THE ERROR ZETA FUNCTION

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Abstract. This paper investigates a new special function referred to as the error zeta function. Derived as a fractional generalization of hypergeometric zeta functions, the error zeta function is shown to exhibit many properties analogous to its hypergeometric counterpart. These new properties are treated in detail, including an intimate connection to generalized Bernoulli numbers and a pre-functional equation satisfied by the error zeta function.

1. INTRODUCTION

In [3] the authors investigated an interesting generalization of the Riemann zeta function based on its integral representation:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx. \tag{1.1}$$

In particular, the denominator $e^x - 1$ in (1.1) was replaced by an arbitrary Taylor remainder $e^x - T_{N-1}(x)$, where N is a positive integer and $T_N(x)$ is the Taylor polynomial of e^x at the origin having degree N. This defines a family of higher-order functions of zeta-type referred to as *hypergeometric zeta functions*:

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}\, dx \qquad (N = 1, 2, ...). \tag{1.2}$$

Observe that $\zeta_1(s) = \zeta(s)$. In the same paper, we developed the analytic continuation of $\zeta_N(s)$ to the entire complex plane, except for N simple poles at $s = 1, 0, -1, \cdots, 2-N$, and established many properties analogous to those satisfied by Riemann's zeta function, including a pre-functional equation and results involving generalized Bernoulli numbers.

In this present work, we investigate a continuous version of hypergeometric functions by generalizing definition (1.2) to all positive values of N. Since

$$e^x - T_{N-1}(x) = \frac{x^N [{}_1F_1(1, N+1; x)]}{\Gamma(N+1)}, \tag{1.3}$$

where ${}_1F_1(a, b; x)$ is the hypergeometric series

$${}_1F_1(a, b; x) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \tag{1.4}$$

definition (1.2) can be rephrased as

$$\zeta_N(s) = \frac{\Gamma(N+1)}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s-2}}{{}_1F_1(1, N+1; x)} dx. \tag{1.5}$$

We now take (1.5) as our new definition of $\zeta_N(s)$, which is formally defined for *all* positive values of N.

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We focus in particular on the half-integer case $N = 1/2$ where the error function $\mathrm{erf}(x)$ makes its appearance, hence motivating the title of this paper. Since

$$ {}_1F_1(1, 1/2; x) = \frac{\sqrt{\pi}}{2\sqrt{x}}\; e^x \mathrm{erf}(\sqrt{x}), \tag{1.6} $$

where $\mathrm{erf}(x)$ is defined by

$$ \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \tag{1.7} $$

it follows from a change of variable that

$$ \zeta_{1/2}(s) = \frac{2\Gamma(3/2)}{\sqrt{\pi}\Gamma(s-1/2)} \int_0^\infty \frac{x^{s-3/2}e^{-x}}{\mathrm{erf}(\sqrt{x})} dx = \frac{2}{\Gamma(s-1/2)} \int_0^\infty \frac{x^{2(s-1)}e^{-x^2}}{\mathrm{erf}(x)} dx. \tag{1.8} $$

It is discovered that this fractional hypergeometric zeta function of order $N = 1/2$, which we now refer to as the *error zeta function*, shares many of the same properties found in hypergeometric zeta functions of integer order, including a pre-functional equation valid for $\Re(s) < 0$ (see Theorem 4.1):

$$ \zeta_{1/2}(s) = \frac{\Gamma(1/2)\Gamma(1-(s-1/2))}{\cos[\pi(s-1/2)]} \sum_{k=1}^\infty r_k^{2s-2} \left\{ \cos[2(s-1)(\pi-\theta_k)] + \cos[2(s-1)\theta_k] \right\}. $$

Here, $z_k = r_k e^{i\theta_k}$ are the non-zero complex zeros of the error function $\mathrm{erf}(z)$ arranged by increasing length. We apply the pre-functional equation above to obtain a bound on error zeta (see Theorem 4.2):

$$ \left|\zeta_{1/2}(s)\right| < \left| \frac{\Gamma(1/2)\Gamma(1-(s-1/2))(e^{|\Im(s)\theta_1|} + e^{|3\pi\Im(s)/4|})}{(2\pi)^{1-s}\cos[\pi(s-1/2)]} \right| \zeta(1-\Re(s), 7/8). $$

However, as with hypergeometric zeta functions of positive integer order, any hope of establishing a functional equation in the future will most likely require knowing the precise locations of these zeros which at the moment is intractable.

