LOG-POLYNOMIAL PERIOD FUNCTIONS FOR HECKE GROUPS

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ABSTRACT. In this article we shall determine the automorphic integrals of positive and negative integral weight associated with the full modular group and some Hecke groups. This will be done by using the Hecke Correspondence. We will also give a characterization of multiplier systems of real weight for Hecke groups.

1. INTRODUCTION

Roughly speaking, Hamburger's Theorem states that the Riemann zeta function is uniquely determined by its functional equation. More precisely,

Theorem 1.1. (Hamburger's Theorem) *Let*

$$R(s) = \pi^{-s}\Gamma(s)\phi(s), \qquad \phi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{1}$$

where $a_n = \mathcal{O}(n^{\gamma}), \quad n \to \infty, \quad \gamma > 0.$
Suppose
(i) there exists a polynomial $p(s)$ such that $p(s)\phi(s)$ is entire and of finite order.
(ii) $\phi\left(\frac{s}{2}\right)$ is also a Dirichlet series convergent in some half plane.
If $R(s)$ satisfies the functional equation

$$R\left(\frac{1}{2} - s\right) = R(s),$$

then $\phi(s) = C\zeta(2s)$ for some constant C.

Hecke gave the following version of the Hamburger's Theorem:

Theorem 1.2. *With $\phi(s)$ and $R(s)$ as in* (1) *, suppose $\left(s - \frac{1}{2}\right)\phi(s)$ is entire and of finite order. If $R\left(\frac{1}{2} - s\right) = R(s)$, then $\phi(s) = C\zeta(2s)$ for some constant C.*

The significance of Hecke's version is that we are not assuming that $\phi\left(\frac{s}{2}\right)$ is a Dirichlet series. Suppose we assume that only $\phi(s)$ is a Dirichlet series and that $p(s)\phi(s)$ is entire and of finite order for some polynomial $p(s)$. Does the conclusion of Hamburger's Theorem still hold? Recently, M. Knopp in [12] answered this question in the negative. He showed that there are infinitely many linearly independent solutions of the functional equation by showing the existence of infinitely many automorphic integrals. In fact he was able to generalize this result to the following more general setup.

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Suppose $\lambda > 0$ and define

$$R_\lambda(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\,\phi(s), \quad \phi(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Also the functional equation generalizes to

$$R_\lambda(k-s) = Ce^{\pi ik/2} R_\lambda(s), \quad |C| = 1.$$

Knopp showed that there are infinitely many linearly independent solutions of this functional equation when $\lambda \geq 2$ and k is any real number. It is the objective of this article to investigate the case when $\lambda = 2\cos\pi\theta$, where $0 < \theta < \frac{1}{2}$. This will be done by deteriming the necessary conditions for the existence of automorphic integrals corresponding to the Dirichlet series. To this end, following M. Knopp, we apply the Hecke correspondence. This was first used by Riemann, developed by Hecke in [7] and then by Bochner in [1]. Here we state it as generalized by Weil in [20], and following Weil we shall refer to it in the sequel as Hecke's Lemma.

Lemma 1.1. (Riemann-Hecke-Bochner-Weil Correspondence) *Let f and g be continuous functions on $(0,\infty)$ such that*

$$f(y),\ g(1/y) = \mathcal{O}\left(e^{-ay}\right), \quad y \to \infty, \quad a > 0$$
$$f(y),\ g(1/y) = \mathcal{O}\left(y^{-b}\right), \quad y \to 0, \quad b > 0.$$

Throughout, we shall write $s = \sigma + it$ where σ and t are real numbers. Define

$$\Phi(s) = \int_0^\infty f(y)\,y^{s-1}dy \quad \text{and} \quad \Psi(s) = \int_0^\infty g(y)\,y^{s-1}dy.$$

Assume that for some $\sigma_0 > b$, $\sigma_o' < -b$,

$$\Phi(\sigma_0 + it) = \mathcal{O}\left(|t|^{-2}\right) \quad \text{and} \quad \Psi(\sigma_0' + it) = \mathcal{O}\left(|t|^{-2}\right) \text{ as } |t| \to \infty.$$

Let $Q(s)$ be a rational function which vanishes at infinity and let $s_1, \cdots, s_m$ be the poles of Q. Assume that $\sigma_0' \leq Re(s_\nu) \leq \sigma_0$ for each ν.
Then the following are equivalent:
(A) $\quad f(y) - g(y) = \sum_{\nu=1}^m Res\left(Q(s)\,y^{-s}, s_\nu\right);$
(B) $\quad \Phi(s) - Q(s)$ *and* $\Psi(s) - Q(s)$ *can be continued to the same entire function $F(s)$ which is bounded in every vertical strip.*

Here $Res\left(Q(s)\,y^{-s}, s_\nu\right)$ is the residue of $Q(s)\,y^{-s}$ at s_ν.

Remark 1.1 Let $Q(z) = \sum_{\nu=1}^m \left\{\frac{b(\nu,0)}{(z-s_\nu)^{n_\nu}} + \cdots + \frac{b(\nu,n_\nu-1)}{(z-s_\nu)}\right\}$, be a rational function with $b(\nu,0) \neq 0$. Then it can easily be seen that

$$\sum_{\nu=1}^m Res\left(Q(z)\,y^{-z}, s_\nu\right) = \sum_{\nu=1}^m \sum_{j=0}^{n_\nu-1} \frac{(-1)^j}{j!} b(\nu, n_\nu - j - 1)\,(\log y)^j\,y^{-s_\nu}; \quad (2)$$

which is a sum of the form

$$q(z) = \sum_{j=1}^m \sum_{t=0}^{n_j} c(j,t)\,(\log z)^t\,z^{\gamma_j}.$$

Knopp has called these functions *log-polynomial sums*. He used these sums to show that the conditions in Hamburger's Theorem cannot be relaxed without losing uniqueness (Theorems 1 and 2 in). In[1] , Bochner introduced a more general class of functions which he called *residual functions* and showed that log-polynomial sums are instances of residual functions (Lemma 1 in [1]).

A log-polynomial sum $q(z)$ is called a Log-Polynomial Period Function(LPPF) of weight $2k$ and multiplier system v for Hecke group $G(\lambda), \lambda = 2\cos\pi\theta, \theta = \frac{1}{p}$, p is an integer greater than 2, if there is exists a function F defined and holomorphic in the complex upper half plane satisfying

$$e^{-2\pi i\kappa}F(z+\lambda) = F(z) \quad \text{and} \quad v(T)\, z^{-2k}F\left(-\frac{1}{z}\right) = F(z) + q(z),$$

with $e^{2\pi i\kappa} = v(S_\lambda)$.

In this paper we will characterize, completely, LPPFs of weight $2k$ for the following cases; (1) $k \geq 1$ and $\kappa = 0$, (2) $k > 0$ and $\kappa \neq 0$, (3) $k \geq 0, 2k \in Z$ and $\kappa = 0$, and (4) $k \leq 0$ and $\kappa > 0$.

Hecke's Lemma is most frequently stated in a manner that illustrates the underlying group and the weight of the modular form. We record this in a slightly more general form in the following corollary.

Corollary 1.1. *Let $\lambda > 0$ and k be a real number. Let a_n and b_n be complex numbers such that*

$$a_n,\ b_n = \mathcal{O}(n^c) \quad \text{as} \quad n \to \infty, \qquad \text{for some } c > 0.$$

Define

$$\phi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad \psi(s) = \sum_{n=1}^{\infty} b_n n^{-s};$$

$$\Phi_1(s) = \left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s)\,\phi(s), \qquad \Psi_1(s) = \left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s)\,\psi(s);$$

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz/\lambda}, \qquad G(z) = \sum_{n=0}^{\infty} b_n e^{2\pi inz/\lambda}.$$

Then the following are equivalent:
(A) $\quad z^{-2k}G(-1/z) = F(z)$;
(B) $\quad \Phi_1(s) + \frac{a_0}{s} + \frac{e^{-\pi ik}b_0}{2k-s}$ *is entire and*

$$\Phi_1(s) = e^{-\pi ik}\Psi_1(2k-s).$$

Proof: Put $f(y) = F(iy) - a_0,\ g(y) = e^{-\pi ik}y^{-2k}\{G(i/y) - b_0\}$ and $Q(s) = -\frac{a_0}{s} + \frac{e^{-\pi ik}b_0}{s-2k}$. From (2) we obtain

$$Res\left(Q(s)\,y^{-s},0\right) + Res\left(Q(s)\,y^{-s},2k\right) = -a_0 + e^{-\pi ik}b_0y^{-2k}.$$

Thus (A) is equivalent to

$$f(y) - g(y) \;=\; \sum_{\nu=1}^{m} Res\left(Q(s)\,y^{-s}, s_\nu\right).$$

It is clear that

$$\Phi(s) = \int_0^\infty f(y)y^{s-1}dy = \Phi_1(s)$$

and
$$\Psi(s)=\int_0^\infty g(y)y^{s-1}dy=e^{-\pi ik}\Psi_1(2k-s).$$
By Stirling's formula Φ and Ψ satisfy the growth condition in Hecke's Lemma. We now apply Hecke's Lemma and analytic continuation.

The following corollary of Hecke's Lemma contains the correspondence as used by Knopp in [12] when applied to automorphic integrals. As Knopp pointed out in [12], the residual functions of Bochner that are associated with automorphic integrals and Dirichlet series are log-polynomial sums. In this paper we shall apply this corollary in the case when $\lambda_1=\lambda_2=\lambda$ and $G=\bar{v}(T)F$.

Corollary 1.2. *Let $\lambda_1,\lambda_2>0$, k a real number, γ_j and $c(j,t)$ be complex numbers and let t,j,m, and n_j be nonnegative integers. Let a_n and b_n be complex numbers such that*
$$a_n,\ b_n=\mathcal{O}(n^c)\quad as\quad n\to\infty,\qquad for\ some\ c>0.$$
Define
$$\phi(s)=\sum_{n=1}^\infty a_n n^{-s},\qquad \psi(s)=\sum_{n=1}^\infty b_n n^{-s};$$
$$\Phi_2(s)=\left(\frac{2\pi}{\lambda_1}\right)^{-s}\Gamma(s)\phi(s),\qquad \Psi_2(s)=\left(\frac{2\pi}{\lambda_2}\right)^{-s}\Gamma(s)\psi(s);$$
$$F(z)=\sum_{n=0}^\infty a_n e^{2\pi iz/\lambda_1},\qquad G(z)=\sum_{n=0}^\infty b_n e^{2\pi iz/\lambda_2};$$
$$Q(s)=\sum_{j=1}^m\sum_{t=0}^{n_j}\frac{d(j,t)}{(s-\gamma_j)^{n_j-t+1}}-\frac{a_0}{s}+\frac{e^{-\pi ik}b_0}{s-2k},\quad q(z)=\sum_{j=1}^m\sum_{t=0}^{n_j}c(j,t)(\log z)^t z^{\gamma_j}.$$
Then the following are equivalent:
(A) $\quad z^{-2k}G(-1/z)=F(z)+q(z)$ *;*
(B) $\quad \Phi_2(s)-Q(s)$ *and* $\Psi_2(s)-Q(s)$ *can be continued to the same entire function which is bounded in every lacunary vertical strip:*
$$\sigma_1\le\sigma\le\sigma_2,\quad |Im(s)|\ge t_0>0,$$
and they satisfy the functional equation
$$\Phi_2(s)=e^{-\pi ik}\Psi_2(2k-s).$$

Proof: Put $f(y)=F(iy)-a_0$ and $g(y)=e^{-\pi ik}y^{-2k}(G(i/y)-b_0)$. Again by (2), we see that (A) becomes
$$f(y)-g(y)=-a_0+e^{-\pi ik}b_0y^{-2k}-q(iy)=\sum_{\nu=1}^m Res\left(Q(s)y^{-s},s_\nu\right).$$
(Note that $d(j,t)$ can be calculated from (2).) The rest of the argument is the same as that of the proof of Corollary 1.1.

Observe that in Corollary 1.2, if we take $G=\bar{v}(T)F$ and $\lambda_1=\lambda_2=\lambda$, then the term $z^{\gamma_j}(\log z)^{n_j}$ in the log-polynomial sum gives rise to a pole of order n_j+1 at γ_j for the Dirichlet series (with the Γ-factor), and conversely. Thus the determination of Dirichlet series with functional equation amounts to finding log-polynomial sums which are period functions in the sense introduced by Knopp.

