Real Analysis II

Chapter 8. Elementary Functions

8.1 Hyperbolic Functions

Let

$$U(x) = \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt \qquad (-\infty\ <\ x\ \infty)$$

$U(x)$ has the following properties.

1) $U(-x) = -U(x)$ for all$x \in \mathbf{R}$.

Proof. Let $u = -t$. Then

$$U(-x) = \int_0^{-x} \frac{1}{\sqrt{1+t^2}}\,dt = \int_0^x \frac{1}{\sqrt{1+(-u)^2}}(-1)du = -U(x)$$

2)

$$\lim_{x\to\infty} U(x) = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} U(x) = -\infty.$$

Proof. Clearly , $U(x) \geq 0$ for all $x \geq 0$. Hence $U(x) < 0$ for all $x < 0$. For all $t \geq 1$,

$$\frac{1}{\sqrt{1+t^2}} \ \geq\ \frac{1}{\sqrt{2t^2}} = \frac{1}{\sqrt{2}t}.$$

It follows that $\int_1^\infty \frac{1}{\sqrt{1+t^2}}\,dt$ diverges and hence $\lim_{x\to\infty} U(x) = \infty$. The second limit follows by replacing x by $-x$ and using 1).

3)

$$U'(x) = \frac{1}{\sqrt{1+x^2}} \qquad \text{for all } x \in \mathbf{R}$$

Proof. This is the Fundamental Theorem of Calculus.

4) U is continuous and 1-1 on $(-\infty, \infty)$.

Proof. A differentiable function is continuous. From 3) we note that U is strictly increasing. Hence it is one-to-one.

5) If S is the inverse of U and if $b = S(a)$, then $a = U(b)$ and

$$S'(a) = \sqrt{1 + \left[S(a)\right]^2}$$

Proof. Use the chain rule at $x = a$ and the fact that $U(S(x)) = x$ to get $U'(S(a)) \cdot S'(a) = 1$. Solve for $S'(a)$ and use 3).

6) $S''(x) = S(x)$ for all $x \in \mathbf{R}$

Proof. Differentiate both sides of 5) with respect to a.

7) Define

$$C(x) = \sqrt{1 + \left[S(x)\right]^2}.$$

Then $C(x) = S'(x)$ and $C'(x) = S(x)$ for all $x \in \mathbf{R}$.

Proof. Follows from 5).

Definition. We define $\sinh^{-1}(x)$ to be $U(x)$. That is,

$$\sinh^{-1}(x) = \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt \qquad (-\infty \ < \ x \,\infty)$$

Then

$$S(x) = \sinh x \qquad \text{and} \qquad C(x) = \cosh x$$

are the **hyperbolic sine** and **hyperbolic cosine** functions.

Remark. (i) Many of the properties of the hyperbolic sine and hyperbolic cosine functions can be deduced from properties 1 to 7 above. For example we can easily see that

$$\cosh^2 x - \sinh^2 x = 1$$

(ii) The other hyperbolic trig functions can be obtained from these by taking ratios:

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}$$
$$\text{sech } x = \frac{1}{\cosh x} \quad \text{csch } x = \frac{1}{\sinh x}$$

8.2 The Exponential Function

We define $E(x) = C(x) + S(x) \qquad (-\infty < x < \infty)$

Remarks $E(x)$ has the following properties.

1) E is continuous on $\mathbf{R}$ and $E(0) = 1$.

Proof. $S(x)$ is continuous and hence $C(x) = \sqrt{1 + (S(x))^2}$ is also continuous. $U(x) = \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt$ implies $U(0) = 0$ and so $S(0) = S(U(0)) = 0$. But then $C(0) = \sqrt{1 + \left[S(0)\right]^2} = 1$.

2) $E(-x) = C(x) - S(x)$ for all $x \in \mathbf{R}$.