This paper is organized as follows. In section 2, we define the error zeta function, establish its convergence on a right half-plane, and develop its series representation. In section 3, we reveal its analytic continuation to the entire complex plane, except at a countably infinite number of poles, and calculate the residues of these poles in terms of fractional Bernoulli numbers. In section 4, we establish a pre-functional equation for the error zeta function that is valid on a left half-plane. Lastly, section 5 (Appendix) establishes a bound on the zeros of the error function claimed in section 4 and provides a list of the first ten of these zeros.

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2. PRELIMINARIES

In this section we formally define the error zeta function, establish a domain of convergence, and demonstrate its series representation.

Definition 2.1. The *fractional hypergeometric zeta function* is defined for all real $N > 0$ as

$$ \zeta_N(s) = \frac{\Gamma(N+1)}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s-2}}{{}_1F_1(1, N+1; x)} dx. \tag{2.1} $$

Remark 2.1. Observe that for $N = 0$, (2.1) reduces to $\zeta_0(s) \equiv 1$. In this paper we shall avoid discussion of this trivial case.

Lemma 2.1. *$\zeta_N(s)$ is absolutely convergent and hence analytic for $\sigma = \Re(s) > 1$.*

Proof. Since the integrand ${}_1F_1(1, N+1; x)$ is increasing on $[0, \infty)$ and grows asymptotically as

$$ {}_1F_1(1, N+1; x) \sim \frac{\Gamma(N+1)e^x}{x^N}(1 + \mathrm{O}(1/x)), \tag{2.2} $$

it follows that there exists a positive constant K such that on $[K, \infty)$,

$$ {}_1F_1(1, N+1; x) \geq \frac{\Gamma(N+1)e^x}{x^N} \tag{2.3} $$

Moreover, ${}_1F_1(1, N+1; x) \geq 1$ on $[0, 1]$. Therefore, for $\Re(s) > 1$,

$$ \begin{aligned} |\zeta_N(s)| &\leq \frac{\Gamma(N+1)}{|\Gamma(s+N-1)|}\left[\int_0^K \left|x^{s-2}\right| dx + \frac{1}{\Gamma(N+1)}\int_K^\infty \left|x^{s-N-2}e^{-x}\right| dx\right] \\ &\leq \frac{\Gamma(N+1)}{|\Gamma(s+N-1)|}\left[\int_0^K \left|x^{\Re(s)-2}\right| dx + \frac{1}{\Gamma(N+1)}\int_K^\infty \left|x^{\Re(s)-N-2}e^{-x}\right| dx\right] \\ &\leq \infty. \end{aligned} $$

This establishes the lemma. □

We now focus on the error zeta function ($N = 1/2$). Using its expression in (1.8), we can develop a series representation for error zeta:

Lemma 2.2. *For $\sigma = \Re(s) > 1$, we have*

$$ \zeta_{1/2}(s) = \sum_{n=1}^{\infty} f_n(1/2, s), \tag{2.4} $$

where

$$ f_n(1/2, s) = \frac{2}{\Gamma(s-1/2)}\int_0^\infty x^{2(s-1)}e^{-x^2}\mathrm{erfc}^{n-1}(x)\, dx. \tag{2.5} $$

Proof. Since $\mathrm{erf}(x) = 1 - \mathrm{erfc}(x)$, where $\mathrm{erfc}(x)$ is the complementary error function, and $\mathrm{erfc}(x) < 1$ for all $x > 0$, we can rewrite the integrand in (1.8) as a geometric series:

$$ \frac{x^{2(s-1)}e^{-x^2}}{\mathrm{erf}(x)} = \frac{x^{2(s-1)}e^{-x^2}}{1 - \mathrm{erfc}(x)} = x^{2(s-1)}e^{-x^2}\sum_{n=0}^{\infty}\mathrm{erfc}^n(x) $$

The lemma now follows by reversing the order of integration and summation because of Dominated Convergence Theorem:

$$ \begin{aligned} \zeta_{1/2}(s) &= \frac{2}{\Gamma(s-1/2)}\int_0^\infty x^{2(s-1)}e^{-x^2}\sum_{n=0}^{\infty}\mathrm{erfc}^n(x)\, dx \\ &= \sum_{n=1}^{\infty}\left[\frac{2}{\Gamma(s-1/2)}\int_0^\infty x^{2(s-1)}e^{-x^2}\mathrm{erfc}^{n-1}(x)\, dx\right]. \end{aligned} $$

□

Remark 2.2. The series representation given by (2.4) above reduces formally to the harmonic series at $s = 1$. This is because

$$ f_n(1/2, 1) = \frac{2}{\Gamma(1/2)}\int_0^\infty e^{-x^2}\mathrm{erfc}^{n-1}(x)\, dx = \left[-\frac{\mathrm{erfc}^n(x)}{n}\right]_0^\infty = \frac{1}{n} $$

It follows that $\zeta_{1/2}(1) = \sum_{n=1}^{\infty} 1/n$ formally represents the harmonic series. This reveals our motivation for normalizing (2.1) as we did with a suitable gamma factor in defining $\zeta_{1/2}(s)$.

