. *This fact and the functional equation imply that all other zeroes are in the vertical strip* $0 \le \sigma \le 1$. *This is known as the critical strip. The Riemann Hypothesis, one of the most famous open problems of the last 14 decades, states that all of the nontrivial zeroes of* $\zeta(s)$ *lie on the line* $\sigma = 1/2$.

Next we derive the Euler Product Formula for $\zeta(s)$. For the remainder of our discussion p will be used exclusively to denote a prime number.

Theorem 6. *If* $Re(s) = \sigma > 1$, *then* $\zeta(s)$ *has following infinite product expansion*

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}, \tag{11}$$

where the product is taken over all primes p. *This assumption will be used throughout our discussion.*

Proof: Let $X > 0$ be a positive integer. Consider the product $\prod_{p\le X}\left(1-p^{-s}\right)^{-1}$. We expand each term of this product in to power series to get

$$\left(1 - p^{-s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots.$$

Substituting in the above product and multiplying out the terms (note that we have an absolutely convergent series), we get

$$\prod_{p\le X}\left(1-p^{-s}\right)^{-1}=\prod_{p\le X}\sum_{k=0}^{\infty}\frac{1}{p^{ks}}=\sum_{k=1}^{\infty}\frac{1}{n_k^s},$$

where the n_k are those integers for which their maximum prime divisor is less than or equal to X. But every integer less than or equal to X is an n_k for some k, so it follows that

$$\left|\zeta(s)-\prod_{p\le X}\left(1-p^{-s}\right)^{-1}\right|\le\frac{1}{(X+1)^\sigma}+\frac{1}{(X+2)^\sigma}+\cdots.$$

We now let $X\to\infty$ and observe that

$$\frac{1}{(X+1)^\sigma}+\frac{1}{(X+2)^\sigma}+\cdots$$

is the tail end of a convergent series for $\sigma>1$ and hence approaches 0, as $X\to\infty$. This proves the theorem.

One of the many consequences of Euler's product formula (11) is a proof for the infinitude of primes. For if there were a finite number of primes then the product in (11) would be finite for $s=1$ which in term would imply that $\lim_{s\to1}\zeta(s)$ is finite. Since $\zeta(s)$ has a pole of order 1 at $s=1$, we have a contradiction. Here is another consequence of (11).

Corollary 2. *$\zeta(s)\neq0$ for $Re(s)=\sigma>1$.*

Proof: This follows from Euler Product formula (11) and the fact that for $\sigma>1$, $1-p^{-s}\neq0$ for all primes p. (Observe that the product in (11) cannot diverge to 0.)

The following theorem is critical in the proof of the Prime Number Theorem that will be considered in the next section.

Theorem 7. *$|\zeta(1+it)|\neq0$ for $Re(s)=\sigma\ge1$.*

We first prove

Lemma 1. *For $s=\sigma+it$, $\sigma>1$, and $t\neq0$, we have*

$$\left|\zeta^3(\sigma)\zeta^4(s)\zeta(s+it)\right|\ge1.$$

Proof: Let $\rho=e^{i\phi}$, ϕ real. Then $\rho^{1/2}+\rho^{-1/2}=2\cos(\phi/2)$ and hence

$$0\le\left(\rho^{1/2}+\rho^{-1/2}\right)^4=\rho^2+\rho^{-2}+4(\rho+\rho^{-1})+6=2Re(\rho^2)+8Re(\rho)+6.$$

Thus

$$Re(\rho^2)+4Re(\rho)+3\ge0. \tag{12}$$

Assume for now that $Re(s)=\sigma>1$ and take the logarithm of both sides of (11) to get

$$\log\zeta(s)=-\sum_p\log\left(1-p^{-s}\right)=\sum_p\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{p^{ms}}=\sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

where

$$a_n=\begin{cases}\frac{1}{m}, & n=p^m\\ 0, & \text{otherwise.}\end{cases}$$

Here we have applied the power series expansion of $\log(1-x)$ and rearranged the double sum. From this we conclude that

$$\log\zeta(\sigma) = \sum_{n=1}^{\infty} a_n n^{-\sigma}, \qquad \log\zeta(s) = \sum_{n=1}^{\infty} \left(a_n n^{-\sigma}\right) n^{-it}, \qquad \log\zeta(s+it) = \sum_{n=1}^{\infty} \left(a_n n^{-\sigma}\right) n^{-2it}.$$

Note that $a_n n^{-\sigma} \geq 0$. Put $\rho = n^{-it} = e^{-it\log n}$. Then $|\rho| = 1$ and by (12) we have

$$Re(n^{-2it}) + 4Re(n^{-it}) + 3 \geq 0.$$

Multiplying this inequality by $a_n n^{-\sigma} \geq 0$, noting the fact that $Re\big(a_n n^{-\sigma} n^{-it}\big) = (a_n n^{-\sigma}) Re\big(n^{-it}\big)$, and adding the resulting inequality over n, we get

$$\sum_{n=1}^{\infty} \left\{ Re\big(a_n n^{-\sigma} n^{-2it}\big) + 4Re\big(a_n n^{-\sigma} n^{-it}\big) + 3a_n n^{-\sigma} \right\} \geq 0.$$

Thus we have

$$Re\Big(\log\zeta(\sigma+2it) + 4\log\zeta(s) + 3\log\zeta(\sigma)\Big) \geq 0$$

and exponentiation yields

$$\exp\Big(Re\Big(\log\zeta(\sigma+2it) + 4\log\zeta(s) + 3\log\zeta(\sigma)\Big)\Big) \geq 1.$$

This implies that

$$\exp\left(Re(\log\zeta(\sigma+2it)\right) \cdot \exp\left(4\log\zeta(s)\right) \cdot \exp\left(3\log\zeta(\sigma)\right)) \geq 1.$$

Since $|z| = \exp\big(Re\big(\log(z)\big)\big)$ holds for all z, we conclude that

$$\left|\zeta^3(\sigma)\zeta^4(s)\zeta(s+it)\right| \geq 1,$$

as desired.