Other applications of Hecke's Lemma can be found in [20]. Among other things Weil shows that Hecke's Lemma can be used to obtain modular integrals with polynomial period functions. In other words, Eichler's integrals can be obtained from modular forms by manipulating the functional equations of the Dirichlet series and then applying Hecke's Lemma.

In the next two sections we shall be dealing with log-polynomial sums as period functions for the discrete Hecke groups. In Section 2 we will discuss multiplier systems and define automorphic integrals with log-polynomial period functions, and in Section 3 we will characterize those periods of positive weight greater than or equal to 2 and negative integral weight. In some cases we will obtain the complete result for all positive weight (See Theorems 3.1 and 3.3) and give the general form of these periods for nonintegral negative weights.

2. MULTIPLIER SYSTEMS AND LOG-POLYNOMIAL PERIOD FUNCTIONS

2.1. **Multiplier Systems for the Hecke Groups.**

In what follows, k will be a fixed real number. We shall denote by $\mathbf{Z}$, $\mathbf{C}$, and $\mathcal{H}$, the set of integers, the set of complex numbers and the upper half-plane, respectively. For $z \in \mathbf{C}$, we assume that $-\pi \leq \arg z < \pi$ and define

$$(cz+d)^r = |cz+d|^r \exp\{ri \arg(cz+d)\}, \tag{3}$$

where r, c, d are real numbers.

Let $\lambda > 0$. *The Hecke group* $G(\lambda)$ is the group generated by

$$S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$G(\lambda)$ acts on $\mathcal{H}$ by $Mz = \frac{az+b}{cz+d}$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda)$ and $z \in \mathcal{H}$. In this case we identify M with its negative $-M$ and consider the elements of $G(\lambda)$ as fractional linear transformations.

A *multiplier system in weight* $2k$ *for* $G(\lambda)$ is a complex-valued function v defined on $G(\lambda)$ such that the following two properties hold:

$$|v(M)| = 1 \ \ \forall M \in G(\lambda), \tag{4}$$

$$v(M_3)(c_3z+d_3)^{2k} = v(M_1)v(M_2)(c_1M_2z+d_1)^{2k}(c_2z+d_2)^{2k}, \tag{5}$$

for all $M_1,\ M_2 \in G(\lambda),\ M_3 = M_1M_2,\ M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$, $j = 1,2,3$, and $\forall z \in \mathcal{H}$.

Remark 2.1 (i) Condition (5) is sometimes called the *consistency condition.*
(ii) By taking $M_1 = M_2 = I$ in (5), we get $v(I) = v(I)^2$ and by (4) we conclude that $v(I) = 1$. If we take $M_1 = M_2 = -I$ in (5), we get $v(-I)(-1)^{2k} = \pm 1$.

Lemma 2.1. *Let v be a multipier system in weight $2k$ for $G(\lambda)$ such that $v(-I)(-1)^{2k} = 1$. Then, with $v(S_\lambda) = e^{2\pi i\kappa}$, $0 \leq \kappa < 1$, we have*

(i) $v(T) = \pm e^{-\pi ik}$, $\quad (ii)$ $v\left(TS_\lambda^{-1}T\right) = e^{2\pi i(k-\kappa)}$, $\quad (iii)$ $v\left(-TS_\lambda^{-1}T\right) = e^{-2\pi i\kappa}$.

Proof : (i) Let $M_1 = M_2 = T$. Then $M_3 = T^2 = -I$ and hence for any $z \in \mathcal{H}$, we have

$$1 = v(-I)(-1)^{2k} = v(T)^2 z^{2k}\left(-\frac{1}{z}\right)^{2k}. \tag{6}$$

If we let $z = i$ in (6), we get $v(T)^2 i^{4k} = 1$, that is , $\left(v(T)\, i^{2k}\right)^2 = 1$ and (i) follows from (3).

(ii) Let $M_1 = T$ and $M_2 = S_\lambda^{-1}T = \begin{pmatrix} -\lambda & -1 \\ 1 & 0 \end{pmatrix}$. Then $M_3 = TS_\lambda^{-1}T = \begin{pmatrix} -1 & 0 \\ -\lambda & -1 \end{pmatrix}$ and hence

$$v\left(TS_\lambda^{-1}T\right)(-\lambda z - 1)^{2k} = v(T)\, v\left(S_\lambda^{-1}T\right)\left(S_\lambda^{-1}Tz\right)^{2k} z^{2k}. \tag{7}$$

On the other hand,

$$v\left(S_\lambda^{-1}T\right) z^{2k} = v\left(S_\lambda^{-1}\right) v(T)\, z^{2k}. \tag{8}$$

Substituting (8) in (7) yields

$$v\left(TS_\lambda^{-1}T\right)(-\lambda z - 1)^{2k} = v(T)^2\, v\left(S_\lambda^{-1}\right)\left(S_\lambda^{-1}Tz\right)^{2k} z^{2k}. \tag{9}$$

Dividing both sides of (9) by $(-\lambda z - 1)^{2k}$, we get

$$v\left(TS_\lambda^{-1}T\right) = v(T)^2\, v\left(S_\lambda^{-1}\right) \frac{z^{2k}\left(-\lambda - \frac{1}{z}\right)^{2k}}{(-\lambda z - 1)^{2k}}. \tag{10}$$

By (3),

$$\begin{aligned} &\frac{z^{2k}\left(-\lambda - \frac{1}{z}\right)^{2k}}{(-\lambda z - 1)^{2k}} \\ &= \frac{|z|^{2k}\, | -\lambda - \frac{1}{z} |^{2k} \exp\left\{2ki\left(\arg z + \arg\left(-\lambda - \frac{1}{z}\right)\right)\right\}}{| -\lambda z - 1|^{2k} \exp\left\{2ki \arg(-\lambda z - 1)\right\}} \\ &= \exp\left\{2ki\left(\arg z + \arg\left(-\lambda - \frac{1}{z}\right) - \arg(-\lambda z - 1)\right)\right\}. \end{aligned} \tag{11}$$

For $z \in \mathcal{H}$, we have $0 < \arg z < \pi$, $0 < \arg\left(-\lambda - \frac{1}{z}\right) < 0$ and $-\pi < \arg(-\lambda z - 1) < 0$. Thus,

$$0 < \arg z + \arg\left(-\lambda - \frac{1}{z}\right) - \arg(-\lambda z - 1) < 3\pi. \tag{12}$$

On the other hand,

$$\begin{aligned} \arg z + \arg\left(-\lambda - \frac{1}{z}\right) - \arg(-\lambda z - 1) &\equiv \arg\left(\frac{z\left(-\lambda - \frac{1}{z}\right)}{-\lambda z - 1}\right) \\ &\equiv \arg 1 \equiv 0 \,(\mathrm{mod} 2\pi). \end{aligned} \tag{13}$$

From (12) and (13) we obtain

$$\arg z + \arg\left(-\lambda - \frac{1}{z}\right) - \arg(-\lambda z - 1) = 2\pi. \tag{14}$$

Thus using (14) and (11) in (10) we conclude that

$$v\left(TS_\lambda^{-1}T\right) = v(T)^2\, v\left(S_\lambda^{-1}\right) e^{4\pi i k}.$$

Since $v\left(S_\lambda^{-1}\right) = v(S_\lambda)^{-1} = e^{-2\pi i\kappa}$ and since $v(T)^2 e^{2\pi i k} = 1$, (ii) follows.

(iii) Let $M_1 = -I$ and $M_2 = TS_\lambda^{-1}T$. Then $M_3 = M_1M_2 = -TS_\lambda^{-1}T = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$, and for any $z \in \mathcal{H}$ we have

$$v\left(-TS_\lambda^{-1}T\right)(\lambda z+1)^{2k} = v(-I)\, v\left(TS_\lambda^{-1}T\right)(-1)^{2k}(-\lambda z-1)^{2k}.$$

Since $v(-I)(-1)^{2k} = 1$, we see that

$$v\left(-TS_\lambda^{-1}T\right) = v\left(TS_\lambda^{-1}T\right)\frac{(-\lambda z-1)^{2k}}{(\lambda z+1)^{2k}}.$$

By an argument similar to that in (ii) above, we can show that

$$\frac{(-\lambda z-1)^{2k}}{(\lambda z+1)^{2k}} = e^{-2\pi ik}$$

and (iii) follows from this and (ii). This completes the proof of the lemma.

Remark 2.2 It is well-known (see [7] and [3]) that the only discrete Hecke groups are those for which

$$\lambda \geq 2 \quad \text{or} \quad \lambda = 2\cos(\pi/p)\,,\ p \in \mathbf{Z},\ \ p \geq 3.$$

When

$$\lambda = 2\cos(\pi/p)\,,\ p \in \mathbf{Z},\ p \geq 3, \tag{15}$$

there are two relations between the generators S_λ and T of $G(\lambda)$; namely,

$$T^2 = -I \quad \text{and} \quad (S_\lambda T)^p = -I. \tag{16}$$

The following lemma gives the relationship between the weight $2k$, the multiplier system v and λ. It also generalizes a similar relation for the case of the full modular group given by Rademacher and Zuckerman in [16].

Lemma 2.2. *Let v be a multiplier system in weight $2k$ for $G(\lambda)$ such that $v(-I)(-1)^{2k} = 1$ and let λ be given by (15).*
(i) If $v(T) = e^{-\pi ik}$ or p is even, then $(p-2)k - 2p\kappa$ is an even integer.
(ii) If $v(T) = -e^{-\pi ik}$ and p is odd, then $(p-2)k - 2p\kappa$ is an odd integer.

Proof : Let $V_n = (S_\lambda T)^n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$. Then by (16), $V_p = -I = V_1V_{p-1}$, and we have, for $z \in \mathcal{H}$,

$$\begin{aligned} 1 &= v(-I)(-1)^{2k} \\ &= v(V_p)(-1)^{2k} = v(V_1)\,v(V_{p-1})(\gamma_1V_{p-1}z+\delta_1)^{2k}(\gamma_{p-1}z+\delta_{p-1})^{2k} \\ &= v(V_1)\,v(V_{p-1})(V_{p-1}z)^{2k}(\gamma_{p-1}z+\delta_{p-1})^{2k}, \end{aligned}$$

since $V_1 = S_\lambda T = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$ and hence $\gamma_1 = 1$ and $\delta_1 = 0$. But $V_{p-1} = V_1V_{p-2}$. Hence we have

$$\begin{aligned} v(V_{p-1})(\gamma_{p-1}z+\delta_{p-1})^{2k} &= v(V_1)\,v(V_{p-1})(\gamma_1V_{p-2}z+\delta_1)^{2k}(\gamma_{p-2}z+\delta_{p-2})^{2k} \\ &= v(V_1)\,v(V_{p-2})(V_{p-2}z)^{2k}(\gamma_{p-2}z+\delta_{p-2})^{2k}. \end{aligned}$$

Thus

$$1 = v(V_1)^2(V_{p-1}z)^{2k}(V_{p-2}z)^{2k}\,v(V_{p-2})(\gamma_{p-2}z+\delta_{p-2})^{2k}.$$

Proceeding inductively, we obtain

$$1 = v(V_1)^p (V_{p-1}z)^{2k} (V_{p-2}z)^{2k} \cdots (V_1 z)^{2k} z^{2k}. \tag{17}$$

If we put $z = e^{\pi i/p}$, then we have $V_{p-1}z = V_{p-2}z = \cdots = V_1 z = z = e^{\pi i/p}$. Substituting this in (17) and observing the fact that $v(V_1) = v(S_\lambda T) = v(S_\lambda) v(T) = e^{2\pi i\kappa} v(T)$, we get

$$v(T)^p e^{2\pi ip\kappa} e^{2\pi ik} = 1. \tag{18}$$

(i) If $v(T) = e^{-\pi ik}$ or p is an even integer, then $v(T)^p = e^{-\pi ikp}$ and (18) yields

$$e^{-\pi ipk + 2\pi ip\kappa + 2\pi ik} = 1;$$

that is, $(p-2)k - 2p\kappa$ is an even integer.

(ii) If $v(T) = -e^{-\pi ik}$ *and* p is an odd integer, then $v(T)^p = -e^{-\pi ikp}$ and (18) gives

$$e^{-\pi ipk + 2\pi ip\kappa + 2\pi ik} = -1;$$

that is, $(p-2)k - 2p\kappa$ is an odd integer. This completes the proof.

