

# HYPERGEOMETRIC BERNOULLI POLYNOMIALS AND APPELL SEQUENCES

ABDUL HASSEN AND HIEU D. NGUYEN

ABSTRACT. There are two analytic approaches to Bernoulli polynomials  $B_n(x)$ : either by way of the generating function  $ze^{xz}/(e^z - 1) = \sum B_n(x)z^n/n!$  or as an Appell sequence with zero mean. In this article we discuss a generalization of Bernoulli polynomials defined by the generating function  $z^N e^{xz}/(e^z - T_{N-1}(z))$ , where  $T_N(z)$  denotes the  $N$ -th Maclaurin polynomial of  $e^z$ , and establish an equivalent definition in terms of Appell sequences with zero moments in complete analogy to their classical counterpart. The zero-moment condition is further shown to generalize to Bernoulli polynomials generated by the confluent hypergeometric series.

## 1. INTRODUCTION

There are two analytic approaches to Bernoulli polynomials. One approach, initiated by Euler, defines them through the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}, \quad (1.1)$$

where  $B_n(x)$  denotes the Bernoulli polynomial of degree  $n$ . Here are the first few:

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= x - 1/2, \\ B_2(x) &= x^2 - x + 1/6, \\ B_3(x) &= x^3 - 3x^2/2 + x/2. \end{aligned}$$

The corresponding Bernoulli numbers are given by  $B_n = B_n(0)$ .

A second approach to Bernoulli polynomials is to define them as an Appell sequence with zero mean:

$$B_0(x) = 1, \quad (1.2)$$

$$B'_n(x) = nB_{n-1}(x), \quad (1.3)$$

$$\int_0^1 B_n(x) dx = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}. \quad (1.4)$$

F. T. Howard in [4] considered the following generalization of Bernoulli polynomials:

$$\frac{z^2 e^{xz}/2}{e^z - 1 - z} = \sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!}, \quad (1.5)$$

and more generally,

$$\frac{z^N e^{xz}/N!}{e^z - T_{N-1}(z)} = \sum_{n=0}^{\infty} B_n(N, x) \frac{z^n}{n!}, \quad (1.6)$$

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where  $T_N(z) = \sum_{n=0}^N z^n/n!$  and  $N$  is any positive integer. For the cases  $N = 1$  and  $N = 2$ , (1.6) reduces to (1.1) and (1.5), respectively. Here are the first few polynomials when  $N = 2$ :

$$\begin{aligned} B_0(2, x) &= 1, \\ B_1(2, x) &= x - 1/3, \\ B_2(2, x) &= x^2 - 2x/3 + 1/18, \\ B_3(2, x) &= x^3 - x^2 + x/6 + 1/90. \end{aligned}$$

It was shown in [4], [5], and [6] that the polynomials  $B_n(N, x)$ , referred to here as *hypergeometric Bernoulli polynomials*, and their corresponding numbers,  $B_n(N) = B_n(N, 0)$ , share many number-theoretic properties with their classical counterpart, including its intimate connection with Riemann's zeta function. Indeed, a theory of hypergeometric zeta functions has been developed by us in [3], where these functions are shown to take on values expressed in terms of hypergeometric Bernoulli numbers at sufficiently large negative integers.

In this paper we establish the following equivalent definition of hypergeometric Bernoulli polynomials in terms of Appell sequences with zero moments, in complete analogy to (1.2)-(1.4):

$$B_0(N, x) = 1, \tag{1.7}$$

$$B'_n(N, x) = nB_{n-1}(N, x), \tag{1.8}$$

$$\int_0^1 (1-x)^{N-1} B_n(N, x) dx = \begin{cases} 1/N & n = 0 \\ 0 & n > 0 \end{cases}. \tag{1.9}$$

The integral condition (1.9) can be extended even further to resemble the beta function. Toward this end, we consider a generalization of Bernoulli polynomials based on the confluent hypergeometric series  ${}_1F_1(M, M+N, z)$ , first discussed by K. Dilcher [2]:

$$\frac{e^{xz}}{{}_1F_1(M, M+N, z)} = \sum_{n=0}^{\infty} B_n(M, N, x) \frac{z^n}{n!}. \tag{1.10}$$

We end by demonstrating that the polynomials  $B_n(M, N, x)$  satisfy the following integral condition:

$$\int_0^1 x^{M-1} (1-x)^{N-1} B_n(M, N, x) dx = \begin{cases} \frac{\Gamma(M)\Gamma(N)}{\Gamma(M+N)} & n = 0, \\ 0 & n > 0. \end{cases} \tag{1.11}$$

This paper is organized as follows. In section 2, we review Appell sequences and demonstrate that the two analytic definitions of hypergeometric Bernoulli polynomials are equivalent. In section three, we generalize the integral condition to Bernoulli polynomials derived from the confluent hypergeometric series.