Proof of Theorem 7. From the above lemma we have, for $\sigma > 1$,

$$\Big|(\sigma-1)\zeta(s+it)\Big|\left|\left(\frac{\zeta(s)}{\sigma-1}\right)\right|^4\Big|\big((\sigma-1)\zeta(\sigma)\big)\Big|^3 \geq 1. \tag{13}$$

Now suppose $|\zeta(1+it)| = 0$ for $t \neq 0$. Since $\zeta(s)$ has a simple pole at $s = 1$, we have

$$\lim_{\sigma\to 1}(\sigma-1)\zeta(\sigma) = 1. \tag{14}$$

Note also that

$$\zeta'(1+it) = \lim_{\sigma\to 1} \frac{\zeta(\sigma+it) - \zeta(1+it)}{(\sigma+it)-(1+it)} = \lim_{\sigma\to 1} \frac{\zeta(\sigma+it)}{\sigma-1} \tag{15}$$

exists and is finite. Since $\zeta(s+it) = \zeta(\sigma+2it)$, we see that

$$\lim_{\sigma\to 1}(\sigma-1)\zeta(s+it) = 0. \tag{16}$$

Taking the limit as $\sigma \to 1^+$ for the expression on the left side of (13) and using (14), (15) and (16), we conclude that the limit of the left hand side of (13) is 0, in contradiction to the inequality stated there.

We conclude this section with an application of the Euler-Maclaurin summation formula:

Theorem 8. Euler-Maclaurin Summation Formula *If f has a continuous derivative f' on the interval $[a,b]$, where $0 < a < b$, then*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \int_a^b \left(\{x\} - \frac{1}{2}\right) f'(x)dx - \left(\{b\} - \frac{1}{2}\right) f(b) + \left(\{a\} - \frac{1}{2}\right) f(a), \tag{17}$$

where $[x]$ *denotes the greatest integer less than or equal to* x *and* $\{x\} = x - [x]$ *denotes the fraction part of* x.

For the proof of Euler-Maclaurin Summation Formula see [20], page 13 and also [1], page 54. If we let $f(x) = x^{-s}$, $\Re(s) > 1$ and let a, b be integers, then $\{a\} = \{b\} = 0$ and $f'(x) = -sx^{-s-1}$. The Euler-Maclaurin summation formula yields

$$\begin{aligned}
\sum_{a<n\le b} \frac{1}{n^s} &= \int_a^b \frac{1}{x^s}dx - s\int_a^b \frac{\{x\}-1/2}{x^{s+1}}dx \; - \; \frac{1}{2p^s} + \frac{1}{2b^s} \\
&= \int_a^b \frac{1}{x^s}dx - s\int_a^b \frac{\{x\}-1}{x^{s+1}}dx \; - \frac{s}{2}\int_a^b \frac{1}{x^{s+1}}dx \; - \; \frac{1}{2p^s} + \frac{1}{2b^s} \\
&= \frac{1}{s-1}\left(\frac{1}{p^{s-1}} - \frac{1}{b^{s-1}}\right) - s\int_a^b \frac{\{x\}-1}{x^{s+1}}dx - \frac{s}{2}\int_a^b \frac{1}{x^{s+1}}dx \; - \; \frac{1}{2p^s} + \frac{1}{2b^s} \\
&= \frac{1}{s-1}\left(\frac{1}{p^{s-1}} - \frac{1}{b^{s-1}}\right) \; + s\int_a^b \frac{1-\{x\}}{x^{s+1}}dx + \frac{1}{2}\left(\frac{1}{b^s} - \frac{1}{a^s}\right) \; - \; \frac{1}{2p^s} + \frac{1}{2b^s} \\
&= \frac{1}{s-1}\left(\frac{1}{a^{s-1}} - \frac{1}{b^{s-1}}\right) \; + s\int_a^b \frac{1-\{x\}}{x^{s+1}}dx + \frac{1}{b^s} - \frac{1}{a^s}.
\end{aligned}$$

We now let $b \to \infty$ and add a^{-s} to both sides to obtain

$$\sum_{n\ge a} \frac{1}{n^s} = \frac{1}{(s-1)a^{s-1}} \; + \; s\int_a^\infty \frac{1-\{x\}}{x^{s+1}}dx. \tag{18}$$

Note that with $a = 1$, (18) gives another representation of ζ:

$$\zeta(s) = \frac{1}{s-1} \; + \; s\int_1^\infty \frac{1-\{x\}}{x^{s+1}}dx.$$

Since the integral is convergent and holomorphic in $\sigma > 0$, this representation also gives an analytic continuation of ζ to the region $\sigma > 0$ with a simple pole at $s = 1$ and having residue 1 there. Equation (18) will be used in the proof of Theorem 13 in the next section. For some asymptotic formulas involving $\zeta(s)$ that follow from the Euler-Maclaurin summation formula, see [1], page 55.

4. The Prime Number Theorem

Let $\pi(x)$ denote the number of primes less than or equal to x. Euclid proved that there are infinitely many prime numbers. Euclid's proof of the infinitude of primes can be found in any introductory level number theory book (e.g. [6], [19]). Thus clearly $\lim_{x\to\infty} \pi(x) = \infty$. The question then becomes: how does $\pi(x)$ behave at infinity? In other words, how rapidly does it go to infinity? To answer this question we introduce the following notations.

Definition 3. *Let* $f,\ g\ : R \to R$ *be functions such that* $g(x) \ge 0$. *Then we say,*

(1) $f(x) = \mathcal{O}(g(x))$ *if there exists* $R > 0$ *and* $M > 0$ *such that* $|f(x)| \le Mg(x)$ *for all* $x > R$,
(2) $f(x) = o(g(x))$ *if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$,

(3) $f(x) \sim g(x)$ *if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

If f and g satisfy definition 3, we say that they are asymptotic. The following facts can easily be proved and will be used freely.

(1) $\mathcal{O}(\mathcal{O}\,(g(x))) = \mathcal{O}(g(x))$,
(2) $\mathcal{O}(g(x)) \pm \mathcal{O}(g(x)) = \mathcal{O}(g(x))$,
(3) $\mathcal{O}(g(x)) \pm o(g(x)) = \mathcal{O}(g(x))$,
(4) $(\mathcal{O}\,(g(x)))^2 = \mathcal{O}\big((g(x))^2\big)$.