2.2. **Log-Polynomial Period Functions.**

A log-polynomial sum is a function of the form

$$q(z) = \sum_{j=0}^{n} z^{\gamma_j} \sum_{t=0}^{m_j} c(j,t) (\log z)^t, \tag{19}$$

where $\gamma_1, \ldots, \gamma_n$ and the coefficients $c(j,t)$ are complex numbers, n, j, m_j and t are nonnegative integers. Here z^α is defined by $e^{\alpha \log z}$, where $\log z$ is the principal branch of the logarithm function.

A log-polynomial sum $q(z)$ is said to be a *log-polynomial period function (LPPF) of weight* $2k$ *and multiplier system* v *for the Hecke group* $G(\lambda)$, if there exists a function F defined and holomorphic in $\mathcal{H}$ such that:

$$e^{-2\pi i\kappa} F(z+\lambda) = F(z), \tag{20}$$

$$\bar{v}(T) z^{-2k} F\left(-\frac{1}{z}\right) = F(z) + q(z), \tag{21}$$

where $e^{2\pi i\kappa} = v(S_\lambda)$.

A function F satisfying (20) and (21) is called an *automorphic integral of weight* $2k$ *and multiplier system* v *for* $G(\lambda)$, if it has an exponential series expansion:

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i(n+\kappa)z/\lambda}, \tag{22}$$

where $a_n \in \mathbf{C}$ and satisfy the growth condition

$$a_n = \mathcal{O}(n^c) \quad \text{as} \quad n \to \infty, \qquad c > 0. \tag{23}$$

In this case we say that q *is the log-polynomial period function of the automorphic integral* F.

If we use the *slash operator* $|_v^{2k}$ defined by

$$\left(F \,|_v^{2k}\, M\right)(z) = \bar{v}(M) (cz+d)^{-2k} F(Mz),$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda)$ and $z \in \mathcal{H}$, then (20) and (21) become

$$F \mid_v^{2k} S_\lambda = F \qquad \text{and} \qquad F \mid_v^{2k} T = F + q, \tag{24}$$

respectively. It can be shown easily that the consistency condition (5) for a multiplier system v in weight $2k$ for $G(\lambda)$ is equivalent to

$$F \mid_v^{2k} M_1 M_2 = \left(F \mid_v^{2k} M_1\right) \mid_v^{2k} M_2, \tag{25}$$

where $M_1, M_2 \in G(\lambda)$. From now on we shall write $F \mid M$ for $F \mid_v^{2k} M$.

Remark 2.3 The assumptions made in Lemmas 2.1 and 2.2 are no restrictions of generality in the sense that if $v(-I)(-1)^{2k} = -1$, then there is no nontrivial LPPF corresponding to v. To see this, suppose that q is an LPPF of weight $2k$ and multiplier system v for $G(\lambda)$. Then there exists F satisfying (20) and (21). From the second equation in (24) we get,

$$F \mid (-I) = F \mid T^2 = (F \mid T) \mid T = (F+q) \mid T = F \mid T + q \mid T = F + q + q \mid T.$$

That is,

$$\bar{v}(-I)(-1)^{2k} F(z) = F(z) + q(z) + \bar{v}(T) z^{-2k} q\left(-\frac{1}{z}\right),$$

and hence

$$\left(\bar{v}(-I)(-1)^{2k} - 1\right) F(z) = q(z) + \bar{v}(T) z^{-2k} q\left(-\frac{1}{z}\right). \tag{26}$$

Let us suppose that $v(-I)(-1)^{2k} = -1$. Then (26) reduces to

$$-2F(z) = q(z) + \bar{v}(T) z^{-2k} q\left(-\frac{1}{z}\right). \tag{27}$$

Since $q(z) + \bar{v}(T) z^{-2k} q\left(-\frac{1}{z}\right)$ is a log-polynomial sum, (27) implies that F is of the form

$$F(z) = \sum_{j=0}^{J} z^{\delta_j} \sum_{t=0}^{m_j} d(j,t) (\log z)^t.$$

If F is given by the above sum, then $F(n\lambda)$ is well-defined for each $n \in \mathbf{Z}$, $n \neq 0$. From the fact that $F(z+\lambda) = e^{2\pi i\kappa} F(z)$, we deduce that $F(z+n\lambda) = e^{2\pi i n\kappa} F(z)$. Thus,

$$\lim_{z \to 0} F(z) = e^{-2\pi i n\kappa} F(n\lambda).$$

This implies that (i) $Re(\delta_j) > 0$ or (ii) $Re(\delta_j) = 0$ and $m_j = 0$. Suppose that (i) or (ii) holds. Let $L = \lim_{z\to 0} F(z)$. Then the above limit yields $F(n\lambda) = Le^{2\pi i n\kappa}$. This in turn implies that F is bounded as $n \to \infty$, and so $Re(\delta_j) = 0$ with $m_j = 0$. In other words, (i) cannot occur. As we shall see in Section 3, Lemma 3.7, $\lim_{z\to 0} F(z)$ does not exist if F is of the form

$$F(z) = \sum_{j=0}^{J} d_j z^{iu_j},$$

where $u_1, \ldots, u_J$ are real numbers. Hence $F \equiv a$, where a is a constant.

If $\kappa \neq 0$, then $F(z+\lambda) = e^{2\pi i\kappa} F(z)$ does not hold for a nonzero constant function F. Hence, if $\kappa \neq 0$, $F \equiv 0$ and consequently, $q \equiv 0$. If $\kappa = 0$, and $F \equiv a$, then $F \mid T = F + q$ implies that

$$q(z) = a\left(\bar{v}(T)\, z^{-2k} - 1\right).$$

Because they are periods of constant automorphic integrals, we call these functions q the trivial period functions.

Thus the multiplier systems associated to nontrivial LPPF's are those for which

$$v(-I)(-1)^{2k} = 1, \tag{28}$$

thereby justifying the assumptions made in Lemmas 2.1 and 2.2. Henceforth we shall assume that *if v is a multiplier system of weight $2k$ for $G(\lambda)$, then (28) holds.* Note that (28) now yields

$$q \;+\; q \mid T \;= 0. \tag{29}$$

If $\lambda \geq 2$, then $T^2 = -I$ is the only relation between the generators S_λ and T of $G(\lambda)$. Consequently, (29) is the only condition we have on an LPPF. In fact, if $r(z)$ is any log-polynomial sum, and if we define q by

$$q(z) = r(z) - \bar{v}(T)\, z^{-2k} r\left(-\frac{1}{z}\right),$$

then q satisfies (29). In [9] and [12], Knopp has shown for these groups that if q satisfies (29), then it is a period of an automorphic integral F of weight $2k$ and multiplier system v.

It is customary to denote by q_M the period function associated with $M \in G(\lambda)$. Thus $F \mid M = F + q_M$ and from (25) we deduce that

$$q_{M_1 M_2} = q_{M_1} \mid M_2 + q_{M_2}. \tag{30}$$

Clearly, $q_I = q_{-I} \equiv 0$. When λ is given by (15), we have $(S_\lambda T)^p = -I$. Repeated application of (30) yields

$$q + q \mid (S_\lambda T) + q \mid (S_\lambda T)^2 + \cdots + q \mid (S_\lambda T)^{p-1} = 0. \tag{31}$$

In the next section we shall make use of (29) and (31) to determine the LPPF's of positive weight for the discrete Hecke groups. Then we shall apply Bol's Theorem to obtain log-polynomial period functions of negative integral weight.

3. LOG-POLYNOMIAL PERIOD FUNCTIONS FOR THE DISCRETE HECKE GROUPS

3.1. **Preliminaries.** Throughout this section, k will be a fixed real number and v will be a multiplier system in weight $2k$ for the Hecke group $G(\lambda)$, where λ is given by (15). Let us write $\lambda = 2\cos\pi\theta$ with $\theta = 1/p$. We define $V_n = (S_\lambda T)^n$ and

$$M_n = V_n T = (S_\lambda T)^n\, T = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad \text{for} \quad n = 1, \ldots, p.$$

A straightforward induction shows that

$$a_n = -\frac{\sin n\pi\theta}{\sin \pi\theta}, \qquad b_n = -\frac{\sin (n+1)\pi\theta}{\sin \pi\theta},$$
$$c_n = -\frac{\sin (n-1)\pi\theta}{\sin \pi\theta}, \qquad d_n = -\frac{\sin n\pi\theta}{\sin \pi\theta}. \tag{32}$$

In what follows we shall assume that $k > 0$. Suppose q is an LPPF of weight $2k$ and multiplier system v for $G(\lambda)$. Combining (29) and (31), we obtain

$$q \mid M_{p-1} + q \mid M_{p-2} + \cdots + q \mid M_1 - q = 0. \tag{33}$$

From (32) we see that $b_n \neq 0$ and $d_n \neq 0$ for $n = 1, \dots, p-2$. Thus we have

$$\lim_{z\to 0} (q \mid M_n)(z) = \lim_{z\to 0} \bar{v}(M_n)(c_n z + d_n)^{-2k} q\left(\frac{a_n z + b_n}{c_n z + d_n}\right) = L_n, \tag{34}$$

where

$$L_n = \bar{v}(M_n)\, d_n^{-2k} q\left(\frac{b_n}{d_n}\right),\ n = 1, \dots, p-2.$$

Combining (33) and (34), we get

$$lim_{z\to 0}(q \mid M_{p-1} - q)(z) = L,$$

where

$$L = -\sum_{n=1}^{p-2} L_n = -\sum_{n=1}^{p-2} \bar{v}(M_n)\, d_n^{-2k} q\left(\frac{b_n}{d_n}\right). \tag{35}$$

Since $M_{p-1} = (S_\lambda T)^{p-1} T = T S_\lambda^{-1} T = \begin{pmatrix} -1 & 0 \\ -\lambda & -1 \end{pmatrix}$, and since by consistency condition (5) for v we have

$$\bar{v}\left(T S_\lambda^{-1} T\right)(-\lambda z - 1)^{-2k} = \bar{v}\left(-T S_\lambda^{-1} T\right)(\lambda z + 1)^{-2k} = v(S_\lambda)(\lambda z + 1)^{-2k}$$

(where in the last equality we have used Lemma 2.1 (iii)), we see that

$$(q \mid M_{p-1})(z) = v(S_\lambda)(\lambda z + 1)^{-2k} q\left(\frac{z}{\lambda z + 1}\right).$$

Consequently,

$$lim_{z\to 0}\left\{v(S_\lambda)(\lambda z + 1)^{-2k} q\left(\frac{z}{\lambda z + 1}\right) - q(z)\right\} = L.$$

We now replace z by $1/z$ in the last limit to obtain

$$\lim_{z\to\infty}\left\{v(S_\lambda)\left(\frac{z+\lambda}{z}\right)^{-2k} q\left(\frac{1}{z+\lambda}\right) - q\left(\frac{1}{z}\right)\right\} = L. \tag{36}$$

On the other hand, from (32) we also see that $a_n \neq 0$ and $c_n \neq 0$, for each $n = 2, \dots, p-1$. Since $k > 0$, it then follows that for $n = 2, \dots, p-1$,

$$\lim_{z\to\infty} (q \mid M_n)(z) = \lim_{z\to\infty} \bar{v}(M_n)(c_n z + d_n)^{-2k} q\left(\frac{a_n z + b_n}{c_n z + d_n}\right) = 0. \tag{37}$$

Using (33) and (37), we obtain

$$lim_{z\to\infty}(q \mid M_1 - q)(z) = 0.$$

Since $M_1 = (S_\lambda T)\, T = -S_\lambda$ and $v\,(S_\lambda) = v\,(-S_\lambda)$, the above limit becomes

$$\lim_{z\to\infty} \{\bar{v}\,(S_\lambda)\, q\,(z+\lambda) - q\,(z)\} = 0. \tag{38}$$

Remark 3.1: Let $q\,(z)$ be a log-polynomial sum given by

$$q\,(z) = \sum_{j=0}^{n} z^{\gamma_j} \sum_{t=0}^{m_j} c\,(j,t)\,(\log z)^t\,,$$

where $\gamma_1, \ldots, \gamma_n$ and the coefficients $c\,(j,t)$ are complex numbers and m_j and t are nonnegative integers. It is no loss of generality to assume that

$$Re\,(\gamma_1) \le \cdots \le Re\,(\gamma_{n_0}) \le 0 < Re\,(\gamma_{n_0+1}) \le \cdots \le Re\,(\gamma_n)\,,$$

with the m_j's satisfying the condition

$$m_j \le m_l \;\text{ whenever }\; j < l \;\text{ and }\; Re\,(\gamma_j) = Re\,(\gamma_l)\,.$$