## 2. APPELL SEQUENCES

A sequence of polynomials  $\{P_n(x)\}$  is called an *Appell sequence* if it satisfies

$$\begin{aligned} P_0(x) &= 1, \\ P'_n(x) &= nP_{n-1}(x). \end{aligned}$$

Moreover, if  $\{P_n(x)\}$  is generated by the function

$$G(x, z) = \sum_{n=0}^{\infty} P_n(x) \frac{z^n}{n!}, \tag{2.1}$$

and  $G(x, z)$  is a real analytic function in  $x$ , then

$$\frac{\partial G}{\partial x} = zG(x, z).$$

It follows that

$$G(x, z) = e^{xz} g(z), \tag{2.2}$$

where the function  $g(z)$  is arbitrary unless additional constraints are given. We are now ready to state our main result.

**Theorem 2.1.** *The two definitions of hypergeometric Bernoulli polynomials  $\{B_n(N, x)\}$  given by the generating function in (1.6) and the Appell sequence in (1.7)-(1.9) are equivalent.*

*Proof.* Assume first that  $\{B_n(N, x)\}$  is the Appell sequence given by (1.7)-(1.9). Then define  $G(x, z)$  to be its generating function:

$$G(x, z) = \sum_{n=0}^{\infty} B_n(N, x) \frac{z^n}{n!}. \quad (2.3)$$

The series in (2.3) converges uniformly in  $x$  on  $[0, 1]$  for any  $|z| < 1/2$ . This follows from the following bound on hypergeometric Bernoulli numbers (see [3]):

$$|B_n(N)| \leq C \frac{n!}{(2\pi)^n}, \quad (2.4)$$

where  $C$  is a sufficiently large positive constant. Since

$$B_n(N, x) = \sum_{k=0}^n \binom{n}{k} B_k(N) x^{n-k}, \quad (2.5)$$

we can bound  $B_n(N, x)$  on  $[0, 1]$  using (2.4):

$$\begin{aligned} \left| B_n(N, x) \frac{z^n}{n!} \right| &\leq z^n \sum_{k=0}^n \binom{n}{k} \frac{|B_k(N)|}{n!} \\ &\leq C z^n \sum_{k=0}^n \binom{n}{k} \frac{k!}{n! (2\pi)^k} \\ &\leq \frac{C}{2\pi} z^n \sum_{k=0}^n \binom{n}{k} \\ &\leq \frac{C}{2\pi} (2z)^n. \end{aligned}$$

It follows for  $|z| < 1/2$  that

$$\left| \sum_{n=0}^{\infty} B_n(N, x) \frac{z^n}{n!} \right| \leq \frac{C}{2\pi} \sum_{n=0}^{\infty} (2z)^n < \infty. \quad (2.6)$$

Thus  $G(x, z)$  converges uniformly in  $x$  on  $[0, 1]$ .

From (2.2) we have  $G(x, z) = e^{xz} g(z)$ . The function  $g(z)$  can now be found by integrating (2.3) with weight  $(1-x)^{N-1}$ :

$$\int_0^1 (1-x)^{N-1} g(z) e^{xz} dx = \int_0^1 (1-x)^{N-1} \sum_{n=0}^{\infty} B_{N,n}(x) \frac{z^n}{n!} dx, \quad (2.7)$$

Through induction on  $N$  and integration by parts, the left hand side above yields the closed formula

$$\int_0^1 (1-x)^{N-1} g(z) e^{xz} dx = (N-1)! \frac{e^z - T_{N-1}(z)}{z^N} g(z). \quad (2.8)$$

Now equate (2.7) and (2.8) and reverse the order of integration and summation to obtain

$$(N-1)! \frac{e^z - T_{N-1}(z)}{z^N} g(z) = \sum_{n=0}^{\infty} \left( \int_0^1 (1-x)^{N-1} B_{N,n}(x) dx \right) \frac{z^n}{n!}. \quad (2.9)$$

But every integral except the first on the right hand side above must vanish because of (1.9). Therefore

$$(N-1)! \frac{e^z - T_{N-1}(z)}{z^N} g(z) = \frac{1}{N},$$

which yields

$$g(z) = \frac{z^N/N!}{e^z - T_{N-1}(z)}. \quad (2.10)$$

By analytic continuation, (2.10) holds for all  $z \in \mathbb{C}$ . Hence,  $\{B_n(N, x)\}$  satisfies (1.6) is desired.