Theorem 9. *(The Prime Number Theorem - PNT)*

$$\lim_{x\to\infty} \frac{\pi(x)}{\left(\frac{x}{\log x}\right)} = 1. \tag{19}$$

The proof of PNT is one of the crowning achievements of modern mathematics. The effort made in proving it had tremendous impact upon the development of complex analysis in the 19th and 20th centuries. Among the principal contributers to the proof of the PNT were Legendre, Gauss, Tchebychev, Riemann, Dirichlet, Hadamard, and De la Valle Poussin. Each of these mathematicians used the methods of analysis. In 1949 Erdös and Selberg gave a proof that is elementary in the sense that it does not use the methods of analysis. For a brief summary of the history of the theorem and its impact see the excellent and readable paper of Bateman and Diamond [3]. The first proof of the theorem appeared in 1896, given independently by Hadamard and De la Valle Poussin. In this section we present the proof of Newman [14] (See also [15], Chapter 7).

In 1796, Adrien-Marie Legendre conjectured that

$$\pi(x) \sim \frac{x}{\log x - B}, \tag{20}$$

where B=1.08... is a certain constant close to 1.

Carl Friedrich Gauss, based on the computational evidence available to him and on some heuristic reasoning, was able to arrive at his own approximating function. We state Guass's conjecture as follows.

If we define

$$Li(x) = \int_3^x \frac{dt}{\log t}, \tag{21}$$

then

$$\pi(x) \sim Li(x). \tag{22}$$

One can show that Li(x) has the following expansion.

$$Li(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + \frac{2!x}{(\log x)^3} + \frac{3!x}{(\log x)^4} + \cdots + \frac{n!x}{(\log x)^{n+1}}\left(1 + \epsilon(x)\right), \tag{23}$$

where $\epsilon(x) \to 0$ as $x \to \infty$.

The Russian mathematician Pafnuty L'vovich Tchebyshev attempted to prove PNT in two papers from 1848 and 1850. In fact, Tchebychev proved the following two statements about distribution of primes. Note that these statements are weaker than PNT.

Theorem 10. *(Tchebychev) For $Li(x)$ as in (21) we have*

$$\varliminf_{x\to\infty} \frac{\pi(x)}{Li(x)} \leq 1 \leq \varlimsup_{x\to\infty} \frac{\pi(x)}{Li(x)}, \tag{24}$$

and

$$c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x}, \tag{25}$$

where $c_1 = 0.92\cdots$ and $c_2 = 1.105\cdots$. Here $\varliminf$ and $\varlimsup$ are limit inferior and limit superior.

Following Remark 3 (below), we will prove a weak form of (25) in which the inequalities hold with $c_1 = (\log 2)/3$ and $c_2 = 8\log 2$. To this end, we need to define the following functions.

Definition 4. (1) $\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^\alpha \\ 0, & \text{else,} \end{cases}$

(2) $\vartheta(x) = \sum_{p\leq x} \log p$,

(3) $\Psi(x) = \sum_{p^m \leq x} \log p$.

Remark 2. *Note that*

$$\Psi(x) = \sum_{n\leq x} \Lambda(n) = \sum_{p^m\leq x} \left[\frac{\log x}{\log p}\right] \log p$$

and that

$$\Psi(x) = \vartheta(x) \ + \ \vartheta(x^{1/2}) \ + \ \vartheta(x^{1/3}) \ + \ \cdots .$$

The following theorem gives the connections between the above functions and the Prime Number Theorem.

Theorem 11. *Let*

$$L_1 = \varlimsup_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}}, \qquad L_2 = \varlimsup_{x\to\infty} \frac{\vartheta(x)}{x}, \qquad L_3 = \varlimsup_{x\to\infty} \frac{\Psi(x)}{x}, \tag{26}$$

$$l_1 = \varliminf_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}}, \qquad l_2 = \varliminf_{x\to\infty} \frac{\vartheta(x)}{x}, \qquad l_3 = \varliminf_{x\to\infty} \frac{\Psi(x)}{x}. \tag{27}$$

Then

$$l_1 = l_2 = l_3 \qquad \text{and} \qquad L_1 = L_2 = L_3.$$

Proof: As pointed out in the preceeding remark, we have

$$\Psi(x) = \vartheta(x) + \vartheta(x^{1/2}) \ + \ \vartheta(x^{1/3}) \ + \ \cdots .$$

Thus $\vartheta(x) \leq \Psi(x)$. Also

$$\Psi(x) = \sum_{p\leq x} \left[\frac{\log x}{\log p}\right] \log p \leq \sum_{p\leq x} \log x = \pi(x)\log x.$$

Hence we have

$$\frac{\vartheta(x)}{x} \leq \frac{\Psi(x)}{x} \leq \frac{\pi(x)\log x}{x}.$$

Taking lim sup we get $L_2 \le L_3 \le L_1$. To complete the proof of $L_1 = L_2 = L_3$, it suffices to show $L_2 \ge L_1$. To this end, let $0 < \alpha < 1$ and $x > 1$. Then

$$\vartheta(x) \ge \sum_{x^\alpha < p \le x} \log p \ge \alpha \log x \sum_{x^\alpha < p \le x} 1 = \alpha \log x \, (\pi(x) - \pi(x^\alpha)) \ge \alpha \log x \, (\pi(x) - x^\alpha) \, ,$$

since $\pi(x^\alpha) \le x^\alpha$. Thus we have

$$\frac{\vartheta(x)}{x} \ge \alpha \frac{\pi(x) \log x}{x} - \alpha \frac{\log x}{x^{1-\alpha}}.$$

Since $\lim_{x\to\infty} \frac{\log x}{x^{1-\alpha}} = 0$, we conclude that

$$\overline{\lim}_{x\to\infty} \frac{\vartheta(x)}{x} \ge \alpha \overline{\lim}_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}}.$$

Thus $L_2 \ge \alpha L_1$ and taking the limit as $\alpha \to 1^-$, we conclude that $L_2 \ge L_1$.

Similar arguments can be used to show $l_1 = l_2 = l_3$ and we leave this to the read as an exercise.