Thus any log-polynomial sum can be expressed in the form:

$$q\,(z) = \sum_{j=1}^{N}\sum_{t=0}^{m_j} a\,(j,t)\, z^{-\beta_j}\,(\log z)^t \;+\; \sum_{l=1}^{M}\sum_{t=0}^{n_l} b\,(l,t)\, z^{\alpha_l}\,(\log z)^t\,, \tag{39}$$

where

$$\begin{aligned} 0 \le Re\,(\beta_1) \le \cdots \le Re\,(\beta_N)\,;\;\; m_j \le m_l \;\text{ if }\; Re\,(\beta_j) = Re\,(\beta_l) \;\;\; (j<l)\,; \\ 0 < Re\,(\alpha_1) \le \cdots \le Re\,(\alpha_M)\,;\;\; n_j \le n_l \;\text{ if }\; Re\,(\alpha_j) = Re\,(\alpha_l) \;\;\; (j<l)\,. \end{aligned} \tag{40}$$

The following notation will be helpful for further investigation of the limits in (36) and (38). Let $z, \alpha \in \mathbf{C}$, $z \ne 0$,$z \ne -\lambda$, and $\nu \in \mathbf{Z}$. Then we define

$$\begin{aligned} \tau\,(z) &= \frac{z\log z - (z+\lambda)\log\,(z+\lambda)}{\log z}, \\ f_{\alpha,\nu}\,(z) &= C_1\left(\frac{z+\lambda}{z}\right)^{\alpha}\left(\frac{\log\,(z+\lambda)}{\log z}\right)^{\nu} - 1, \\ g_{\alpha,\nu}\,(z) &= C_2\left(\frac{z+\lambda}{z}\right)^{\alpha-2k}\left(\frac{\log\,(z+\lambda)}{\log z}\right)^{\nu} - 1, \\ \phi_{\alpha,\nu}\,(z) &= C_1\,(z+\lambda)^{\alpha}\,(\log\,(z+\lambda))^{\nu} - z^{\alpha}\,(\log z)^{\nu}\,, \\ \psi_{\alpha,\nu}\,(z) &= C_2\left(\frac{z+\lambda}{z}\right)^{-2k}(z+\lambda)^{\alpha}\,(\log\,(z+\lambda))^{\nu} - z^{\alpha}\,(\log z)^{\nu}\,. \end{aligned}$$

Note then that

$$\phi_{\alpha,\nu}\,(z) = z^{\alpha}\,(\log z)^{\nu}\, f_{\alpha,\nu}\,(z) \qquad\text{and}\qquad \psi_{\alpha,\nu}\,(z) = z^{\alpha}\,(\log z)^{\nu}\, g_{\alpha,\nu}\,(z)\,.$$

Thus if $q\,(z)$ is given by (39), then we can rewrite $C_1 q\,(z+\lambda) - q\,(z)$ as

$$\begin{aligned} C_1 q\,(z+\lambda) - q\,(z) &= \sum_{j=1}^{N}\sum_{t=0}^{m_j} a\,(j,t)\,\phi_{-\beta_j,t}\,(z) + \sum_{l=1}^{M}\sum_{t=0}^{n_l} b\,(l,t)\,\phi_{\alpha_l,t}\,(z) \\ &= \phi_{\alpha_M,n_M}\,(z)\, F\,(z)\,, \end{aligned}$$

where $C_1 = \bar{v}(S_\lambda) = e^{-2\pi i\kappa}$ and

$$F(z) = \sum_{j=1}^{N}\sum_{t=0}^{m_j} a(j,t)\,\frac{\phi_{-\beta_j,t}(z)}{\phi_{\alpha_M,n_M}(z)} \quad + \quad \sum_{l=1}^{M}\sum_{t=0}^{n_l} b(l,t)\,\frac{\phi_{\alpha_l,t}(z)}{\phi_{\alpha_M,n_M}(z)}. \tag{41}$$

Similarly,

$$\begin{aligned} &C_2\left(\frac{z+\lambda}{z}\right)^{-2k} q\left(\frac{1}{z+\lambda}\right) - q\left(\frac{1}{z}\right) = \\ &\quad \sum_{j=1}^{N}\sum_{t=0}^{m_j} a(j,t)(-1)^t\,\psi_{\beta_j,t}(z) \quad + \quad \sum_{l=1}^{M}\sum_{t=0}^{n_l} b(l,t)(-1)^t\,\psi_{-\alpha_l,t}(z) \\ &= \quad \psi_{\beta_N,m_N}(z)\,G(z), \end{aligned}$$

where $C_2 = \bar{v}\left(-TS_\lambda^{-1}T\right) = e^{2\pi i\kappa}$ and

$$H(z) = \sum_{j=1}^{N}\sum_{t=0}^{m_j} a(j,t)(-1)^t\,\frac{\psi_{\beta_j,t}(z)}{\psi_{\beta_N,m_N}(z)} \quad + \quad \sum_{l=1}^{M}\sum_{t=0}^{n_l} b(l,t)(-1)^t\,\frac{\psi_{-\alpha_l,t}(z)}{\psi_{\beta_N,m_N}(z)}. \tag{42}$$

If we combine (36) with the equation before (41), and (40) with the equation before (42), we have proved the following result.

Proposition 3.1. *If q is an LPPF of weight $2k$ and multiplier system v for $G(\lambda)$, given by (39) and (40), then*

$$\lim_{z\to\infty} \phi_{\alpha_M,n_M}(z)\,F(z) = 0 \quad \text{and} \quad \lim_{z\to\infty} \psi_{\beta_N,m_N}(z)\,H(z) = L, \tag{43}$$

where F and H are given by (41) and (42), respectively, and L is as in (35).

3.2. **Log-Polynomial Period Functions of Positive Weight.** In this section we shall show that $\lim_{z\to\infty} F(z) = 0$ and $\lim_{z\to\infty} H(z) = 0$ are both impossible. In doing so we will arrive at the necessary and sufficient conditions for the boundedness and existence of the limit at infinity for the functions ϕ_{α_M,n_M} and ψ_{β_N,m_N}. By (40) this will enable us to determine, if it exists, an LPPF of positive weight. To this end, we begin by proving the following limit-lemmas. In Lemmas 3.1 to 3.5, we assume that $\alpha, \beta \in \mathbf{C}$, $\nu, \mu \in \mathbf{Z}$, and we shall say:

I holds if $Re(\alpha) < 0,\ \nu \in \mathbf{Z}$ or $Re(\alpha) = 0,\ \nu < 0$.
II holds if $Re(\alpha) > 0,\ \nu \in \mathbf{Z}$ or $Re(\alpha) = 0,\ \nu > 0$.
III holds if $Re(\alpha) < 1,\ \nu \in \mathbf{Z}$ or $Re(\alpha) = 1,\ \nu < 0$.
IV holds if $Re(\alpha) > 1,\ \nu \in \mathbf{Z}$ or $Re(\alpha) = 1,\ \nu > 0$.
V holds if $\beta \neq 0,\ Re(\alpha) < Re(\beta),\ \nu,\ \mu \in \mathbf{Z}$ or $\beta \neq 0,\ Re(\alpha) = Re(\beta),\ \nu < \mu$.
VI holds if $Re(\alpha) < 0,\ \nu,\ \mu \in \mathbf{Z}$ or $Re(\alpha) = 0,\ \nu < \mu - 1$.

Lemma 3.1.
(a) $\lim_{z\to\infty} \tau(z) = -\lambda$.
(b) $\lim_{z\to\infty} z^\alpha (\log z)^\nu = \begin{cases} 0, & \text{if } I \text{ holds} \\ \infty, & \text{if } II \text{ holds.} \end{cases}$

Proof : This is a routine application of L'Hospital's Rule.

Lemma 3.2. *Suppose $\kappa = 0$. Then $C_1 = 1$ and we have:*

(a) $\lim_{z\to\infty} \phi_{\alpha,\nu}(z) = \begin{cases} 0, & \text{if } III \text{ holds} \\ \infty, & \text{if } IV \text{ holds.} \end{cases}$

(b) *If V or VI holds, then*

$$\lim_{z\to\infty} \frac{\phi_{\alpha,\nu}(z)}{\phi_{\beta,\mu}(z)} = 0.$$

Proof : (a) When $C_1 = 1$, we have

$$\phi_{\alpha,\nu}(z) = z^{\alpha} (\log z)^{\nu} f_{\alpha,\nu}(z),$$

where

$$f_{\alpha,\nu}(z) = \left(\frac{z+\lambda}{z}\right)^{\alpha} \left(\frac{\log(z+\lambda)}{\log z}\right)^{\nu} - 1.$$

But then $lim_{z\to\infty} f_{\alpha,\nu}(z) = 0$. If $Re(\alpha) < 0,\ \nu \in \mathbf{Z}$ or $Re(\alpha) = 0,\ \nu \le 0$, then $z^{\alpha}(\log z)^{\nu}$ is bounded as $z \to \infty$ and hence $\lim_{z\to\infty} \phi_{\alpha,\nu}(z) = 0$.

Next suppose $Re(\alpha) > 0$ and $\nu \in \mathbf{Z}$. Then $\lim_{z\to\infty} z^{-\alpha}(\log z)^{-\nu} = 0$ and by L'Hospital's Rule, we have

$$\lim_{z\to\infty} \phi_{\alpha,\nu}(z) = \lim_{z\to\infty} \frac{f_{\alpha,\nu}(z)}{z^{-\alpha}(\log z)^{-\nu}} = \lim_{z\to\infty} \frac{\frac{d}{dz}(f_{\alpha,\nu}(z))}{\frac{d}{dz}\left(z^{-\alpha}(\log z)^{-\nu}\right)}.$$

But,

$$\frac{d}{dz}(f_{\alpha,\nu}(z)) = \frac{1}{z^2}\left(\frac{z+\lambda}{z}\right)^{\alpha-1}\left(\frac{\log(z+\lambda)}{\log z}\right)^{\nu}\left\{-\alpha\lambda + \frac{\nu\tau(z)}{\log z}\right\} \tag{44}$$

and

$$\frac{d}{dz}\left(z^{-\alpha}(\log z)^{-\nu}\right) = z^{-\alpha-1}(\log z)^{-\nu}\left\{-\alpha - \frac{\nu}{\log z}\right\}. \tag{45}$$

Thus,

$$\begin{aligned} \lim_{z\to\infty} \phi_{\alpha,\nu}(z) &= \lim_{z\to\infty} (z+\lambda)^{\alpha-1}(\log(z+\lambda))^{\nu}\left(\frac{-\alpha\lambda + \nu\tau(z)/\log z}{-\alpha - \nu/\log z}\right) \\ &= \lambda \lim_{z\to\infty} (z+\lambda)^{\alpha-1}(\log(z+\lambda))^{\nu}, \end{aligned}$$

since $\alpha \neq 0$. By Lemma 3.1 (b), we then have

$$\lim_{z\to\infty} \phi_{\alpha,\nu}(z) = \begin{cases} 0, & \text{if } Re(\alpha) < 0,\ \nu \in \mathbf{Z} \text{ or } Re(\alpha) = 0,\ \nu \le 0 \\ 0, & \text{if } 0 < Re(\alpha) < 1,\ \nu \in \mathbf{Z} \text{ or } Re(\alpha) = 1,\ \nu < 0 \\ \infty, & \text{if } Re(\alpha) > 1,\ \nu \in \mathbf{Z} \text{ or } Re(\alpha) = 1,\ \nu > 0. \end{cases}$$

It remains to consider the case when $Re(\alpha) = 0,\ \nu > 0$. If $\alpha \neq 0$, then (44) and (45) are applicable and we have the desired result. So suppose $\alpha = 0$ and $\nu > 0$. In this case we have

$$\frac{d}{dz}(f_{0,\nu}(z)) = \left(\frac{\log(z+\lambda)}{\log z}\right)^{\nu} \frac{\nu\tau(z)}{z(z+\lambda)\log z} \tag{46}$$

and

$$\frac{d}{dz}\left((\log z)^{-\nu}\right) = \frac{-\nu}{z}(\log z)^{-\nu-1}.$$

Hence

$$\lim_{z\to\infty} \phi_{0,\nu}(z) = -\lim_{z\to\infty} \frac{\tau(z)(\log z)(\log(z+\lambda))^{\nu-1}}{z+\lambda} = 0.$$

This completes the proof of (a).