Now assume on the other hand that  $\{B_n(N, x)\}$  is defined by (1.6). To establish properties (1.7)-(1.9), we can of course employ uniqueness of power series expansions to argue that the only polynomials satisfying such properties must be  $\{B_n(N, x)\}$ . However, we shall instead give a more explicit proof. To this end, we first differentiate (1.6) with respect to  $x$  and re-index our series (since  $B'_0(x) = 0$ ) to obtain

$$\frac{z^{N+1} e^{xz}/N!}{e^z - T_{N-1}(z)} = \sum_{n=1}^{\infty} B'_n(N, x) \frac{z^n}{n(n-1)!} \quad (2.11)$$

On the other hand,

$$\frac{z^{N+1} e^{xz}/N!}{e^z - T_{N-1}(z)} = z \sum_{n=0}^{\infty} B_n(N, x) \frac{z^n}{n!} = \sum_{n=1}^{\infty} B_{n-1}(N, x) \frac{z^n}{(n-1)!}. \quad (2.12)$$

Equating (2.11) and (2.12) then yields the derivative property (1.8).

To establish the other two conditions, we rewrite (1.6) as

$$e^{xz} = \frac{e^z - T_{N-1}(z)}{z^N/N!} \sum_{n=0}^{\infty} B_n(N, x) \frac{z^n}{n!} = \left( \sum_{m=0}^{\infty} \frac{z^m}{(N+1)_m} \right) \left( \sum_{n=0}^{\infty} B_n(N, x) \frac{z^n}{n!} \right),$$

where the Pochhammer symbol  $(N)_m = N(N+1)\dots(N+m-1)$  refers to a rising factorial starting at  $N$ . It follows from equating series coefficients that

$$\frac{x^n}{n!} = \sum_{m=0}^n \frac{B_m(N, x)}{m!(N+1)_{n-m}},$$

or, equivalently,

$$x^n = N! \sum_{m=0}^n \binom{n}{m} \frac{(n-m)!}{(N+n-m)!} B_m(N, x). \quad (2.13)$$

Now set  $n = 0$  in the above equation to obtain property (1.7). Next, define hypergeometric Bernoulli numbers by  $B_n(N) = B_n(N, 0)$ , or equivalently,

$$\frac{z^N/N!}{e^z - T_{N-1}(z)} = \sum_{n=0}^{\infty} B_n(N) \frac{z^n}{n!}. \quad (2.14)$$

Next

$$e^{xz} \sum_{n=0}^{\infty} B_n(N) \frac{z^n}{n!} = \sum_{n=0}^{\infty} B_n(N, x) \frac{z^n}{n!}$$

follows by comparison of 2.14 with 1.6. Then equating series coefficients yields

$$B_n(N, x) = \sum_{k=0}^n \binom{n}{k} B_k(N) x^{n-k}. \quad (2.15)$$

Therefore by integrating (2.15) with weight  $(1-x)^{N-1}$  we obtain

$$\int_0^1 (1-x)^{N-1} B_n(N, x) dx = \sum_{k=0}^n \binom{n}{k} B_k(N) \int_0^1 (1-x)^{N-1} x^{n-k} dx. \quad (2.16)$$

To simplify the right hand side above we first evaluate the beta integral that appears in terms of factorials (valid for non-negative exponents):

$$\int_0^1 (1-x)^{N-1} x^{n-k} dx = \frac{(N-1)!(n-k)!}{(N+n-k)!}. \quad (2.17)$$

Then equating (2.16) and (2.17) produces our result:

$$\begin{aligned} \int_0^1 (1-x)^{N-1} B_n(N, x) dx &= (N-1)! \sum_{k=0}^n \binom{n}{k} \frac{(n-k)!}{(N+n-k)!} B_k(N) \\ &= \begin{cases} 1/N & n=0, \\ 0 & n>0. \end{cases} \end{aligned}$$