Remark 3. *In view of the above theorem, note that PNT follows if we can show that $l_2 = L_2 = 1$. The main goal of the remainder of this paper is to prove this fact.*

We now prove a weaker form of (25) mentioned above. To this end, for any positive integer n, let

$$N = \binom{2n}{n} = \frac{(2n)!}{n!n!} = \frac{(n+1)(n+2)\cdots 2n}{1 \cdot 2 \cdots n}.$$

Then

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} > \binom{2n}{n} = N.$$

Since $\binom{2n}{k} \le \binom{2n}{n}$ for each k, we also have

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \le \sum_{k=0}^{2n} \binom{2n}{n} = N(2n+1).$$

We note that

$$\frac{N}{2} = \frac{1}{2}\binom{2n}{n} = \frac{(2n)!}{(n!)^2}\frac{n}{2n} = \frac{(2n-1)!}{n!(n-1)!} = \binom{2n-1}{n-1}$$

Thus $N/2$ is divisible by every prime in the range $n < p \le 2n-1$ and hence divisible by their product. In particular, we have

$$\prod_{n<p\le 2n} p \le N.$$

Combining the preceeding inequalities, we have proved

$$\prod_{n<p\le 2n} p \le N < 2^{2n} \le N(2n+1). \tag{28}$$

We also note that

$$\prod_{n<p\le 2n} p \ge \prod_{n<p\le 2n} n = n^{\pi(2n)-\pi(n)} \quad \text{and} \quad 2^n \le \prod_{k=1}^{n} \frac{k+n}{k} = \frac{(n+1)(n+2)\cdots 2n}{1 \cdot 2 \cdots n} = N. \tag{29}$$

On the other hand, if p^α is the largest power of a prime p that divides $n!$, then

$$\alpha = \sum_{m=1}^{\infty} \left[\frac{n}{p^m}\right].$$

(For the proof of this, see [1], page 67.) Since $N = \frac{(2n)!}{n!n!}$, any prime that divides N must divide $(2n)!$ and hence is less than $2n$. Thus if we write

$$N = \prod_{p \le 2n} p^{\alpha(p)},$$

then

$$\alpha(p) = \sum_{m=1}^{\infty} \left(\left[\frac{2n}{p^m}\right] - 2\left[\frac{n}{p^m}\right]\right).$$

Note that $p^m > 2n$ (and hence $\left[\frac{2n}{p^m}\right] = 0$) if and only if $m > (\log 2n)/\log p$. Thus the above sum contains no more than $\left[\frac{\log 2n}{\log p}\right]$ terms. Also each term is either 0 or 1. Hence $\alpha(p) \le \left[\frac{\log 2n}{\log p}\right] \le \frac{\log 2n}{\log p}$. But then $p^{\alpha(p)} \le 2n$ and we have proved that

$$N \le (2n)^{\pi(2n)}. \tag{30}$$

From (28), (29) and (30) we get

$$2^n \le (2n)^{\pi(2n)} \quad \text{and} \quad n^{\pi(2n)-\pi(n)} \le 2^{2n}. \tag{31}$$

Taking the logarithm of the first inequality in (31) yields

$$n \log 2 \le \pi(2n) \cdot \log(2n).$$

For any real $x \ge 3$, let $n = [x/2]$. Then $n \le x/2 < n+1$ and hence

$$\frac{x}{2}\log 2 \le (n+1)\log 2 = n\log 2 + \log 2 \le \pi(2n)\log(2n) + \log 2 \le \pi(x)\log(x) + \log 2 \le \frac{3}{2}\pi(x)\log x,$$

since $2\log 2 \le \pi(x)\log x$ for all $x \ge 3$. Thus

$$c \cdot \frac{x}{\log x} \le \pi(x) \tag{32}$$

for any $x \ge 3$ and $c = (\log 2)/3 = 0.231049\cdots$. This is much weaker than (25).

Taking the logarithm of the second inequality in (31) and dividing the resulting inequality by $\log n$, we get

$$\pi(2n) - \pi(n) \le \frac{2n}{\log n} \cdot \log 2.$$

Now let $x \ge 6$ be any real number, and let $n = [x/2]$. Then $\pi(x/2) = \pi(n)$ and $\pi(x) \le \pi(2n) + 1$. Thus

$$\pi(x) - \pi(x/2) \le \pi(2n) - \pi(n) + 1 \le \frac{2n}{\log n}\log 2 + 1 = \frac{2n\log 2 + \log n}{\log n} \le \frac{x\log 2 + \log(x/2)}{\log(x/2-1)}.$$

For $x \ge 6$, it can be shown that

$$\frac{x\log 2 + \log(x/2)}{\log(x/2-1)} \le \frac{2x\log 2}{\log(x/2)}.$$

We use this in the above inequality to conclude that

$$\left(\pi(x) - \pi\left(\frac{x}{2}\right)\right)\log\left(\frac{x}{2}\right) \le 2x\log 2,$$

which implies

$$\pi(x) - \pi\left(\frac{x}{2}\right) \leq \frac{2x\log 2}{\log x - \log 2}.$$

From this and the fact that $\pi(x/2) \leq x/2$, we conclude that

$$\begin{aligned}
\pi(x)\log x - \pi\left(\frac{x}{2}\right)\log\left(\frac{x}{2}\right) &= \pi(x)\log x - \pi\left(\frac{x}{2}\right)\log x + \pi\left(\frac{x}{2}\right)\log 2 \\
&\leq \left(\pi(x) - \pi\left(\frac{x}{2}\right)\right)\log x + \left(\frac{\log 2}{2}\right)x \\
&\leq x\left(\frac{2\log 2\log x}{\log x - \log 2} + \frac{\log 2}{2}\right) \leq (4\log 2)x,
\end{aligned}$$

for $x \geq 2^3 = 8$. One can easily check that the last inequality is valid for $0 < x \leq 8$.

Now fix y and let $x = y/2^m$ for $m = 0, 1, \cdots, N$ in the above inequality and sum over m to get

$$\pi(y)\log y - \pi\left(\frac{y}{2^{N+1}}\right)\log\left(\frac{y}{2^{N+1}}\right) \leq (8\log 2)y\left(1 - \left(\frac{1}{2}\right)^{N+1}\right).$$

Taking large N yields

$$\pi(y) \leq d\frac{y}{\log y}, \tag{33}$$

where $d = 8\log 2 = 5.54518\cdots$. Again this is a weaker form of (25).

We can improve on the constants occuring in (32) and (33). Taking the logarithm in (28), we get

$$\log\left(\prod_{n<p\leq 2n} p\right) \leq \log N < 2n\log 2,$$

which yields

$$\vartheta(2n) - \vartheta(n) < 2n\log 2.$$

Put $n = 2^m$ and add the result for $m = 0,\ 1, \cdots, k$, to obtain

$$\sum_{m=0}^{k}\{\vartheta(2^{m+1}) - \vartheta(2^m)\} < \sum_{m=0}^{k}\left(2^{m+1}\log 2\right) < 2^{k+2}\log 2.$$

From this we conclude that

$$\frac{\vartheta(2^{k+1})}{2^k} < 4\log 2.$$

Finally for any $x > 2$, choose k so that $2^k < x \leq 2^{k+1}$ and apply the last inequality to obtain

$$\frac{\vartheta(x)}{x} \leq 4\log 2 = c_2.$$

This implies that $L_2 \leq 4\log 2 = 2.77529\cdots = c_2$.