(b) Once again we apply L'Hospital's Rule. We need to consider four cases.
Case 1. $\alpha \neq 0$ and $\beta \neq 0$. From (44), we see that

$$\lim_{z\to\infty} \frac{\phi_{\alpha,\nu}(z)}{\phi_{\beta,\mu}(z)} = \lim_{z\to\infty} \frac{f_{\alpha,\nu}(z)}{f_{\beta,\mu}(z)} \frac{z^{\alpha}(\log z)^{\nu}}{z^{\beta}(\log z)^{\mu}} = \frac{\alpha}{\beta} \lim_{z\to\infty} z^{\alpha-\beta}(\log z)^{\nu-\mu}.$$

We now apply Lemma 3.1 (b) for the desired conclusions.

Case 2. $\alpha = 0$ and $\beta \neq 0$. From (44) and (46) we deduce that

$$\begin{aligned} &\lim_{z\to\infty} \frac{f_{0,\nu}(z)}{f_{\beta,\mu}(z)} \\ =\;& \lim_{z\to\infty} \frac{\left(\left(\frac{\log(z+\lambda)}{\log z}\right)^{\nu} \frac{\nu\tau(z)}{z(z+\lambda)\log z}\right)}{\left(\frac{1}{z^2}\left(\frac{z+\lambda}{z}\right)^{\beta-1}\left(\frac{\log(z+\lambda)}{\log z}\right)^{\mu}\left\{-\beta\lambda + \frac{\mu\tau(z)}{\log z}\right\}\right)} \\ =\;& \lim_{z\to\infty} \nu \left(\frac{z+\lambda}{z}\right)^{-\beta}\left(\frac{\log(z+\lambda)}{\log z}\right)^{\nu-\mu-1} \frac{\tau(z)}{\log(z+\lambda)} \frac{1}{(-\beta\lambda + \mu\tau(z)/\log z)} \\ =\;& 0. \end{aligned}$$

If V holds, then $z^{\alpha-\beta}(\log z)^{\nu-\mu}$ is bounded and hence

$$\lim_{z\to\infty} \frac{\phi_{0,\nu}(z)}{\phi_{\beta,\mu}(z)} = \lim_{z\to\infty} z^{\alpha-\beta}(\log z)^{\nu-\mu} \frac{f_{0,\nu}(z)}{f_{\beta,\mu}(z)} = 0.$$

Case 3. $\alpha \neq 0$ and $\beta = 0$. We should note that when $\beta = 0$ we must assume that $\mu \neq 0$ for the obvious reason that $\phi_{0,0} \equiv 0$. Write

$$\begin{aligned} \frac{\phi_{\alpha,\nu}(z)}{\phi_{\beta,\mu}(z)} &= \frac{z^{\alpha}(\log z)^{\nu-\mu} f_{\alpha,\nu}(z)}{f_{0,\mu}(z)} \\ &= \frac{h(z) f_{\alpha,\nu}(z)}{f_{0,\mu}(z)}, \end{aligned}$$

where $h(z) = z^{\alpha}(\log z)^{\nu-\mu}$.

If $Re(\alpha) < Re(\beta) = 0$ or $Re(\alpha) = 0,\ \nu < \mu$, then $\lim_{z\to\infty} h(z) = 0$ and applying L'Hospital's Rule, we get

$$\lim_{z\to\infty} \frac{\phi_{\alpha,\nu}(z)}{\phi_{0,\mu}(z)} = \lim_{z\to\infty} \left(\frac{h'(z) f_{\alpha,\nu}(z)}{f'_{0,\mu}(z)} + \frac{h(z) f'_{\alpha,\nu}(z)}{f'_{0,\mu}(z)}\right).$$

But,

$$h'(z) = z^{\alpha-1}(\log z)^{\nu-\mu}\left\{\alpha + \frac{\nu-\mu}{\log z}\right\},$$

and using (46), we get

$$\frac{h'(z) f_{\alpha,\nu}(z)}{f'_{0,\mu}(z)} = g_1(z) z^{\alpha}(\log z)^{\nu-\mu+1}(z+\lambda) f_{\alpha,\nu}(z),$$

where

$$g_1(z) = \frac{1}{\nu\tau(z)}\left(\frac{log(z+\lambda)}{\log z}\right)^{1-\mu}\left\{\alpha + \frac{\nu-\mu}{\log z}\right\}.$$

From (44) and (46) we get

$$\frac{h(z)\, f'_{\alpha,\nu}(z)}{f'_{0,\mu}(z)} = g_2(z)\, z^{\alpha} (\log z)^{\nu-\mu+1},$$

where

$$g_2(z) = \frac{1}{\nu\tau(z)} \left(\frac{z+\lambda}{z}\right)^{\alpha} \left(\frac{log(z+\lambda)}{\log z}\right)^{\nu-\mu+1} \left\{-\alpha\lambda + \frac{\nu\tau(z)}{\log z}\right\}.$$

Since $\lim_{z\to\infty} g_1(z) = -\frac{\alpha}{\nu\lambda}$, $\lim_{z\to\infty} g_2(z) = \frac{\alpha}{\nu}$, and by L'Hospital's Rule,

$$\lim_{z\to\infty} (z+\lambda)\, f_{\alpha,\nu}(z) = \lim_{z\to\infty} \frac{f_{\alpha,\nu}(z)}{\frac{1}{z+\lambda}} = \lim_{z\to\infty} \frac{f'_{\alpha,\nu}(z)}{\frac{-1}{(z+\lambda)^2}} = -\lim_{z\to\infty} f'_{\alpha,\nu}(z)\,(z+\lambda)^2 = \alpha\lambda,$$

we have

$$\lim_{z\to\infty} \{g_1(z)(z+\lambda) f_{\alpha,\nu}(z) + g_2(z)\} = \frac{-(\alpha)^2+\alpha}{\nu} = \frac{\alpha(1-\alpha)}{\nu} \neq 0.$$

(Note that $Re(\alpha) < 0$ or $Re(\alpha) = 0,\ \alpha \neq 0$.) Thus if VI holds, then

$$\begin{aligned}
\lim_{z\to\infty} \frac{\phi_{\alpha,\nu}(z)}{\phi_{0,\mu}(z)} &= \lim_{z\to\infty} \left(\frac{h'(z)\, f_{\alpha,\nu}(z)}{f'_{0,\mu}(z)} + \frac{h(z)\, f'_{\alpha,\nu}(z)}{f'_{0,\mu}(z)}\right) \\
&= \lim_{z\to\infty} z^{\alpha} (\log z)^{\nu-\mu+1} \{g_1(z)(z+\lambda) f_{\alpha,\nu}(z) + g_2(z)\} \\
&= \frac{\alpha(1-\alpha)}{\nu} \lim_{z\to\infty} z^{\alpha} (\log z)^{\nu-\mu+1} \\
&= 0.
\end{aligned}$$

Case 4. $\alpha = 0$ and $\beta = 0$. Then

$$\lim_{z\to\infty} \frac{f_{0,\nu}(z)}{f_{0,\mu}(z)} = \frac{\nu}{\mu}.$$

In this case we must have $\nu \neq 0$ and $\mu \neq 0$ and hence, if $\nu < \mu$, we have

$$\lim_{z\to\infty} \frac{\phi_{0,\nu}(z)}{\phi_{0,\mu}(z)} = 0.$$

This completes the proof of the lemma.

The following three lemmas are immediate consequences of the above two lemmas.

Lemma 3.3. *Suppose $\kappa = 0$. Then $C_2 = 1$ and we have*

(*a*) $\lim_{z\to\infty} \psi_{\alpha,\nu}(z) = \begin{cases} 0, & \text{if } III \text{ holds} \\ \infty, & \text{if } IV \text{ holds.} \end{cases}$

(*b*) *Suppose $\beta \neq 2k$. If V holds, then*

$$\lim_{z\to\infty} \frac{\psi_{\alpha,\nu}(z)}{\psi_{\beta,\mu}(z)} = 0.$$

(*c*) *If $Re(\alpha) < 2k,\ \nu, \mu \in \mathbf{Z}$ or if $Re(\alpha) = 2k,\ \nu < \mu - 1$, then*

$$\lim_{z\to\infty} \frac{\psi_{\alpha,\nu}(z)}{\psi_{2k,\mu}(z)} = 0.$$

Lemma 3.4. *Suppose* $\kappa > 0$. *Then*
(*a*) $\lim_{z\to\infty} \phi_{\alpha,\nu}(z) = \begin{cases} 0, & \text{if } I \text{ holds} \\ \infty, & \text{if } II \text{ holds.} \end{cases}$
(*b*) *If* V *holds or if* $\beta = 0, Re(\alpha) < 0, \nu, \mu \in \mathbf{Z}$ *or if* $\beta = 0, Re(\alpha) = 0, \nu < \mu$, *then*

$$\lim_{z\to\infty} \frac{\phi_{\alpha,\nu}(z)}{\phi_{\beta,\mu}(z)} = 0.$$

Lemma 3.5. *Suppose* $\kappa > 0$. *Then*
(*a*) $\lim_{z\to\infty} \psi_{\alpha,\nu}(z) = \begin{cases} 0, & \text{if } I \text{ holds} \\ \infty, & \text{if } II \text{ holds.} \end{cases}$
(*b*) *If* V *holds or if* $\beta = 0, Re(\alpha) < 0, \nu, \mu \in \mathbf{Z}$ *or if* $\beta = 0, Re(\alpha) = 0, \nu < \mu$, *then*

$$\lim_{z\to\infty} \frac{\psi_{\alpha,\nu}(z)}{\psi_{\beta,\mu}(z)} = 0.$$

Lemma 3.6. *Let* $\alpha_1, \ldots, \alpha_N$ *be distinct complex numbers and* $P_1(x), \cdots, P_N(x)$ *be polynomials. If*

$$\sum_{j=1}^{N} P_j(x)\, e^{\alpha_j x} \equiv 0, \quad \textit{then} \quad P_j \equiv 0 \quad \textit{for all } j.$$

Proof : The proof is a simple induction on N. For $N = 1$, the lemma is trivial. So suppose the lemma is true for N and assume that $\sum_{j=1}^{N+1} P_j(x)\, e^{\alpha_j x} = 0$, where $\alpha_1, \ldots, \alpha_{N+1}$ are distinct. Put $\beta_j = \alpha_j - \alpha_{N+1}$. Then, dividing the above equation by $e^{\alpha_{N+1} x}$, we get

$$P_{N+1}(x) + \sum_{j=1}^{N} P_j(x)\, e^{\beta_j x} \equiv 0.$$

If $P_{N+1}(x) \equiv 0$, then the rest of the polynomials are also zero by the induction hypothesis. So suppose $P_{N+1}(x) \neq 0$ and let m be the degree of P_{N+1}. If we differentiate the last equation $m + 1$ times, we obtain

$$\sum_{j=1}^{N} Q_j(x)\, e^{\beta_j x} \equiv 0, \quad \text{where} \quad Q_j(x) = \sum_{r=0}^{m+1} \begin{pmatrix} m+1 \\ r \end{pmatrix} \beta_j^r P_j^{(m+1-r)}(x).$$

Here $P^{(t)}(x)$ is the t^{th} derivative of $P(x)$.

Since β_j are distinct we have, by the induction assumption, that $Q_j(x) \equiv 0$ for each $j = 1, \ldots, N$.

However, no nonzero polynomial P satisfies a nontrivial differential equation of the form $\sum_{r=0}^{n} c_r P^{(r)}(x) \equiv 0$, with $c_r \neq 0$ for all r. For if $P(x) = a_m x^m + \cdots + a_0$, $a_m \neq 0$, is a polynomial, then

$P^{(r)}(x) = m(m-1)\cdots(m-r+1)a_m x^{m-r} + \cdots + r!a_r$ and hence the leading coefficient of $\sum_{r=0}^{n} c_r P^{(r)}(x)$ is $c_0 a_m$, which is nonzero. Therefore, P cannot satisfy the differential equation.

Thus $P_j \equiv 0$ for each $j = 1, \ldots, N$ and hence $P_{N+1} \equiv 0$. This completes the proof of the lemma.