3. ANALYTIC CONTINUATION

In this section we develop the analytic continuation of $\zeta_N(s)$ to the entire complex plane by manipulating its integral definition, previously done for hypergeometric zeta functions of positive integer order in [3]. In addition, for $\zeta_{1/2}(s)$, we shall demonstrate a second method via contour integration to perform the analytic continuation in one stroke.

Theorem 3.1. *For $\Re(s) > 0$,*

$$\zeta_N(s) = \frac{\Gamma(N+1)\Gamma(s-1)}{\Gamma(s+N-1)} + \frac{\Gamma(N+1)}{\Gamma(s+N-1)} \int_0^\infty \left(\frac{1}{{}_1F_1(1, N+1; x)} - e^{-x} \right) x^{s-2}\, dx. \tag{3.1}$$

Proof. For $\Re(s) > 1$, we can first rewrite (2.1) as

$$\begin{aligned} \frac{\Gamma(s+N-1)}{\Gamma(N+1)}\zeta_N(s) &= \int_0^1 \frac{x^{s-2}}{{}_1F_1(1, N+1; x)}\, dx + \int_1^\infty \frac{x^{s-2}}{{}_1F_1(1, N+1; x)}\, dx \\ &= \int_0^1 \left(\frac{1}{{}_1F_1(1, N+1; x)} - e^{-x} \right) x^{s-2}\, dx + \int_0^1 x^{s-2} e^{-x} dx \\ &\quad + \int_1^\infty \frac{x^{s-2}}{{}_1F_1(1, N+1; x)}\, dx. \end{aligned} \tag{3.2}$$

Then rewrite

$$\int_0^1 x^{s-2} e^{-x} dx = \Gamma(s-1) - \int_1^\infty x^{s-2} e^{-x} dx. \tag{3.3}$$

This simplifies (3.2) to

$$\frac{\Gamma(s+N-1)}{\Gamma(N+1)}\zeta_N(s) = \Gamma(s-1) + \int_0^\infty \left(\frac{1}{{}_1F_1(1, N+1; x)} - e^{-x} \right) x^{s-2}\, dx, \tag{3.4}$$

as desired. Now observe that the integral appearing in (3.4) is absolutely convergent for all $\Re(s) > 0$, hence (3.4) defines an analytic function for all $\Re(s) > 0$, except for the simple pole due to $\Gamma(s-1)$ at $s = 1$. This establishes (3.1). □

Remark 3.1. Observe from (3.1) that besides having a pole at $s = 1$, $\zeta_N(s)$ also has a zero at $s = 1 - N$ if $0 < N < 1$. Morever, this process of analytic continuation can be repeated to extend the domain of $\zeta_N(s)$ to $\Re(s) > -1$ so that a second pole emerges at $s = 0$ and a second zero at $s = -N$. Therefore, it seems that the error zeta function has an infinite number of poles since each application produces an additional pole. On the other hand, each application also produces an additional zero. For hypergeometric zeta functions defined by (1.2) for positive integers N, infinitely many of these poles and zeros overlap, resulting in a net finite number of poles (cf. [3]). We will have more to say about this behavior in our second approach using contour integration (see Theorem 3.3).

The main advantage in using (3.2) to analytically continue $\zeta_N(s)$ is that it reveals the behavior of $\zeta_N(s)$ near the distinguished pole $s = 1$. The following theorem is valid for all positive real values of N and generalizes the corresponding result stated in [3] for hypergeometric zeta functions of positive integer order.

Theorem 3.2.

$$\lim_{s\to 1} \left[\zeta_N(s) - \frac{N}{s-1} \right] = \log\Gamma(N+1) - N\frac{\Gamma'(N)}{\Gamma(N)}. \tag{3.5}$$