To prove $l_3 = \underline{\lim}_{x\to\infty}\frac{\Psi(x)}{x} \geq c_1 = .6932$, we observe that

$$\Psi(2n) = \sum_{p\leq 2n}\left[\frac{\log 2n}{\log p}\right]\log p = \sum_{p\leq 2n} M_p\log p = \log\left(\prod_{p\leq 2n} p^{M_p}\right) \geq \log N,$$

where $M_p = [\frac{\log 2n}{\log p}]$. On the other hand, $2^{2n} \le (2n+1)N$ implies $N \ge \frac{2^{2n}}{2n+1}$ and hence

$$\log N \ge 2n \log 2 - \log(2n+1).$$

Therefore we have

$$\Psi(2n) \ge \log N \ge 2n \log 2 - \log(2n+1).$$

For $x > 2$, let $n = \left[\frac{x}{2}\right]$. Then $n \ge 1$, $n > \frac{x}{2} - 1$, and $2n \le x$. Hence

$$\Psi(x) \ge \Psi(2n) \ge 2n \log 2 - \log(2n+1) \ge 2(\frac{x}{2} - 1) \log 2 - \log(x+1) = (x-2)\log 2 - \log(x+1),$$

and dividing the first and last terms by x, we obtain

$$\frac{\Psi(x)}{x} \ge \frac{(x-2)\log 2}{x} - \frac{\log(x+1)}{x}.$$

Taking lim inf and noting that $\lim_{x\to\infty} \frac{\log(x+1)}{x} = 0$, we conclude that

$$l_3 \ge \log 2 = .6932\cdots,$$

as desired.

We now return to the proof of $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$. We first prove the following theorem.

Theorem 12. *Let*

$$\rho(x) = \sum_{p \le x} \frac{\log p}{p}. \tag{34}$$

Then the following statements are equivalent.

(1) $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$.

(2) $\lim_{x\to\infty} (\rho(x) - \log x)$ *exists.*

Proof: We will show (2) ⇒ (1). Note that this is enough for our main objective here. We leave it to the reader to prove (1) ⇒ (2).

Let $E = \lim_{x\to\infty} \Big(\rho(x) - \log x\Big)$. We observe that

$$n\left\{\rho(n) - \rho(n-1)\right\} = \begin{cases} \log p, & \text{if } n = p \\ 0, & \text{else,} \end{cases}$$

and hence we have

$$\begin{aligned} \vartheta(x) &= \sum_{p \le x} \log p \\ &= \sum_{2 \le n \le x} \Big\{\rho(n) - \rho(n-1)\Big\} n \\ &= \sum_{2 \le n \le x} \Big\{\Big(\rho(n) - \log n\Big) - \Big(\rho(n-1) - \log(n-1)\Big)\Big\} n + \sum_{2 \le n \le x} n \log\left(\frac{n}{n-1}\right). \end{aligned}$$

Let

$$F_1(x) = \sum_{2 \le n \le x} \left\{\Big(\rho(n) - \log n\Big) - \Big(\rho(n-1) - \log(n-1)\Big)\right\} n,$$

and

$$F_2(x) = \sum_{2\le n\le x} n\log\left(\frac{n}{n-1}\right).$$

Claim

$$F_1(x) = o(x) \qquad \text{and} \qquad F_2(x) = x + \mathcal{O}(\log x).$$

Note then that

$$\vartheta(x) = F_1(x) + F_2(x) = o(x) + x + \mathcal{O}(\log x) = x + o(x)$$

and hence

$$\lim_{x\to\infty}\frac{\vartheta(x)}{x} = 1.$$

To prove the claim, we rewrite F_1 as

$$\begin{aligned}
F_1(x) &= \sum_{2\le n\le x} n\Big\{\rho(n)-\log n\Big\} - \sum_{2\le n\le x} n\Big\{\rho(n-1)-\log(n-1)\Big\}\\
&= \sum_{2\le n\le x} n\Big\{\rho(n)-\log n\Big\} - \sum_{1\le n\le x-1} (n+1)\Big\{\rho(n)-\log(n)\Big\}\\
&= \Big(\rho([x])-\log([x])\Big)[x] + \sum_{2\le n\le x-1} n\Big\{\rho(n)-\log n\Big\}\\
&\quad - \sum_{1\le n\le x-1} n\Big\{\rho(n)-\log(n)\Big\} - \sum_{1\le n\le x-1}\Big\{\rho(n)-\log(n)\Big\}\\
&= \Big(\rho([x])-\log([x])\Big)\Big([x]+1\Big) - \sum_{1\le n\le x}\Big\{\rho(n)-\log(n)\Big\}.
\end{aligned}$$

Let $\delta(x) = \rho(x) - \log x - E$. Then by assumption, $\lim_{x\to\infty}\delta(x) = 0$. That is $\delta(x) = o(1)$ and we have

$$\begin{aligned}
F_1(x) &= \Big(\delta([x])+E\Big)\Big([x]+1\Big) - \sum_{1\le n\le x}\Big(\delta(n)+E\Big)\\
&= \Big(\delta([x])+E\Big)\Big([x]+1\Big) - [x]\cdot E - \sum_{1\le n\le x}\delta(n)\\
&= E + \delta([x])\Big([x]+1\Big) - \sum_{1\le n\le x}\delta(n).
\end{aligned}$$

But

$$\sum_{1\le n\le x}\delta(n) = o(x) \;\text{ and }\; \delta([x])\left([x]+1\right) = o(x).$$

Therefore,

$$F_1(x) = E + o(x) = o(x).$$

Also

$$\begin{aligned}
F_2(x) &= \sum_{2\le n\le x} n\log\left(\frac{n}{n-1}\right) = \sum_{2\le n\le x} n\log\left(1+\frac{1}{n-1}\right)\\
&= 2\log 2 + \sum_{3\le n\le x} n\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{1}{n-1}\right)^k\\
&= 2\log 2 + \sum_{3\le n\le x}\left\{\frac{n}{n-1} + \sum_{k=2}^{\infty} n\frac{(-1)^{k+1}}{k(n-1)^k}\right\}.
\end{aligned}$$