Proof. $-x = U(S(-x))$ implies $x = -U(S(-x)) = U(-S(-x))$. Apply S to get $S(x) = -S(-x)$ and hence $S(-x) = -S(x)$. Clearly $C(-x) = \sqrt{1 + \left[S(-x)\right]^2} = \sqrt{1 + \left[S(x)\right]^2} = C(x)$. Thus $E(-x) = C(-x) + S(-x) = C(x) - S(x)$.

3) $E(x)E(-x) = 1$.

Proof. Using the definition and 2), we have

$$\begin{aligned} E(x) \cdot E(-x) &= (C(x) + S(x))(C(x) - S(x)) \\ &= (C(x))^2 - (S(x))^2 = 1. \end{aligned}$$

4) $E(x) > 0$ for all $x \in \mathbf{R}$.

Proof. From 3), we conclude that $E(x)$ is never zero. Since it is continuous, it must always be positive or always negative. Since $E(0) = 1$, the result follows.

5) $E'(x) = E(x)$ for all $x \in \mathbf{R}$.

Proof. $E'(x) = C'(x) + S'(x) = S(x) + C(x) = E(x)$.

6) E is increasing and hence one-to-one on $\mathbf{R}$.

Proof. Follows from 4) and 5).

7) $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbf{R}$.

Proof. Fix y. Let $F(x) = E(x + y)E(-x)$. Then

$$\begin{aligned} F''(x) &= E'(x + y)E(-x) + E(x + y)E'(-x)(-1) \\ &= E(x + y)E(-x) - E(x + y)E(-x) = 0 \end{aligned}$$

and hence $F(x)$ is a constant. But then $F(x) = F(y)$ for all $x \in \mathbf{R}$. In other words,

$$E(x + y)E(-x) = E(y).$$

The result follows from 3).

8.3 The Logarithmic Function

Recall that $E(x)$ is continuous and one-to-one on $\mathbf{R}$ and its range is $(0, \infty)$. We define $L(x)$ to be the inverse of $E(x)$. Thus

$$E(L(x)) = x \quad \text{for all} \quad x \in (-\infty, \infty)$$

and

$$L(E(x)) = x \quad \text{for all} \quad x \in (0, \infty).$$

Remark. $L(x)$ has the following properties.

1) $L(1) = 0$.

Proof. $E(0) = 1$ implies $L(1) = L(E(0)) = 0$.

2) $L(xy) = L(x) + L(y)$ for all $x, y \in (0, \infty)$.

Proof. Let $a = L(x)$ and $b = L(y)$. Then $E(a+b) = E(a)E(b)$ implies $a+b = L[E(a)E(b)]$. But $E(a) = x$ and $E(b) = y$. Hence $L(x) + L(y) = L(xy)$.

3) $L(x^n) = nL(x)$ for all $x \in (0, \infty)$ and all $n \in \mathbf{N}$.

Proof. Use 2) with $y = x$ and induction.

4) For all $x \in (0, \infty)$, $L\left(\frac{1}{x}\right) = -L(x)$.

Proof. This follows from $0 = L(1) = L\left(x \cdot \frac{1}{x}\right) = L(x) + L\left(\frac{1}{x}\right)$

5) For all $x, y \in (0, \infty)$, $L\left(\frac{x}{y}\right) = L(x) - L(y)$.

Proof. Immediate from 2) and 4).

6) For all $x \in (0, \infty)$, $L'(x)\frac{1}{x}$.

Proof. Use chain rule and $E(L(x)) = x$.