Proof. From (3.1) we have by the Dominated Convergence Theorem that

$$\begin{aligned}\lim_{s\to 1}\left[\zeta_N(s)-\frac{\Gamma(N+1)\Gamma(s-1)}{\Gamma(s+N-1)}\right] &= \frac{\Gamma(N+1)}{\Gamma(N)}\int_0^\infty\left(\frac{1}{{}_1F_1(1,N+1;x)}-e^{-x}\right)\frac{1}{x}\,dx\\ &= \frac{\Gamma(N+1)}{\Gamma(N)}\left(\gamma+\frac{\log\Gamma(N+1)}{N}\right) \qquad (3.6)\\ &= N\gamma+\log\Gamma(N+1), \qquad (3.7)\end{aligned}$$

where γ is Euler's constant. It follows from (3.7) that

$$\begin{aligned}\lim_{s\to 1}\left[\zeta_N(s)-\frac{N}{s-1}\right] &= \lim_{s\to 1}\left[\zeta_N(s)-\frac{\Gamma(N+1)\Gamma(s-1)}{\Gamma(s+N-1)}\right]-\lim_{s\to 1}\left[\frac{N}{s-1}-\frac{\Gamma(N+1)\Gamma(s-1)}{\Gamma(s+N-1)}\right]\\ &= N\gamma+\log\Gamma(N+1)-(N\gamma+N\Gamma'(N)/\Gamma(N))\\ &= \log\Gamma(N+1)-N\Gamma'(N)/\Gamma(N),\end{aligned}$$

as desired. □

Remark 3.2. Observe that (3.5) is analogous to the following classic result for $\zeta(s)$ (cf. [11]):

$$\lim_{s\to 1}\left[\zeta(s)-\frac{1}{s-1}\right]=-\frac{\Gamma'(1)}{\Gamma(1)}=\gamma\approx 0.577. \qquad (3.8)$$

We now take a different approach and follow Riemann (cf. [9]) by using contour integration to develop the analytic continuation of $\zeta_{1/2}(s)$. This will not only allow us to make precise our earlier statement about $\zeta_{1/2}(s)$ having an infinite number of poles but also make explicit the role of the zeros of the error function in determining the values of $\zeta_{1/2}(s)$ at negative integers.

To this end consider the contour integral

$$I(s)=\frac{1}{2\pi i}\int_\gamma \frac{2(-w)^{2s-1}e^{-w^2}}{\sqrt{\pi}\,\mathrm{erf}(w)}\frac{dw}{w}, \qquad (3.9)$$

where the contour γ is taken to be along the real axis from ∞ to $\delta>0$, then counterclockwise around the circle of radius δ, and lastly along the real axis from δ to ∞ (cf. Figure 1). Moreover, we let $-w$ have argument $-\pi$ backwards along ∞ to δ and argument π when going to ∞. Also, we choose the radius δ to be sufficiently small so that there are no roots of $\mathrm{erf}(w)=0$ inside the circle of radius δ besides the trivial root $z_0=0$. This follows from the fact that $z_0=0$ is an isolated zero. It is then clear from this assumption that $I(s)$ must converge for all complex s and therefore defines an entire function. Also, since we are most interested in the properties of $I(s)$ in the limiting case when $\delta\to 0$, we will write $I(s)$ to denote $\lim_{\delta\to 0} I(s)$. No confusion should arise from this abuse of notation.

We begin by evaluating $I(s)$ at integer values of s. To this end, we decompose it as follows:

$$\begin{aligned}I(s) &= \frac{1}{2\pi i}\int_\infty^\delta \frac{2e^{(2s-1)(\log x-\pi i)-w^2}}{\sqrt{\pi}\,\mathrm{erf}(w)}\frac{dx}{x}\\ &\quad+\frac{1}{2\pi i}\int_{|w|=\delta}\frac{2(-w)^{2s-1}e^{-w^2}}{\sqrt{\pi}\,\mathrm{erf}(w)}\frac{dw}{w} \qquad (3.10)\\ &\quad+\frac{1}{2\pi i}\int_\delta^\infty \frac{2e^{(2s-1)(\log x+\pi i)-w^2}}{\sqrt{\pi}\,\mathrm{erf}(w)}\frac{dx}{x}.\end{aligned}$$

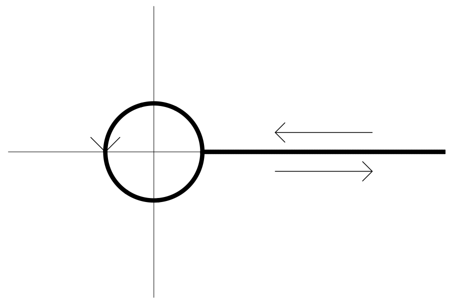

FIGURE 1. Contour γ.