Since

$$\left|\sum_{k=2}^{\infty} n \frac{(-1)^{k+1}}{k(n-1)^k}\right| \leq \frac{n}{2}\sum_{k=2}^{\infty}\left(\frac{1}{n-1}\right)^k = \frac{n}{2(n-1)(n-2)} = \mathcal{O}\left(\frac{1}{n}\right),$$

we have

$$\begin{aligned} F_2(x) &= 2\log 2 + \sum_{3\leq n\leq x}\left\{\frac{n}{n-1} + \mathcal{O}\left(\frac{1}{n}\right)\right\} \\ &= 2\log 2 + \sum_{3\leq n\leq x}\left(1+\frac{1}{n-1}\right) + \mathcal{O}\left(\sum_{3\leq n\leq x}\frac{1}{n}\right) \\ &= 2\log 2 + ([x]-2) + \mathcal{O}(\log x) \\ &= [x] + \mathcal{O}(\log x) \\ &= x + \mathcal{O}(\log x), \end{aligned}$$

as claimed and we have completed the proof of the theorem.

To complete the proof of PNT we must prove

Theorem 13. $\lim_{x\to\infty}(\rho(x) - \log x)$ *exists.*

We shall present the beautiful proof of D. J. Newman [14]. His proof depends on the following convergence theorem, due to Ingham and dating back to 1929. The proof by Ingham uses Fourier analysis while Newman's proof of the convergence theorem uses only the theory of complex analysis. For comments upon and reviews of Newman's proof of PNT, see the articles by D. Zagier [22] and J. Korevaar [11].

Theorem 14. (Convergence Theorem) *Let* $\{a_n\}$ *be a bounded sequence of complex numbers. For* $Re(s) = \sigma > 1$, *assume that*

$$F(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

is analytic in an open set containing the region $Re(s) \geq 1$. *Then the series* $\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ *converges for* $\Re(s) \geq 1$.

We shall return to the proof of the convergence theorem later. Let us assume its validity for now and use it to proof Theorem 13.

Proof of Theorem 13. Define

$$f(s) = \sum_{n=1}^{\infty}\frac{\rho(n)}{n^s} = \sum_{n=1}^{\infty}\left(\sum_{p\leq n}\frac{\log p}{p}\right)\frac{1}{n^s}.$$

Clearly $\rho(n) \leq n$ and hence the series defining $f(s)$ converges absolutely for $\sigma > 2$. We rewrite f as

$$f(s) = \sum_{p}\frac{\log p}{p}\left(\sum_{n\geq p}^{\infty}\frac{1}{n^s}\right) \tag{35}$$

and use (18) with $a = p$ to obtain

$$\sum_{n\geq p}^{\infty}\frac{1}{n^s} = \frac{1}{(s-1)p^{s-1}} + s\int_p^{\infty}\frac{1-\{t\}}{t^{s+1}}\,dt, \tag{36}$$

where $\{t\} = t - [t]$ is the fractional part of the real number t. Define

$$A_p(s) = \frac{1}{p^s} - \frac{1}{p^s-1} + \frac{s(s-1)}{p}\int_p^\infty \frac{1-\{t\}}{t^{s+1}}\,dt = -\frac{1}{p^s(p^s-1)} + \frac{s(s-1)}{p}\int_p^\infty \frac{1-\{t\}}{t^{s+1}}\,dt,$$

so that

$$\sum_{n\geq p}^{\infty} \frac{1}{n^s} = \frac{p}{s-1}\left(\frac{1}{p^s-1} + A_p(s)\right).$$

Using this in (35), we get

$$f(s) = \frac{1}{s-1}\left(\sum_p \frac{\log p}{p^s-1} + \sum_p A_p(s)\,\log p\right). \tag{37}$$

Clearly $A_p(s)$ is analytic in $Re(s) = \sigma > 0$(one can appeal to the Dominated Convergence Theorem to justify differentiating inside the integral to see this) and is bounded there by

$$\frac{1}{p^\sigma(p^\sigma-1)} + \frac{|s(s-1)|}{\sigma p^{\sigma+1}}.$$

For $p \geq 5$ and $\sigma > 1/2$, we note that $p^\sigma - 1 > p^\sigma/2$ and hence

$$|A_p(s)\,\log p| \leq \frac{2\log p}{p^{2\sigma}} + \frac{|s(s-1)|}{\sigma}\frac{\log p}{p^{\sigma+1}}.$$

Clearly

$$\sum_p \frac{\log p}{p^{2\sigma}} < \infty \quad \text{and} \quad \sum_p \frac{\log p}{p^{\sigma+1}} < \infty \quad \text{for } \sigma > \frac{1}{2}.$$

By the Weierstrass M-test we conclude that the series defined by

$$A(s) = \sum_p A_p(s)\,\log p$$

is analytic in $Re(s) > \frac{1}{2}$ and by (37) we may express $f(s)$ as

$$f(s) = \frac{1}{s-1}\left(\sum_p \frac{\log p}{p^s-1} \;+\; A(s)\right). \tag{38}$$

From the Euler Product Formula (11) for $\zeta(s)$, (yes! finally $\zeta(s)$ is coming to the scene), we have $\zeta(s) = \prod_p (1-p^{-s})^{-1}$. Upon logarithmic differentiation, we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log p}{p^s-1}, \quad \text{valid for } \sigma > 1. \tag{39}$$

Using (39) in (38) we see that

$$f(s) = \frac{1}{s-1}\left(-\frac{\zeta'(s)}{\zeta(s)} \;+\; A(s)\right). \tag{40}$$

By Theorem 7, $|\zeta(1+it)| \neq 0$ and from Euler's Product for $\zeta(s)$, we know that $\zeta(s) \neq 0$ for $\sigma > 1$. Since $\zeta(s)$ has a simple pole at $s = 1$, we see that $(s-1)\zeta(s) \neq 0$ in $\sigma \geq 1$. We also observe that

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + g(s),$$

where $g(s)$ is analytic at $s = 1$. Note then that

$$f(s) = \frac{1}{s-1}\left(\frac{1}{s-1} + g(s) + A(s)\right).$$

Thus $f(s)$ is meromorphic in an open set containing $\Re(s) \geq 1$, holomorphic there except at $s = 1$, and with the principal part

$$\frac{1}{(s-1)^2} + \frac{c}{s-1}$$

at $s = 1$, wheres c is a complex number.