Here, we have applied the recurrence relation (see [5] and [6])

$$\sum_{k=0}^n \binom{n}{k} \frac{(n-k)!}{(N+n-k)!} B_k(N) = \begin{cases} 1/N! & n=0, \\ 0 & n>0, \end{cases}$$

which follows from (2.13). This completes our proof.  $\square$

### 3. THE CONFLUENT HYPERGEOMETRIC SERIES

Observe that the generating function in (1.6) can be expressed in terms of the confluent hypergeometric function:

$$\frac{z^N e^{xz}/N!}{e^z - T_{N-1}(z)} = \frac{e^{xz}}{{}_1F_1(1, N+1, x)}. \quad (3.1)$$

Here,  ${}_1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$  is the confluent hypergeometric function. We can therefore use (3.1) to define hypergeometric Bernoulli polynomials in two continuous parameters, a generalization introduced by K. Dilcher [2]:

$$\frac{e^{xz}}{{}_1F_1(M, M+N, z)} = \sum_{n=0}^{\infty} B_n(M, N, x) \frac{z^n}{n!}. \quad (3.2)$$

For  $M=0$  and  $N$  a positive integer, we have  $B_n(0, N, x) = B_n(N, x)$ , where  $B_n(N, x)$  are hypergeometric Bernoulli polynomials defined by (1.6). For  $M=s+1$  and  $N=r+1$ , where  $r$  and  $s$  are nonnegative integers, we have  $B_n(s-1, r-1, x) = B_n^{(r,s)}(x)$ , where  $B_n^{(r,s)}(x)$  are Bernoulli-Padé polynomials defined using Padé approximants of  $e^x$  (see [1] and [2]).

We conclude by demonstrating that these hypergeometric Bernoulli polynomials  $B_n(M, N, x)$  of order  $(M, N)$  satisfy the following integral condition, which generalizes (1.9):

**Theorem 3.1.** *For real  $M > 0$  and  $N > 0$ ,*

$$\int_0^1 x^{M-1} (1-x)^{N-1} B_n(M, N, x) dx = \begin{cases} \frac{\Gamma(M)\Gamma(N)}{\Gamma(M+N)} & n=0, \\ 0 & n>0. \end{cases} \quad (3.3)$$

*Proof.* Following the proof of Theorem 2.1, we first rewrite (3.2) as

$$e^{xz} = {}_1F_1(M, M+N, x) \sum_{n=0}^{\infty} B_n(M, N, x) \frac{z^n}{n!}. \quad (3.4)$$

Then equating coefficients yields

$$x^n = \frac{\Gamma(M+N)}{\Gamma(M)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(M+n-k)}{\Gamma(M+N+n-k)} B_k(M, N, x). \quad (3.5)$$

Moreover, (2.15) continues to hold for  $B_n(M, N, x)$ :

$$B_n(M, N, x) = \sum_{k=0}^n \binom{n}{k} B_k(M, N) x^{n-k}. \quad (3.6)$$

It follows from (3.5) and (3.6) that

$$\begin{aligned} \int_0^1 x^{M-1}(1-x)^{N-1} B_n(M, N, x) dx &= \sum_{k=0}^n \binom{n}{k} B_k(M, N) \int_0^1 (1-x)^{N-1} x^{M+n-k-1} dx \\ &= \Gamma(N) \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(M+n-k)}{\Gamma(M+N+n-k)} B_k(M, N) \\ &= \begin{cases} \frac{\Gamma(M)\Gamma(N)}{\Gamma(M+N)} & n = 0, \\ 0 & n > 0. \end{cases} \end{aligned}$$

□

*Remark 3.1.* Observe that (3.5) and (3.6) are inversions of each other. Moreover, Theorem 3.1 reduces to the beta function  $B(M, N)$  for the case  $n = 0$ :

$$B(M, N) = \int_0^1 x^{M-1}(1-x)^{N-1} dx = \frac{\Gamma(M)\Gamma(N)}{\Gamma(M+N)}. \quad (3.7)$$

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DEPARTMENT OF MATHEMATICS, ROWAN UNIVERSITY, GLASSBORO, NJ 08028.

*E-mail address:* [hassen@rowan.edu](mailto:hassen@rowan.edu), [nguyen@rowan.edu](mailto:nguyen@rowan.edu)