Corollary 3.1. *If $\gamma_1, \ldots, \gamma_N$ are distinct complex numbers and if*

$$\sum_{j=1}^{N} \sum_{t=0}^{m_j} c\,(j,t)\, z^{\gamma_j}\, (\log z)^t = 0,$$

then

$c\,(j,t) = 0$ *for all j and all t.*

Lemma 3.7. *If $c_1, \ldots, c_N$ are complex numbers not all zero and if $u_1, \ldots, u_N$ are distinct real numbers, then $\lim_{z\to\infty} \sum_{j=1}^{N} c_j z^{iu_j}$ exists if and only if $N = 1$ and $u_1 = 0$.*

Proof : Suppose $N > 1$ and the limit exists and is equal to L. Let us assume that $u_j \neq 0$ for all j. Fix φ and let

$$z_n = r_n e^{i\varphi} \quad \text{where} \quad r_n = e^{\frac{2\pi n}{|u_1|}}.$$

Claim. There exist a subsequence $\{z_{n_m}\}$ of $\{z_n\}$ and complex numbers $\xi_1, \ldots, \xi_N$, independent of φ and of absolute value 1, such that

$$\lim_{m\to\infty} z_{n_m}^{iu_j} = \xi_j e^{-u_j\varphi} \quad \text{for all} \quad j = 1, \ldots, N.$$

Assume the claim for a moment. Then, since $z_n \to \infty$ as $n \to \infty$, we have $z_{n_m} \to \infty$ as $m \to \infty$, and hence

$$L = \lim_{m\to\infty} \sum_{j=1}^{N} c_j z_{n_m}^{iu_j} = \sum_{j=1}^{N} c_j \lim_{m\to\infty} z_{n_m}^{iu_j} = \sum_{j=1}^{N} c_j \xi_j e^{-u_j\varphi}.$$

With $x = \varphi$, $P_0(\varphi) = -L$ and $P_j(\varphi) = c_j\xi_j$ for $j = 1 \cdots N$, and recalling that the u_j are distinct and nonzero, we apply Lemma 3.6 to conclude that $L = c_1 = c_2 = \cdots = c_N = 0$. (Note that $\xi_j \neq 0$.) But this is contrary to the hypothesis of the lemma. Thus if the limit exists, then $N = 1$. In this case if $u_1 \neq 0$, then the above choice of z_n yields $L = c_1\xi_1 e^{-u_1\varphi}$ which is impossible unless $c_1 = L = 0$.

To complete the proof of the lemma we need only prove the claim. To this end, we first observe that when $z_n = r_n e^{i\varphi}$, then

$$z_n^{iu_j} = e^{iu_j \log z_n} = e^{iu_j(\log r_n + i\varphi)} = e^{-u_j\varphi} e^{2\pi i \frac{u_j}{|u_1|} n}.$$

Put $\xi_1 = 1$. Then $z_n^{iu_1} \to \xi_1 e^{-u_1\varphi}$ as $n \to \infty$. Consider the sequence

$$\left\{ e^{2\pi i \frac{u_2}{|u_1|} n} \right\}.$$

Since it is a sequence of numbers with absolute value 1, it has a convergent subsequence, say,

$$e^{2\pi i \frac{u_2}{|u_1|} n_\nu} \to \xi_2 \quad \text{as} \quad \nu \to \infty.$$

We repeat this procedure with the sequence

$$\left\{ e^{2\pi i \frac{u_3}{|u_1|} n_\nu} \right\}.$$

This yields a convergent subsequence, say ,

$$e^{2\pi i \frac{u_3}{|u_1|} n_{\nu_\mu}} \to \xi_3 \quad \text{as} \quad \mu \to \infty.$$

Note then that

$$z_{n_{\nu_\mu}}^{iu_j} \to \xi_j e^{-u_j\varphi} \quad \text{as} \quad \mu \to \infty \ \text{ for } \ j = 1, 2, 3.$$

We proceed in this manner to get the subsequence and the complex numbers of the claim.

We are now in a position to prove a proposition which is essential in determining the LPPF's of positive weight for the discrete Hecke groups.

Proposition 3.2.

(a)
$\lim_{z\to\infty} F(z) \neq 0$ *or the limit does not exist.*
(b) $\lim_{z\to\infty} H(z) \neq 0$ *or the limit does not exist.*

Proof : (a) Suppose, to the contrary, that

$$\lim_{z\to\infty} F(z) = 0. \tag{47}$$

Recall that F is given by

$$F(z) = \sum_{j=1}^{N}\sum_{t=0}^{m_j} a(j,t)\frac{\phi_{-\beta_j,t}(z)}{\phi_{\alpha_M,n_M}(z)} + \sum_{l=1}^{M}\sum_{t=0}^{n_l} b(l,t)\frac{\phi_{\alpha_l,t}(z)}{\phi_{\alpha_M,n_M}(z)}.$$

Since $Re(-\beta_j) \leq 0 < Re(\alpha_M)$, for each j, we have (by Lemma 3.3 (b) if $\kappa = 0$ and by Lemma 3.4 (b) if $\kappa \neq 0$)

$$\lim_{z\to\infty}\frac{\phi_{-\beta_j,t}(z)}{\phi_{\alpha_M,n_M}(z)} = 0,$$

for all j and all t. By the same lemmas, we also have

$$\lim_{z\to\infty}\frac{\phi_{\alpha_l,t}(z)}{\phi_{\alpha_M,n_M}(z)} = 0,$$

for each l and each t such that (i) $Re(\alpha_l) < Re(\alpha_M)$ and any t or (ii) $Re(\alpha_l) = Re(\alpha_M)$, and $t < n_l \leq n_M$. Thus

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\sum_{l=l_0}^{M} b(l,n)\frac{\phi_{\alpha_l,n}(z)}{\phi_{\alpha_M,n}(z)}, \tag{48}$$

where $n = n_l = n_M$ and $Re(\alpha_l) = Re(\alpha_M)$ for all $l \geq l_0$.

Let $\alpha_l - \alpha_M = iy_l$ for $l = l_0, \ldots, M$. Then we have

$$\frac{\phi_{\alpha_l,n}(z)}{\phi_{\alpha_M,n}(z)} = z^{iy_l}\frac{f_{\alpha_l,n}(z)}{f_{\alpha_M,n}(z)}.$$

Note that $Re(\alpha_l) > 0$ for each l, so that $\alpha_l \neq 0$, and hence by (44), we have

$$f_l := \lim_{z\to\infty}\frac{f_{\alpha_l,n}(z)}{f_{\alpha_M,n}(z)} = \begin{cases} \frac{\alpha_l}{\alpha_M}, & \kappa = 0 \\ 1, & \kappa \neq 0. \end{cases} \tag{49}$$

As z^{iy_l} is bounded for each l, (49) yields

$$\lim_{z\to\infty} z^{iy_l}\left(f_l - \frac{f_{\alpha_l,n}(z)}{f_{\alpha_M,n}(z)}\right) = 0,$$

and hence

$$\lim_{z\to\infty}\sum_{l=l_0}^{M} b\left(l,t\right) z^{iy_l}\left(f_l-\frac{f_{\alpha_l,n}\left(z\right)}{f_{\alpha_M,n}\left(z\right)}\right) = 0.$$

The last equation can be rewritten as

$$\lim_{z\to\infty}\left(\sum_{l=l_0}^{M}\left(b\left(l,t\right)f_l\right) z^{iy_l}-\sum_{l=l_0}^{M} b\left(l,t\right) z^{iy_l}\frac{f_{\alpha_l,n}\left(z\right)}{f_{\alpha_M,n}\left(z\right)}\right)=0. \tag{50}$$

Combining (47), (48) and (50), we obtain

$$\lim_{z\to\infty}\sum_{l=l_0}^{M}\left(b\left(l,n\right)f_l\right) z^{iy_l}=0,$$

contrary to Lemma 3.7.

(b) Suppose $\lim_{z\to\infty} H\left(z\right)=0$, where

$$H\left(z\right) = \sum_{j=1}^{N}\sum_{t=0}^{m_j} a\left(j,t\right)\left(-1\right)^t\frac{\psi_{\beta_j,t}\left(z\right)}{\psi_{\beta_N,m_N}\left(z\right)} + \sum_{l=1}^{M}\sum_{t=0}^{n_l} b\left(l,t\right)\left(-1\right)^t\frac{\psi_{-\alpha_l,t}\left(z\right)}{\psi_{\beta_N,m_N}\left(z\right)}.$$

We will consider two cases:

Case 1. $\beta_N\neq 2k$ or $\kappa>0$.

Since $Re\left(-\alpha_l\right)<0\leq Re\left(\beta_N\right)$, by Lemmas 3.3$(b)$ or 3.5(b), we have

$$\lim_{z\to\infty}\frac{\psi_{-\alpha_l,t}\left(z\right)}{\psi_{\beta_N,m_N}\left(z\right)} = 0,$$

for all l and all t. Also

$$\lim_{z\to\infty}\frac{\psi_{\beta_j,t}\left(z\right)}{\psi_{\beta_N,m_N}\left(z\right)} = 0,$$

for all j and t such that (i) $Re\left(\beta_j\right)<Re\left(\beta_N\right)$ and any t or (ii) $Re\left(\beta_j\right)=Re\left(\beta_N\right)$ and $t<m_N$. Thus we have

$$\lim_{z\to\infty}H\left(z\right)=\lim_{z\to\infty}\sum_{j=j_0}^{N}\left(-1\right)^m a\left(j,m\right)\frac{\psi_{\beta_j,m}\left(z\right)}{\psi_{\beta_N,m}\left(z\right)},$$

where $m=m_j=m_N$ and $Re\left(\beta_j\right)=Re\left(\beta_N\right)$ for all $j\geq j_0$.

Let $\beta_j-\beta_N=iu_j$ for $j=j_0,\ldots,N$. Then we have

$$\frac{\psi_{\beta_j,m}\left(z\right)}{\psi_{\beta_N,m}\left(z\right)}=z^{iu_j}\frac{g_{\beta_j,m}\left(z\right)}{g_{\beta_N,m}\left(z\right)}.$$

Since $\beta_N-2k\neq 0$ or $\kappa>0$, we see that

$$g_j:=\lim_{z\to\infty}\frac{g_{\beta_j,m}\left(z\right)}{g_{\beta_N,m}\left(z\right)}=\begin{cases}\frac{\beta_j-2k}{\beta_N-2k}, & \kappa=0\\ 1, & \kappa\neq 0.\end{cases}$$

As in (a) above, we conclude then that

$$\lim_{z\to\infty}\sum_{j=j_0}^{N}(-1)^m\left(a\left(j,m\right)g_j\right)z^{iu_j}=0,$$

contrary to Lemma 3.7.

Case 2. $\beta_N = 2k$ and $\kappa = 0$.

In this case we first observe that $m = m_N > 0$ and that $\lim_{z\to\infty}\frac{H(z)}{\log z} = 0$.

Next we note that

$$\lim_{z\to\infty} z\log z\left(\left(\frac{\log(z+\lambda)}{\log z}\right)^m - 1\right) = \lim_{z\to\infty}\frac{\left(\frac{\log(z+\lambda)}{\log z}\right)^m - 1}{\frac{1}{z\log z}}$$

$$= \lim_{z\to\infty}\frac{m\left(\frac{\log(z+\lambda)}{\log z}\right)^{m-1}\frac{\tau(z)}{z(z+\lambda)\log z}}{-\frac{\log z+1}{(z\log z)^2}} = \lim_{z\to\infty} -m\tau(z)\left(\frac{\log(z+\lambda)}{\log z}\right)^{m-1}\frac{z\log z}{(z+\lambda)\left(\log z+1\right)} = m\lambda.$$

Also

$$\lim_{z\to\infty} z\left(\left(\frac{z+\lambda}{z}\right)^{\alpha-2k}\left(\frac{\log(z+\lambda)}{\log z}\right)^{t} - 1\right) = \lambda\left(\alpha-2k\right).$$

Thus,

$$\begin{aligned}\lim_{z\to\infty}\frac{\psi_{\beta,t}\left(z\right)}{\log z\psi_{2k,m}\left(z\right)} &= \lim_{z\to\infty} z^{\beta-2k}\left(\log z\right)^{t-m}\frac{z\left(\left(\frac{z+\lambda}{z}\right)^{\beta-2k}\left(\frac{\log(z+\lambda)}{\log z}\right)^{t}-1\right)}{z\log z\left(\left(\frac{\log(z+\lambda)}{\log z}\right)^{m}-1\right)}\\ &= \frac{\beta-2k}{m}\lim_{z\to\infty} z^{\beta-2k}\left(\log z\right)^{t-m} = 0,\end{aligned}$$

if $Re\left(\beta\right) < 2k$ or if $Re\left(\beta\right) = 2k$ and $t < m$. Hence

$$\lim_{z\to\infty}\frac{H\left(z\right)}{\log z} = \lim_{z\to\infty}\sum_{j=j_1}^{N}(-1)^m a\left(j,m\right)\frac{\psi_{\beta_j,m}\left(z\right)}{\log z\psi_{2k,m}\left(z\right)},$$

where $m = m_j = m_N$ and $Re\left(\beta_j\right) = 2k$ for all $j \geq j_1$.