We have proved that the function

$$F(s) = f(s) + \zeta'(s) - c\zeta(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where

$$a_n = \sum_{p \leq n} \frac{\log p}{p} - \log n \; - \; c = \rho(n) - \log n \; - \; c,$$

is analytic in an open set containing $\sigma \geq 1$. By the Convergence Theorem we conclude that

$$\sum_{n=1} \frac{a_n}{n}$$

converges. Our theorem now follows if we prove the

Claim:

$$\lim_{n \to \infty} a_n = 0.$$

To prove the claim, let $0 < \epsilon < 1$ be given. By Cauchy criterion for convergence of a sequence, there exists $N_0 > 0$ such that for $K \geq N \geq M > N_0$, we have

$$\sum_{n=N}^{K} \frac{a_n}{n} \; \leq \; \epsilon^2 \tag{41}$$

and

$$\sum_{n=M}^{N} \frac{a_n}{n} \; \geq \; -\epsilon^2. \tag{42}$$

But then for $N \leq n \leq K$ we have

$$a_n - a_N = \sum_{N<p\leq n} \frac{\log p}{p} + \log N - \log n \geq \log\left(\frac{N}{n}\right) \geq \log\left(\frac{N}{K}\right).$$

We now choose K so that $\frac{1}{K}(K - N + 1) \geq \frac{\epsilon}{1+\epsilon}$ and $\log\left(\frac{N}{K}\right) > -\epsilon$. Then, for $N \leq n \leq K$, we have $a_n \geq a_N - \epsilon$. It follows that

$$\sum_{n=N}^{K} \frac{a_n}{n} \geq (a_N - \epsilon) \sum_{n=N}^{K} \frac{1}{n},$$

which is equivalently to

$$a_N - \epsilon \; \leq \; \frac{\sum_{n=N}^{K} \frac{a_n}{n}}{\sum_{n=N}^{K} \frac{1}{n}}.$$

But $\sum_{n=N}^{K} \frac{1}{n} \geq \frac{1}{K}(K-N+1) \geq \frac{\epsilon}{1+\epsilon}$. Therefore,

$$a_N - \epsilon \leq \left(\sum_{n=N}^{K} \frac{a_n}{n}\right)\left(\frac{1+\epsilon}{\epsilon}\right). \tag{43}$$

Combining (41) and (43), we get

$$a_N \leq 2\epsilon + \epsilon^2. \tag{44}$$

On the other hand, for $M \leq n \leq N$, we have

$$a_N - a_n = \sum_{n<p\leq N} \frac{\log p}{p} + \log\left(\frac{n}{N}\right) \geq \log\left(\frac{n}{N}\right) \geq \log\left(\frac{M}{N}\right).$$

We now choose $M < N$ so that $\log\left(\frac{M}{N}\right) \geq -\frac{\epsilon}{1-\epsilon}$ and $\frac{N-M+1}{N} \geq \epsilon$. This gives

$$a_n \leq a_N + \frac{\epsilon}{1-\epsilon}$$

and hence

$$\sum_{n=M}^{N} \frac{a_n}{n} \leq \left(a_N + \frac{\epsilon}{1-\epsilon}\right) \sum_{n=M}^{N} \frac{1}{n}.$$

Assume $a_N + \frac{\epsilon}{1-\epsilon} \leq 0$. Since $\sum_{n=M}^{N} \frac{1}{n} \geq \frac{1}{N}(N-M+1) \geq \epsilon$, the above inequality implies that

$$\sum_{n=M}^{N} \frac{a_n}{n} \leq \left(a_N + \frac{\epsilon}{1-\epsilon}\right)\epsilon.$$

From this and (42) we obtain

$$a_N \geq -\frac{\epsilon(2-\epsilon)}{1-\epsilon}. \tag{45}$$

If $a_N + \frac{\epsilon}{1-\epsilon} > 0$, then

$$a_N > -\frac{\epsilon}{1-\epsilon} \geq -\frac{\epsilon(2-\epsilon)}{1-\epsilon},$$

since $0 < \epsilon < 1$. In either cases, (45) holds.

Combining (44) and (45), we conclude that

$$\lim_{n\to\infty} a_n = 0.$$

Note then that for any real number x, if we let $n = [x]$, then

$$\lim_{x\to\infty} (\rho(x) - \log x) = \lim_{n\to\infty} (\rho(n) - \log n) = \lim_{n\to\infty} (a_n + c) = c$$

and the theorem follows.

The proof of PNT is now complete except for the proof of the Convergence Theorem. We shall present this proof now.

Proof of the Convergence Theorem: Suppose $|a_n| \leq K$ and let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be analytic in an open set containing $\sigma \geq 1$. Since we can replace a_n by $\frac{a_n}{K}$, we may assume $K = 1$. Fix w with $\Re(w) \geq 1$ and let $\epsilon > 0$ be given. Let $R = \max\{\frac{3}{\epsilon}, 1\}$. Then $F(s+w)$ is analytic in $\Re(s) \geq 0$. Hence there exist positive numbers δ and M, depending on R, with $0 < \delta < \frac{1}{2}$ such that $F(s+w)$ is analytic and

$$|F(s+w)| \leq M \ \text{ in } \ -\delta \leq \Re(s), \ \ |s| \leq R. \tag{46}$$

Let Γ be the curve, with counter-clockwise orientation, given by

$$\Gamma = \{s \in \mathcal{C} |\ \Re(s) > -\delta, \ |s| = R\ \} \bigcup \{s \in \mathcal{C} | \Re(s) = -\delta, |s| \leq R\}.$$

Let Γ_r be the portion of Γ for which $\Re(s) = \sigma > 0$ and let Γ_l be the portion for which $\sigma \leq 0$. By the Cauchy residue theorem, we have

$$\int_\Gamma F(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds = 2\pi i \text{Res}\left(F(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right); s=0\right) = 2\pi i F(w). \tag{47}$$

On Γ_r, $F(s+w)$ is given by its series and we may write it as $F(s+w) = S_N(s+w) + r_N(s+w)$, where

$$S_N(s+w) = \sum_{n=1}^{N} \frac{a_n}{n^{s+w}} \qquad \text{and} \qquad r_N(s+w) = \sum_{n=N+1}^{\infty} \frac{a_n}{n^{s+w}}.$$

Note then that $S_N(s+w)$ is entire and hence

$$\int_{|s|=R} S_N(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds = 2\pi i S_N(w).$$