Now, for $s = n/2$, where n is an integer, the two integrations along the real axis in (3.10) cancel and we are left with just the middle integral around the circle of radius δ:

$$I(n) = \frac{1}{2\pi i} \int_{|w|=\delta} \frac{2(-w)^{2s-1} e^{-w^2}}{\sqrt{\pi}\,\mathrm{erf}(w)} \frac{dw}{w}.$$

Since the expression $we^{-w^2}/\mathrm{erf}(w)$ inside the integrand has a removable singularity at the origin, it follows by Cauchy's Theorem that for integers $n \geq 3$,

$$I(n/2) = 0.$$

As for integers $n \leq 2$, we consider the power series expansion

$$\frac{2we^{-w^2}}{\sqrt{\pi}\,\mathrm{erf}(w)} = \sum_{m=0}^{\infty} \frac{B_{1/2,m}}{m!} w^m. \tag{3.11}$$

It now follows from the Residue Theorem that

$$\begin{aligned} I(n) &= \frac{1}{2\pi i} \int_{|w|=\delta} \frac{2(-w)^{2s-1} e^{-w^2}}{\sqrt{\pi}\,\mathrm{erf}(w)} \frac{dw}{w} \\ &= \frac{(-1)^{2n-1}}{2\pi i} \int_{|w|=\delta} \left(\sum_{m=0}^{\infty} \frac{B_{1/2,m}}{m!} w^m \right) \frac{dw}{w^{3-2n}} \\ &= -\frac{B_{1/2,2-2n}}{(2-2n)!}. \end{aligned} \tag{3.12}$$

Remark 3.3. The coefficients $B_{1/2,n}$ generalize the Bernoulli numbers B_n, which arise in the case of the Riemann zeta function. Observe that $B_{1/2,n} = 0$ for n odd since $we^{-w^2}/\mathrm{erf}(w)$ is an even function. For n even, B_n can be found recursively by the relation

$$B_{1/2,0} = 1, \qquad \sum_{m=0}^{n} \frac{(-1)^m n! B_{1/2,2m}}{(2m!)(n-m)!(2(n-m)+1)} = 1 \ \ (n \geq 1).$$

Here are the first few values of $B_{1/2,n}$:

$$\begin{array}{lll} B_{1/2,0} = 1, & B_{1/2,2} = -4/3, & B_{1/2,4} = 64/15, \\ B_{1/2,6} = -256/21, & B_{1/2,8} = -4096/45, & B_{1/2,10} = 81920/33 \end{array}$$

We now express $\zeta_{1/2}(s)$ in terms of $I(s)$. For $\Re(s) = \sigma > 1$, the middle integral in (3.10) goes to zero as $\delta \to 0$. It follows that

$$\begin{aligned} I(s) &= \left(\frac{e^{\pi i(2s-1)} - e^{-\pi i(2s-1)}}{2\pi i}\right)\int_0^\infty \frac{2x^{2(s-1)}e^{-x^2}}{\sqrt{\pi}\mathrm{erf}(x)}dx \\ &= \frac{\sin[\pi(2s-1)]\,\Gamma(s-1/2)\zeta_{1/2}(s)}{\pi^{3/2}} \\ &= \frac{2\cos[\pi(s-1/2)]\sin[\pi(s-1/2)]\,\Gamma(s-1/2)\zeta_{1/2}(s)}{\pi^{3/2}}. \end{aligned}$$

Now, by using the functional equation for the gamma function,

$$\Gamma(1-(s-1/2))\Gamma(s-1/2) = \frac{\pi}{\sin[\pi(s-1/2)]},$$

and the fact that $\Gamma(3/2) = \sqrt{\pi}/2$ we obtain

$$\zeta_{1/2}(s) = \frac{\Gamma(3/2)\Gamma(1-(s-1/2))I(s)}{\cos[\pi(s-1/2)]}. \tag{3.13}$$

Remark 3.4. Equation (3.13) implies that the zeros of $I_N(s)$ at positive integers $n > 1$ are simple since we know by definition that $\zeta_{1/2}(n) > 0$ for $n > 1$.

Here is another consequence of (3.13), which we state as

Theorem 3.3. *$\zeta_{1/2}(s)$ is analytic on the entire complex plane except for simple poles at $\{1, 0, -1, -2, \cdots\}$ whose residues are*

$$\mathrm{Res}\left(\zeta_{1/2}(s), s = n\right) = \frac{\Gamma(3/2)B_{1/2,2-2n}}{(2-2n)!\Gamma(n-1/2)} \qquad (2 - N \le n \le 1). \tag{3.14}$$