We now proceed as in Case 1 and arrive at the same contradiction. This completes the proof of the proposition.

Remark 3.2: It follows from Proposition 3.2 that there are sequences $\{z_\nu\}$ and $\{w_\mu\}$ such that

$$\mid F\left(z_\nu\right)\mid \geq \delta \quad\text{and}\quad \mid H\left(w_\mu\right)\mid \geq \delta,$$

for some positive real number δ. This, together with (43), implies that

$$\lim_{\nu\to\infty}\phi_{\alpha_M,n_M}(z_\nu)=0 \quad \text{and} \quad \{\psi_{\beta_N,m_N}(w_\mu)\} \text{ is bounded.} \tag{51}$$

Remark 3.3: In his first two papers on rational period functions [10] and [11], Knopp has shown that for the full modular group $\Gamma(1)$ the rational functions given by

$$q(z)=\begin{cases} a\left(1-\bar{v}(T)\,z^{-2k}\right), & \text{if } k>1\\ a\left(1-\bar{v}(T)\,z^{-2}\right)+bz^{-1}, & \text{if } k=1\end{cases}$$

are rational period functions. Here we are assuming that $k\in\mathbf{Z}$ which is the case if $\kappa=0$. In fact he showed that these are the only rational period functions with rational poles. It can easily be seen that the functions given above are period functions for the Hecke groups $G(\lambda)$. With this in mind we now state and prove our main results.

Theorem 3.1. *Suppose $q(z)$ given by (39) and (40), is an LPPF of weight $2k$, $k>0$ and multiplier system v for the Hecke group $G(\lambda)$, where $\lambda=2\cos\left(\frac{\pi}{p}\right)$. Suppose also that $v(S_\lambda)=e^{2\pi i\kappa}=1$, i.e., $\kappa=0$.*
(a) If $k>1$, then $q(z)$ is of the form

$$q(z)=a\left(1-\bar{v}(T)\,z^{-2k}\right).$$

(b) If $k=1$, then $q(z)$ reduces to the form

$$q(z)=a\left(1-\bar{v}(T)\,z^{-2}\right)+bz^{-1}.$$

Proof : If $v(S_\lambda)=1$, then $C_1=C_2=1$. Note that if $Re(\alpha)=1$ and $\nu=0$, then from the proof of Lemma 3.3 (a) we have $\lim_{z\to\infty}\phi_{\alpha,\nu}(z)=\lim_{z\to\infty}\lambda(z+\lambda)^{\alpha-1}=\lambda$ if $\alpha=1$, and the limit does not exist otherwise. Thus $\lim_{z\to\infty}\phi_{\alpha,\nu}(z)\neq 0$. From Lemma 3.3 (a) and the first part of (51), we conclude that $Re(\alpha_M)<1$. On the other hand, from Lemma 3.4 (a) and the second part of (51), we see that $Re(\beta_N)<1$ or $Re(\beta_N)=1$, $m_N=0$. Noting the fact that $\psi_{2k,0}\equiv 0$ and using (39), we conclude that $q(z)$ takes the form:

$$q(z)=a+cz^{-2k}+\sum_{j=0}^{K}b_j z^{-1+iu_j}+\sum_{j=1}^{N}z^{-\beta_j}P_j(z)+\sum_{l=1}^{M}z^{\alpha_l}R_l(z),$$

where

$$0\le Re(\beta_1)\le\cdots\le Re(\beta_N)<1,\quad 0<Re(\alpha_1)\le\cdots\le Re(\alpha_M)<1,$$

$$P_j(z)=\sum_{t=0}^{m_j}a(j,t)(\log z)^t,\qquad R_l(z)=\sum_{t=0}^{n_l}b(l,t)(\log z)^t,$$

and $u_0,\ldots,u_K$ are real numbers.

Recall that $\lim_{z\to\infty}\psi_{\beta_j,t}(z)=0$ for all j and t, and that $\lim_{z\to\infty}\psi_{-\alpha_l,t}(z)=0$ for all l and t.

Thus

$$\begin{aligned} L &= \lim_{z\to\infty}\left(\left(\frac{z+\lambda}{z}\right)^{-2k} q\left(\frac{1}{z+\lambda}\right) - q\left(\frac{1}{z}\right)\right) \\ &= \lim_{z\to\infty}\sum_{j=0}^{K} b_j\left(\left(\frac{z+\lambda}{z}\right)^{-2k+1-iu_j} z^{1-iu_j} - z^{1-iu_j}\right) \\ &= \lim_{z\to\infty}\sum_{j=0}^{K} b_j z^{-iu_j} h_j\left(z\right), \end{aligned}$$

where

$$h_j\left(z\right) = z\left(\left(\frac{z+\lambda}{z}\right)^{-2k+1-iu_j} - 1\right).$$

But then $\lim_{z\to\infty} h_j(z) = \left(-2k+1-iu_j\right)\lambda = c_j \neq 0$. We conclude from this and the above limit that

$$\lim_{z\to\infty}\sum_{j=0}^{K} b_j c_j z^{-iu_j} = L,$$

which implies that $b_j = 0$ unless $u_j = 0$. Thus q takes the form :

$$q\left(z\right) = a + bz^{-1} + cz^{-2k} + + \sum_{j=1}^{N} z^{-\beta_j} P_j\left(z\right) + \sum_{l=1}^{M} z^{\alpha_l} R_l\left(z\right), \tag{52}$$

where

$$0 \le Re\left(\beta_1\right) \le \cdots \le Re\left(\beta_N\right) < 1, \qquad 0 < Re\left(\alpha_1\right) \le \cdots \le Re\left(\alpha_M\right) < 1,$$

$$P_j\left(z\right) = \sum_{t=0}^{m_j} a\left(j,t\right)\left(\log z\right)^t \quad \text{and} \quad R_l\left(z\right) = \sum_{t=0}^{n_l} b\left(l,t\right)\left(\log z\right)^t. \tag{53}$$

As an LPPF recall that q should satisfy the relation $q + q \mid T = 0$; that is,

$$q\left(z\right) + \bar{v}\left(T\right) z^{-2k} q\left(\frac{-1}{z}\right) = 0.$$

But,

$$\begin{aligned}
&\bar{v}(T)\, z^{-2k} q\left(\frac{-1}{z}\right) \\
&= a\bar{v}(T)\, z^{-2k} - b\bar{v}(T)\, z^{-2k+1} + c\bar{v}(T)\, z^{-2k}\left(\frac{-1}{z}\right)^{-2k} \\
&\quad + \sum_{j=1}^{N} \bar{v}(T)\, z^{-2k}\left(\frac{-1}{z}\right)^{-\beta_j} P_j\left(\frac{-1}{z}\right) + \sum_{l=1}^{M} \bar{v}(T)\, z^{-2k}\left(\frac{-1}{z}\right)^{\alpha_l} R_l\left(\frac{-1}{z}\right) \\
&= a\bar{v}(T)\, z^{-2k} - b\bar{v}(T)\, z^{-2k+1}(-1)^{-2k} + c\bar{v}(T) \\
&\quad + \sum_{j=1}^{N} z^{-2k+\beta_j} A_j(z) + \sum_{l=1}^{M} z^{-2k-\alpha_l} B_l(z),
\end{aligned}$$

where

$$A_j(z) = \bar{v}(T)(-1)^{-\beta_j} P_j\left(\frac{-1}{z}\right), \quad B_l(z) = \bar{v}(T)(-1)^{\alpha_l} R_l\left(\frac{-1}{z}\right).$$

Consequently,

$$\left(a + (-1)^{-2k} c\bar{v}(T)\right) + (a\bar{v}(T) + c)\, z^{-2k} + bz^{-1} - b\bar{v}(T)\, z^{-2k+1} +$$

$$\sum_{j=1}^{N} z^{-\beta_j} P_j(z) + \sum_{j=1}^{N} z^{-2k+\beta_j} A_j(z) + \sum_{l=1}^{M} z^{\alpha_l} R_l(z) + \sum_{l=1}^{M} z^{-2k-\alpha_l} B_l(z) = 0. \quad (54)$$

Note then that the left-hand side of (54) is a sum of the form

$$\sum_{j=1}^{L} \sum_{t=0}^{r_j} c(j,t)\, z^{\gamma_j} (\log z)^t = 0,$$

where the powers γ_j of z are

$$0, \quad -2k, \quad -1, \quad -2k+1, \quad -\beta_j, \quad -2k+\beta_j, \quad \alpha_l, \quad -2k-\alpha_l. \quad (55)$$

Suppose $k > 1$. Then by (53) the numbers in (55) are all distinct and we apply Corollary 3.1 to conclude that all the coefficients of (54) are zero. In particular, we have

$$a + (-1)^{-2k} c\bar{v}(T) = 0, \quad a\bar{v}(T) + c = 0, \quad b_j = 0 \quad P_j(z) = 0, \quad \text{and} \quad R_l(z) = 0.$$

By Lemma 2.1 we have $\bar{v}(T)^2 (-1)^{-2k} = 1$, and so $c = -a\bar{v}(T)$. Therefore,

$$q(z) = a\left(1 - \bar{v}(T)\, z^{-2k}\right),$$

and (a) is proved.

Next suppose $k = 1$. Then the list in (55) becomes

$$0, \quad -2, \quad -1, \quad -1, \quad -\beta_j, \quad -2+\beta_j, \quad \alpha_l, \quad -2-\alpha_l.$$

Here all powers of z except -1 are distinct and we must have $1 - \bar{v}(T) = 0$ or $b = 0$ and hence q takes the form as stated in (b). This completes the proof of the theorem.

Remark 3.4: Suppose $\kappa = 0$ and λ is given by (15). Then by Lemma 2.2, we have $(p-2)k \in \mathbf{Z}$, which is even when p is even. In particular, for $p = 3$ and

$p = 4$, we see that $k \in \mathbf{Z}$ and hence the assumption that $k \geq 1$ in Theorem 3.1 is no restriction of generality. For $p = 6$, that is, for the group $G\left(\sqrt{3}\right)$, there is one possible case that is not covered by Theorem 3.1, for in this case we can have $2k = 1$ as a weight. In general for weight $2k < 2$ we have the following theorem.

Theorem 3.2. *Under the assumptions of Theorem* 3.1*, if* $1 \leq 2k < 2$*, then* q *reduces to the form:*

$$q(z) = a\left(1 - \bar{v}\left(T\right)z^{-1}\right) + \sum_{j=1}^{N} P_j(z)z^{-\beta_j} - \bar{v}\left(T\right)\sum_{j=1}^{N}\left(-1\right)^{-\beta_j} P_j\left(\frac{-1}{z}\right)z^{-2k+\beta_j},$$

where

$$0 \leq Re\left(\beta_1\right) \leq Re\left(\beta_2\right) \leq \cdots \leq Re\left(\beta_N\right) < 1.$$

If $0 < 2k \leq 1$*, then* q *takes the same form as above with*

$$0 \leq Re\left(\beta_1\right) \leq Re\left(\beta_2\right) \leq \cdots \leq Re\left(\beta_J\right) \leq 2k < Re\left(\beta_{J+1}\right) \leq \cdots \leq Re\left(\beta_N\right) < 1.$$

Proof: If $1 \leq 2k < 2$, then the list in (55) becomes

$$0, \quad -2k, \quad -1, \quad -2k+1, \quad -\beta_j, \quad -2k+\beta_j, \quad \alpha_l, \quad -2k-\alpha_l. \tag{56}$$

In (56) repetition occurs if and only if $-\beta_j = -2k+\beta_r$ for some j and r. By the assumptions made on the real parts of the $\beta's$ it follows that $-\beta_j = -2k+\beta_{N-j+1}$ for $j = 1, \ldots, J$, where $J = N/2$ or $J = (N-1)/2$. We now substitute this in the equation for $q + q \mid T = 0$. In this rearranged sum the powers of z are distinct, so we can apply Corollary 3.1. Thus $q\left(z\right)$ has the form

$$q(z) = a\left(1 - \bar{v}\left(T\right)z^{-2k}\right) + \sum_{j=1}^{N} P_j(z)z^{-\beta_j} - \bar{v}\left(T\right)\sum_{j=1}^{N}\left(-1\right)^{-\beta_j} P_j\left(\frac{-1}{z}\right)z^{-2k+\beta_j}.$$

This completes the proof of the theorem.