On the other hand,

$$\int_{|s|=R} S_N(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds = \int_{\Gamma_r} S_N(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds \ + \ \int_{-\Gamma_r} S_N(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds,$$

where $-\Gamma_r$ is the reflection of Γ_r through the origin. Therefore, we have

$$2\pi i S_N(w) \ = \ \int_{\Gamma_r} S_N(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds \ + \ \int_{-\Gamma_r} S_N(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds.$$

In the second integral we change variable from s to $-s$ to obtain

$$2\pi i S_N(w) \ = \ \int_{\Gamma_r} S_N(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds \ + \ \int_{\Gamma_r} S_N(w-s)N^{-s}\left(\frac{1}{s}+\frac{s}{R^2}\right)ds. \tag{48}$$

Combining (47) and (48), and noting that $F(w+s) - S_N(s+w) = r_N(s)$, we get

$$\begin{aligned} 2\pi i\left(F(w) - S_N(w)\right) \ &= \ \int_{\Gamma_r} \left\{r_N(s+w)N^s - \frac{S_N(w-s)}{N^s}\right\}\left(\frac{1}{s}+\frac{s}{R^2}\right)ds \\ &\quad + \int_{\Gamma_l} F(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds. \end{aligned} \tag{49}$$

To estimate the integrals in (49), we observe the following bounds.

On $|s| = R$, we have

$$\left|\frac{1}{s}+\frac{s}{R^2}\right| = \left|\frac{\overline{s}}{|s|^2}+\frac{s}{R^2}\right| = \frac{2|\sigma|}{R^2}. \tag{50}$$

On $\sigma = -\delta, \ \ |s| \le R$, we have

$$\left|\frac{1}{s} + \frac{s}{R^2}\right| = \left|\frac{R^2 + s^2}{sR^2}\right| \le \frac{1}{\delta}\left(\frac{R^2 + |s|^2}{R^2}\right) \le \frac{2}{\delta}, \tag{51}$$

where we have used the fact that $|s| \ge |\sigma| = \delta$.

Since $|a_n| \le 1$ and $\Re(w) \ge 1$, we have, for $s \in \Gamma_r$,

$$|r_N(s+w)| \le \sum_{n=N+1}^{\infty} \frac{1}{n^{\sigma+1}} \le \int_N^{\infty} \frac{du}{u^{\sigma+1}} du = \frac{1}{\sigma N^{\sigma}} \tag{52}$$

and

$$|S_N(w-s)| \le \sum_{n=1}^{N} n^{\sigma-1} \le N^{\sigma-1} + \int_0^N u^{\sigma-1} du \ = \ N^{\sigma}\left(\frac{1}{N} \ + \ \frac{1}{\sigma}\right). \tag{53}$$

Combining (50), (52) and (53) we obtain, for $s \in \Gamma_r$,

$$\left|\left[r_N(s+w)N^s - \frac{S_N(w-s)}{N^s}\right]\left(\frac{1}{s} + \frac{s}{R^2}\right)\right| \le \left(\frac{1}{\sigma} + \frac{1}{\sigma} + \frac{1}{N}\right)\frac{2\sigma}{R^2} \le \frac{4}{R^2} + \frac{2}{RN}.$$

Noting that the length of Γ_r is πR and using the $M-L$ theorem we have

$$\left|\int_{\Gamma_r}\left[r_N(s+w)N^s - \frac{S_N(w-s)}{N^s}\right]\left(\frac{1}{s} + \frac{s}{R^2}\right) ds\right| \le \frac{4\pi}{R} + \frac{2\pi}{N}. \tag{54}$$

On Γ_l, we write

$$\begin{aligned}\int_{\Gamma_l} F(s+w)N^s\left(\frac{1}{s} + \frac{s}{R^2}\right) ds \quad &= \quad \int_{-R}^{R} F(-\delta+it+w)N^{-\delta+it}\left(\frac{1}{-\delta+it} + \frac{-\delta+it}{R^2}\right) idt \\ &\quad + \int_{C_1\cup C_2} F(s+w)N^s\left(\frac{1}{s} + \frac{s}{R^2}\right) ds,\end{aligned} \tag{55}$$

where

$$C_1 = \{s = \sigma + it \in \Gamma | -\delta \le \sigma \le 0 \text{ and } t > 0\} \quad \text{and} \quad C_2 = \{s = \sigma + it \in \Gamma | -\delta \le \sigma \le 0 \text{ and } t < 0\}.$$

By (46) and (50), with $s = -\sigma + it$, we have

$$\begin{aligned}\left|\int_{C_1\cup C_2} F(s+w)N^s\left(\frac{1}{s} + \frac{s}{R^2}\right) ds,\right| \quad &\le \quad 2\int_{-\delta}^{0} MN^{\sigma}\frac{2|\sigma|}{R^2} d\sigma \\ &= \quad \frac{-4M}{R^2}\int_{-\delta}^{0} \sigma N^{\sigma} d\sigma \\ &= \quad \frac{4M}{R^2}\left(\frac{1 - N^{-\delta} - \delta N^{-\delta}\log N}{\log^2 N}\right) \le \frac{4M}{R^2\log^2 N}.\end{aligned} \tag{56}$$

By (46) and (51), with $s = -\delta + it$, we have

$$\left|\int_{-R}^{R} F(-\delta+it+w)N^{-\delta+it}\left(\frac{1}{-\delta+it} + \frac{-\delta+it}{R^2}\right) idt\right| \le \int_{-R}^{R} MN^{-\delta}\frac{2}{\delta} dt \le \frac{4MR}{\delta N^{\delta}}. \tag{57}$$

We now combine (56) and (57) to get

$$\left|\int_{\Gamma_l} F(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)ds\right| \leq \frac{4MR}{\delta N^\delta}+\frac{4M}{R^2\log^2 N}. \tag{58}$$

Using (54) and (58) in (49) we get

$$|F(w)-S_N(w)| \leq \frac{2}{R}+\frac{1}{N}+\frac{MR}{\delta N^\delta}+\frac{M}{R^2\log^2 N}.$$

Since $R \geq \frac{3}{\epsilon}$, we can take N large enough to conclude that $|F(w)-S_N(w)| < \epsilon$ thereby proving the fact that the infinite series $\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ converges for $\Re(s) \geq 1$. This completes the proof of the Convergence Theorem.

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