Furthermore, for half-integers $(2n+1)/2$ less than 1, we have

$$\zeta_{1/2}((2n+1)/2) = 0 \qquad (n = 0, -1, -2, \cdots) \tag{3.15}$$

Proof. Since $\Gamma(1-(s-1/2))$ has only simple poles at $s = 3/2, 5/2, \cdots$, which are cancelled by the zeros of $I(s)$ at $s = 3/2, 2, 5/2, \cdots$, and $\cos[\pi(s-1/2)]$ has simple zeros at the integers, it follows from (3.13) that $\zeta_{1/2}(s)$ is analytic on the whole plane except for simple poles at $s = 1, 0, -1, -2, \cdots$. Recalling the fact that $\Gamma(3/2) = \sqrt{\pi}/2$, it follows from (3.12) that the residue of $\zeta_N(s)$ at these poles are

$$\begin{aligned} \mathrm{Res}\left(\zeta_{1/2}(s), s = n\right) &= \lim_{s\to n}(s-n)\zeta_{1/2}(s) \\ &= \Gamma(3/2)\Gamma(1-(n-1/2))I(n)\lim_{s\to n}\frac{(s-n)}{\cos[\pi(s-1/2)]} \\ &= \frac{\Gamma(3/2)B_{1/2,2-2n}\Gamma(1-(n-1/2))}{\pi(2-2n)!\sin[\pi(n-1/2)]} \\ &= \frac{\Gamma(3/2)B_{1/2,2-2n}}{(2-2n)!\Gamma(n-1/2)} \end{aligned}$$

which proves (3.14). For half-integers $s = (2n+1)/2$, we have from Remark 3.3 that

$$\begin{aligned} \zeta_{1/2}((2n+1)/2) &= \frac{\Gamma(3/2)\Gamma(1-((2n+1)/2-1/2))I((2n+1)/2)}{\cos[\pi((2n+1)/2-1/2)]} \\ &= -\frac{\Gamma(3/2)B_{1/2,1-2n}\Gamma(1-n)}{(2-n)!\cos(\pi n)} \\ &= 0 \end{aligned}$$

which is (3.15). This completes the proof the theorem. $\square$

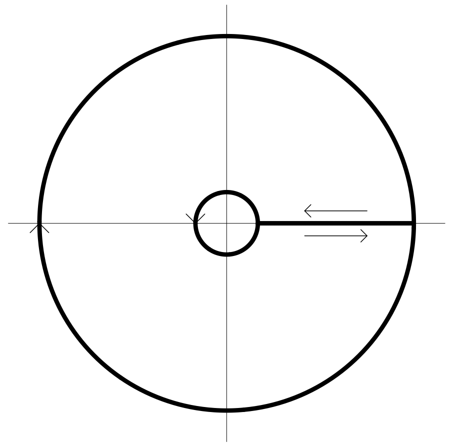

FIGURE 2. Contour γ_M.

4. PRE-FUNCTIONAL EQUATION

In the present section, we discuss a pre-functional equation satisfied by $\zeta_{1/2}(s)$. Let γ_M be the annulus-shaped contour consisting of two concentric circles centered at the origin, the outer circle having radius $(2M+1)\pi$ and the inner circle having radius $\delta < \pi$ (cf. Figure 2). The outer circle is traversed clockwise, the inner circle counterclockwise and the radial segment along the positive real axis is traversed in both directions. Then define

$$I_{\gamma_M}(s) = \frac{1}{2\pi i}\int_{\gamma_M} \frac{2(-z)^{2s-1}e^{-z^2}}{\sqrt{\pi}\mathrm{erf}(z)}\frac{dz}{z}. \tag{4.1}$$

We claim that $I_{\gamma_M}(s)$ converges to $I(s)$ defined by (3.9) as $M \to \infty$ for $\Re(s) < 0$. To prove this, observe that the portion of $I_{\gamma_M}(s)$ around the outer circle tends to zero as $M \to \infty$ on the same domain. This is because on the outer circle defined by $|z| = (2M+1)\pi$ we have that $\left|(-z)^{2s-1}e^{-z^2}/\mathrm{erf}(z)\right|$ is bounded independently of M and s. Therefore,

$$I(s) = \lim_{M\to\infty} I_{\gamma_M}(s). \tag{4.2}$$

On the other hand, we have by residue theory

$$I_{\gamma_M}(s) = \sum_{k=1}^{K}\left[\mathrm{Res}\left(\frac{2(-z)^{2s-2}e^{-z^2}}{\sqrt{\pi}\mathrm{erf}(z)}, z = z_k, \bar{z}_k, -z_k, -\bar{z}\right)\right]. \tag{4.3}$$

Here, $z_k = r_k e^{i\theta_k}$, $\bar{z}_k$, $-z_k$, and $-\bar{z}_k$ are simple roots of $\mathrm{erf}(z) = 0$ (with z_k chosen to be in the first quadrant) and $K = K_M$ is the number of such roots inside γ_M that are in the first quadrant. Moreover, we arrange the roots in ascending order so that $|z_1| < |z_2| < |z_3| < \cdots$, since none of the roots can have the same length (see Appendix). Now, to evaluate the residues, we call upon Cauchy's Integral Formula:

$$\mathrm{Res}\left(\frac{2(-z)^{2s-2}e^{-z^2}}{\sqrt{\pi}\mathrm{erf}(z)}, z = z_k\right) = (-z_k)^{2s-2}e^{-z_k^2}\lim_{z\to z_k}\frac{2(z-z_k)}{\sqrt{\pi}\mathrm{erf}(z)}.$$