Theorem 3.3. *Suppose* $q\left(z\right)$ *given by* (39) *and* (40)*, is an LPPF of weight* $2k$*,* $k > 0$ *and multiplier system* v *for the Hecke group* $G\left(\lambda\right)$*, where* $\lambda = 2\cos\left(\frac{\pi}{p}\right)$*. Suppose also that* $v\left(S_\lambda\right) = e^{2\pi i\kappa} \neq 1$*, i.e.,* $\kappa > 0$*, then* $q\left(z\right) = 0$*.*

Proof : If $v\left(S_\lambda\right) \neq 1$, then $C_1 \neq 1$ and $C_2 \neq 1$. By Lemma 3.4 (a) and the first part of (51), we must have $Re\left(\alpha_M\right) < 0$. But this contradicts (40). Hence it must be that $b\left(l,t\right) = 0$ for all l and all t. On the other hand, combining Lemma 3.5 (a) and the second part of (51), we see that $Re\left(\beta\right) < 0$ or $Re\left(\beta_N\right) = 0, m_N = 0$. Since $Re\left(\beta_N\right) < 0$ is contrary to (40), it must be that $Re\left(\beta_N\right) = 0, m_N = 0$. But then (again by (40)) we have $m_1 = \cdots = m_N = 0$. Consequently $q\left(z\right)$ takes the form

$$q\left(z\right) = \sum_{j=1}^{N} b_j z^{iu_j},$$

where the u_j are distinct real numbers. But then

$$\left(q + q \mid T\right)\left(z\right) = \sum_{j=1}^{N} b_j z^{iu_j} + \sum_{j=1}^{N} b_j' z^{-2k+iu_j} = 0,$$

and Corollary 3.1 implies that $b_j = 0$ for each j and hence $q\left(z\right) = 0$. This completes the proof of the theorem.

3.3. **Log-Polynomial Period Functions of Negative Weight.** Finally we shall determine the LPPF's of negative integral weight for the Hecke groups and give the general form of the LPPF's for nonintegral negative weight. First we will discuss the negative integral weights. For this we shall apply an observation made by G. Bol in [2], which we state as

Lemma 3.8. (Bol's Theorem). *Suppose $r \in \mathbf{Z}, r \geq 0$ and $a, b, c, d \in \mathbf{C}$ such that $ad - bc = 1$. If f is a differentiable function, then*

$$D^{(r+1)}\left((cz+d)^{r} f\left(\frac{az+b}{cz+d}\right)\right) = (cz+d)^{-r-2} f^{(r+1)}\left(\frac{az+b}{cz+d}\right). \tag{57}$$

Proof : Induction on r. See also [8].

We now apply this Lemma to prove

Theorem 3.4. *Suppose q is an LPPF of weight $2k \in \mathbf{Z}, k \leq 0$ and multiplier system v for $G(\lambda)$ where λ is given by (15). Then*

$$q(z) = \begin{cases} a\left((-1)^{-2k}\bar{v}(T)z^{-1} + z^{-2k+1}\right) + p_k(z), & k < 0, \kappa = 0 \\ a\left(z^{-1} + \bar{v}(T)z\right) + b\log z + c, & k = 0, \kappa = 0 \\ p_k(z), & k \leq 0, \kappa > 0, \end{cases}$$

where $a, b, c \in \mathbf{C}$ and $p_k(z)$ is a polynomial of degree at most $-2k$.

Proof : If q is a log-polynomial sum, then it can easily be seen that $q^{(r)}(z)$ is also
a log-polynomial sum for any $r \in \mathbf{Z},\ r > 0$. But then by Bol's Theorem we see that $q^{(-2k+1)}$ is an LPPF of weight $-2k+2$ and multiplier system v for $G(\lambda)$.

Now by Theorem 3.1 (if $\kappa = 0$) and by Theorem 3.3 (if $\kappa > 0$), we obtain

$$q^{(-2k+1)}(z) = \begin{cases} A\left(1 - \bar{v}(T)z^{2k-2}\right), & k < 0, \kappa = 0 \\ A\left(1 - \bar{v}(T)z^{-2}\right) + Bz^{-1}, & k = 0, \kappa = 0 \\ 0, & k \leq 0, \kappa > 0. \end{cases}$$

The theorem now follows from integrating this $(-2k+1)$-times.

Remark 3.5: When $k = 0,\ \kappa = 0$, we have $v(T) = \pm 1$ and $v(S_\lambda) = 1$. Suppose $f(z)$ is an entire modular form of weight r and multiplier system $v = 1$ for the Hecke group $G(\lambda)$ such that f never vanishes in the upper half-plane. Define

$$F(z) = \log f(z).$$

Then

$$F(-1/z) = F(z) + r\log z.$$

Thus

$$q(z) = a\left(z^{-1} + \bar{v}(T)z\right) + b\log z + c$$

is indeed an LPPF of weight 0 for the Hecke group provided such an f exists. That such an f exists was proved by Knopp and Smart in [14] (Theorem 2, page 135).

Next we consider the LPPF's of nonintegral negative weights. Suppose $k < 0$ and $2k$ is not an integer and let q be an LPPF of weight $2k$ and multiplier system v for $G(\lambda)$, where λ is given by (15). Then (36) holds, that is, we have

$$\lim_{z\to\infty}\left\{v(S_\lambda)\left(\frac{z+\lambda}{z}\right)^{-2k} q\left(\frac{1}{z+\lambda}\right) - q\left(\frac{1}{z}\right)\right\} = L, \tag{58}$$

where

$$L = -\sum_{n=1}^{p-2} L_n = -\sum_{n=1}^{p-2} \bar{v}(M_n)\, d_n^{-2k} q\left(\frac{b_n}{d_n}\right)$$

and

$$M_n = V_n T = (S_\lambda T)^n T = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad \text{for} \quad n = 1, \dots, p.$$

Note that (38) does not hold as $-2k > 0$. However, if we multiply both sides of (33) by z^{2k}, we obtain

$$z^{2k}(q \mid M_{p-1})(z) + z^{2k}(q \mid M_{p-2})(z) + \cdots + z^{2k}(q \mid M_1)(z) - z^{2k} q(z) = 0. \tag{59}$$

We now make use of (32)
and take the limit as $z \to \infty$ in (59), to obtain

$$\lim_{z\to\infty} z^{2k}\left\{\bar{v}(S_\lambda)\, q(z+\lambda) - q(z)\right\} = K, \tag{60}$$

where

$$K = -\sum_{n=2}^{p-1} \bar{v}(M_n)\, c_n^{-2k} q\left(\frac{a_n}{c_n}\right).$$

The Limit-Lemmas of Section 3.2 are applicable for any k and we apply them to obtain the following result.

Proposition 3.3. *Suppose $q(z)$, as given by (39) and (40), is an LPPF of weight $2k$, $k < 0$, and multiplier system v for the Hecke group $G(\lambda)$, where λ is given by (15). Suppose also that $v(S_\lambda) = e^{2\pi i \kappa}$, $0 \le \kappa < 1$.*

(i) *If $\kappa = 0$, then*

$$q(z) = A\left(1 - \bar{v}(T)\, z^{-2k}\right) + \sum_{j=1}^{N}\sum_{t=0}^{m_j} a(j,t)\, z^{-\beta_j} (\log z)^t \; + \; \sum_{l=1}^{M}\sum_{t=0}^{n_l} b(l,t)\, z^{\alpha_l} (\log z)^t, \tag{61}$$

where

$$0 < Re(\beta_1) \le \cdots \le Re(\beta_N) < 1; \quad m_j \le m_l \text{ if } Re(\beta_j) = Re(\beta_l)\,(j < l);$$
$$0 < Re(\alpha_1) \le \cdots \le Re(\alpha_M) < -2k + 1; \quad n_j \le n_l \text{ if } Re(\alpha_j) = Re(\alpha_l)\,(j < l).$$

(ii) *If $\kappa > 0$, then*

$$q(z)=\sum_{l=1}^{M}\sum_{t=0}^{n_l}b(l,t)\,z^{\alpha_l}(\log z)^t\,,$$

where

$$0<Re(\alpha_1)\leq\cdots\leq Re(\alpha_M)<-2k;\quad n_j\leq n_l\;\; \textit{if}\;\; Re(\alpha_j)=Re(\alpha_l)\,(j<l)\,.$$

4. Concluding Remarks

As mentioned in Section 1, the Riemann-Hecke-Bochner-Weil correspondence applies to automorphic integrals on the Hecke groups $G(\lambda)$. Suppose that q given is by (39) and (40), that is,

$$q(z)=\sum_{j=1}^{N}\sum_{t=0}^{m_j}a(j,t)\,z^{-\beta_j}(\log z)^t\;+\;\sum_{l=1}^{M}\sum_{t=0}^{n_l}b(l,t)\,z^{\alpha_l}(\log z)^t\,,$$

where

$$\begin{aligned}0\leq Re(\beta_1)\leq\cdots\leq Re(\beta_N)\,;\quad m_j\leq m_l\;\text{ if }\;Re(\beta_j)=Re(\beta_l)\,(j<l)\,;\\ 0<Re(\alpha_1)\leq\cdots\leq Re(\alpha_M)\,;\quad n_j\leq n_l\;\text{ if }\;Re(\alpha_j)=Re(\alpha_l)\,(j<l)\,.\end{aligned}$$

If q is an LPPF, then there exists a function F defined and holomorphic in the upper half-plane satisfying (20), (21) , (22) and (23).

Let us define

$$\phi(s)=\sum_{n=1}^{\infty}a_n n^{-s},\qquad\qquad \Phi(s)=\left(\frac{2\pi}{\lambda_1}\right)^{-s}\Gamma(s)\,\phi(s)\,;$$

$$Q(s)=\sum_{j=1}^{N}\sum_{t=0}^{m_j}\frac{c(j,t)}{(s+\beta_j)^{m_j-t+1}}+\sum_{l=1}^{M}\sum_{t=0}^{n_l}\frac{d(l,t)}{(s-\alpha_l)^{n_l-t+1}}+a_0\left(\frac{1}{s}-\frac{\bar{v}(T)}{2k-s}\right),$$

where

$$c(j,t)=(-1)^j j!a(j,m_j-t+1)\quad\text{and}\quad d(l,t)=(-1)^l l!b(l,n_l-t+1)\,.$$

Then Hecke's Lemma states that $\Phi(s)-Q(s)$ can be continued to an entire function which is bounded in every lacunary vertical strip:

$$\sigma_1\leq\sigma\leq\sigma_2,\quad |Im(s)|\geq t_0>0,$$

and $\Phi(s)$ satisfies the functional equation

$$\Phi(s)=e^{-\pi ik}\bar{v}(T)\,\Phi(2k-s)\,.$$

When $k>0$ and $\kappa>0$, it follows from Theorem 3.3 that $q(z)=0$ and hence $\Phi(s)$ is holomorphic in the entire s-plane except for possible simple poles at $s=0$ and $s=2k$. In fact Theorem 3.3 asserts that the only automorphic integrals with log-polynomial period function are the automorphic forms. If $k\geq 1$ and $\kappa=0$, then from Theorem 3.1 we conclude that $\Phi(s)$ has at worst simple poles at $s=0$ and $s=2k$.

Finally, suppose F_1 and F_2 are two linearly independent automorphic integrals of weight $2k$, $k \geq 1$ and multiplier system v with LPPF's q_1 and q_2, respectively. If $\kappa > 0$, then F_j, $j = 1, 2$ are automorphic forms. If, on the other hand, $\kappa = 0$, then by Theorem 3.1 the q's are of the form
$c\left(1 - \bar{v}\left(T\right) z^{-2k}\right)$. Thus for a suitable choice of constants a and b, $aF_1 + bF_2$ is an automorphic form of weight $2k$.

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