But then

$$\lim_{z\to z_k}\frac{2(z-z_k)}{\sqrt{\pi}\mathrm{erf}(z)} = \frac{1}{e^{-z_k^2}}.$$

It follows that

$$\operatorname{Res}\left(\frac{2(-z)^{2s-2}e^{-z^2}}{\sqrt{\pi}\operatorname{erf}(z)}, z = z_k\right) = (-z_k)^{2s-2}.$$

Therefore,

$$\begin{aligned} I_{\gamma_M}(s) &= \sum_{k=1}^{K}\left[z_k^{2s-2} + \bar{z}_k^{2s-2} + (-z_k)^{2s-2} + (-\bar{z}_k)^{2s-2}\right] \\ &= 2\sum_{k=1}^{K} r_k^{2s-2}\left\{\cos\left[2(s-1)\theta_k\right] + \cos\left[2(s-1)(\pi-\theta_k)\right]\right\}. \end{aligned} \tag{4.4}$$

Since $K \to \infty$ as $M \to \infty$, we have by (4.2) and (4.4),

$$\begin{aligned} I(s) &= \lim_{M\to\infty} I_{\gamma_M}(s) \\ &= 2\sum_{k=1}^{\infty} r_k^{2s-2}\left\{\cos\left[2(s-1)\theta_k\right] + \cos\left[2(s-1)(\pi-\theta_k)\right]\right\}. \end{aligned} \tag{4.5}$$

Combining (3.13) and (4.5) we have proved

Theorem 4.1. *For $\Re(s) < 0$,*

$$\zeta_{1/2}(s) = \frac{\Gamma(1/2)\Gamma(1-(s-1/2))}{\cos\left[\pi(s-1/2)\right]}\sum_{k=1}^{\infty} r_k^{2s-2}\left\{\cos\left[2(s-1)\theta_k\right] + \cos\left[2(s-1)(\pi-\theta_k)\right]\right\}. \tag{4.6}$$

Remark 4.1.

(a) Observe that (4.6) resembles the pre-functional equation for hypergeometric zeta functions:

$$\zeta_N(s) = 2(-1)^{N-1}(N-1)!\Gamma(1-(s+N-1))\sum_{k=1}^{\infty} r_k^{s-1}\cos\left[(s-1)(\pi-\theta_k)\right]. \tag{4.7}$$

(b) Observe that the series appearing in (4.6) vanishes at half integers less than 1. These zeros are the 'trivial' zeros of $\zeta_{1/2}(s)$ described by (3.15).

(c) The first 10 nonzero roots $\{z_k\}$ of $\operatorname{erf}(z) = 0$ are listed in the Appendix.

Lastly we establish a connection between $\zeta_{1/2}(s)$ and the classical zeta function.

Theorem 4.2. *For $\Re(s) < 0$, we have*

$$\left|\zeta_{1/2}(s)\right| < \left|\frac{\Gamma(1/2)\Gamma(1-(s-1/2))(e^{|\Im(s)\theta_1|} + e^{|3\pi\Im(s)/4|})}{(2\pi)^{1-s}\cos\left[\pi(s-1/2)\right]}\right|\zeta(1-\Re(s), 7/8). \tag{4.8}$$

Proof. The argument essentially rests on bounds for the zeros of $\operatorname{erf}(z) = 0$ established in the Appendix. In particular, for each positive integer k, there exists precisely one zero $z_k = r_k e^{i\theta_k}$ whose magnitude is bounded by

$$r_k > \sqrt{2\pi(k-1/8)}. \tag{4.9}$$

It follows that the zeros in the first quadrant satisfying (4.9), their cousins in the other three quadrants, and $z = 0$ exhaust all the roots of $\operatorname{erf}(z) = 0$.

Now, using the fact that the angles $\{\theta_k\}$ are monotonically decreasing to $\pi/4$, we have

$$\begin{gathered} |\cos\left[(s-1)\theta_k\right]| \le e^{|\Im(s)\theta_k|} < e^{|\Im(s)\theta_1|}, \\ |\cos\left[(s-1)(\pi-\theta_k)\right]| \le e^{|\Im(s)(\pi-\theta_k)|} < e^{|3\pi\Im(s)/4|}. \end{gathered}$$

Therefore, by (4.6) and (4.9), for $\Re(s) < 0$, the following bound is achieved:

$$\begin{aligned}
|\zeta_{1/2}(s)| &= \left| \frac{\Gamma(1/2)\Gamma(1-(s-1/2))}{\cos[\pi(s-1/2)]} \sum_{k=1}^{\infty} r_k^{2s-2} \left\{ \cos[2(s-1)\theta_k] + \cos[2(s-1)(\pi-\theta_k)] \right\} \right| \\
&\leq \left| \frac{\Gamma(1/2)\Gamma(1-(s-1/2))(e^{|\Im(s)\theta_1|} + e^{|3\pi\Im(s)/4|})}{\cos[\pi(s-1/2)]} \right| \sum_{k=1}^{\infty} \frac{1}{r_k^{2-2\Re(s)}} \\
&< \left| \frac{\Gamma(1/2)\Gamma(1-(s-1/2))(e^{|\Im(s)\theta_1|} + e^{|3\pi\Im(s)/4|})}{(2\pi)^{1-s}\cos[\pi(s-1/2)]} \right| \sum_{k=1}^{\infty} \frac{1}{(k-1/8)^{1-\Re(s)}} \\
&< \left| \frac{\Gamma(1/2)\Gamma(1-(s-1/2))(e^{|\Im(s)\theta_1|} + e^{|3\pi\Im(s)/4|})}{(2\pi)^{1-s}\cos[\pi(s-1/2)]} \right| \zeta(1-\Re(s), 7/8).
\end{aligned}$$

This completes the proof. □

5. APPENDIX

According to [2], the complex roots of $\operatorname{erf}(z) = 0$, $z = x + iy$, satisfy

$$2xy = 2n\pi - \beta, \quad n = 1, 2, ..., \tag{5.1}$$

where $0 \leq \beta < \pi/4$. Therefore,

$$r^2 = |z|^2 = x^2 + y^2 > 2xy = 2n\pi - \beta > 2n\pi - \pi/4 \tag{5.2}$$

Hence, we may arrange the roots $\{z_k\}$ so that

$$r_k > \sqrt{2\pi(n-1/8)} \tag{5.3}$$

Table 1 lists the first ten zeros of $\operatorname{erf}(z) = 0$. These values were computed using the software program Mathematica.

References

[1] H. M. Edwards, *Riemann's Zeta Function*, Pure and Applied Mathematics Series, Academic Press, 1974.

[2] H. E. Fettis, J. C. Caslin, and K. R. Cramer, *Complex Zeros of the Error Function and of the Complementary Error Function*, Math. Comput. **27** (1973), No. 122, 401-407.

[3] A. Hassen and H. D. Nguyen, *Hypergeometric Zeta Functions*, Preprint, 2005. Available at the Mathematics ArXiv: http://arxiv.org/abs/math.NT/0509637

[4] A.S.B. Holland,*Introduction to the Theory of Entire Functions*, Academic Press, 1973.

[5] F. T. Howard, *A sequence of numbers related to the exponential function*, Duke Math. J. 34 (1967), 599-616.

[6] F. T. Howard,*Some sequences of rational numbers related to the exponential function*, Duke Math. J. **34** (1967), 701-716.

[7] F. T. Howard, *Numbers Generated by the Reciprocal of* $e^x - 1 - x$, Math. Comput. **31** (1977) No. 138, 581-598.

[8] B. E. Peterson, *Riemann Zeta Function*, Lecture Notes, 1996.

[9] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse (On the Number of Prime Numbers less than a Given Quantity)*, 1859, Translated by D. R. Wilkins (1998).

[10] H. E. Salzer, *Complex zeros of the error function*, J. Franklin Inst. **260** (1955), 209-211.

[11] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University Press, 1967.

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k	z_k	r_k	θ_k
1	$1.450616163 + 1.880943000i$	2.375338675	0.9138537276
2	$2.244659274 + 2.616575141i$	3.447457139	0.8617558815
3	$2.839741047 + 3.175628100i$	4.260134158	0.8411783838
4	$3.335460735 + 3.646174376i$	4.941648096	0.8298732386
5	$3.769005567 + 4.060697234i$	5.540276617	0.8226354431
6	$4.158998400 + 4.435571444i$	6.080424469	0.8175670375
7	$4.516319400 + 4.780447644i$	6.576459579	0.8138012859
8	$4.847970309 + 5.101588044i$	7.037685464	0.8108829957
9	$5.158767911 + 5.403332686i$	7.470534818	0.8085489637
10	$5.452192209 + 5.688837465i$	7.879674587	0.8066358482

TABLE 1. First ten nonzero roots of $\operatorname{erf}(z) = 0$ in the first quadrant of the complex plane.