7) For all $x \in (0, \infty)$,

$$L(x) = \int_1^x \frac{1}{t}\, dt.$$

Proof. By the Fundamental Theorem of Calculus and the fact that $L(1) = 0$, we have

$$L(x) = L(x) - L(1) = \int_1^x L'(t)\, dt = \int_1^x \frac{1}{t}\, dt.$$

8)

$$\lim_{h \to 0} \frac{L(1+h)}{h} = 1.$$

Proof. By 6), $L'(1) = 1$. We also have $L(1) = 0$. Thus, by limit definition of derivative, we have

$$1 = \lim_{h \to o} \frac{L(1+h) - L(1)}{h} = \lim_{h \to o} \frac{L(1+h)}{h}.$$

9)

$$E(1) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Proof In 8), replace h by $1/n$ and write

$$\lim_{n \to \infty} \frac{L\left(1 + \frac{1}{n}\right)}{1/n} = 1$$

which implies

$$\lim_{n \to \infty} nL\left(1 + \frac{1}{n}\right) = \lim_{n \to \infty} L\left[\left(1 + \frac{1}{n}\right)^n\right] = 1.$$

Since $E(x)$ is continuous, we see that

$$\lim_{n \to \infty} E\left(L\left[\left(1 + \frac{1}{n}\right)^n\right]\right) = E(1).$$

Definition $E(x)$ is called the **natural exponential function**. $E(1)$ is denoted by e and we write

$$E(x) = e^x \qquad \text{for all } x \in \mathbf{R}.$$

$L(x)$ is called the **natural logarithmic function** and we write

$$L(x) = \log x \qquad \text{for all } x \in (0, \infty).$$

We also define x^a to be

$$x^a = e^{a \log x} \quad x \in (0, \infty) \quad a \in \mathbf{R}.$$

8.4 The Trigonometric Function

Define the real number π by

$$\frac{\pi}{2} = \int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt$$

and define the function $u(x)$ by

$$u(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt.$$

The following properties of $u(x)$ can be proved in the same manner as in the previous section.

1) $u(1) = \pi/2$ and $u(-1) = -\pi/2$.

2) u is continuous on $[-1, 1]$.

3) For all $x \in (-1, 1)$,

$$u'(x) = \frac{1}{\sqrt{1-x^2}}.$$

4) u is a one-to-one function on $[-1, 1]$. Hence it has an inverse.

5) If $s(x)$ is the inverse of u, then $s(x)$ is continuous on $[-\pi/2, \pi/2]$.

6) We have $s(-\pi/2) = -1,\ s(\pi/2) = 1,\ s(0) = 0$.

7) For all $\pi/2 \le x \le \pi/2$, we have

$$s'(x) = \sqrt{1 - (s(x))^2}$$

8) We define $s(y)$ on $\pi/2 < y < 3\pi/2$, by $s(y) = s(x+\pi) = -s(x)$, with $-\pi/2 < x < \pi/2$. Then $s'(x+\pi) = -s'(x) \quad -\pi/2 < x < \pi/2.$ and $(c(x))^2 + (s(x))^2 = 1$.

9) In genreal we can extend $s(x)$ to $\mathbf{R}$ by

$$s(y) = s(x+\pi) = -s(x) \qquad -\infty < x < \infty.$$

10) For all $x \in \mathbf{R}$, we have $s(-x) = -s(x)$.

11) For all $x \in \mathbf{R}$, define $c(x)$ by

$$c(x) = \sqrt{1 - (s(x))^2}.$$

Then $c(0) = 1$, $c(-\pi/20 = 0$, $c(\pi/2) = 0$.

12) For all $x \in \mathbf{R}$, $c(x) = s'(x)$.

13) For all $x \in \mathbf{R}$, $c'(x) = s''(x) = -s(x)$.

14) For all $x, y \in \mathbf{R}$, $s(x + y) = s(x)c(y) + c(x)s(y)$.
15) For all $x, y \in \mathbf{R}$, $c(x + y) = c(x)c(y) - s(x)s(y)$.

Proof of 14 and 15. Fix y and define $F(x)$ by

$$F(x) = s(x + y) - s(x)c(y) - c(x)s(y).$$

Then $F''(x) + F(x) = 0$. Hence

$$\frac{d}{dx}\left(F^2 + (F')^2\right) = 2F'F + 2F'F'' = 2F'(F + F'') = 0$$

and so

$$[F(x)]^2 + (F'(x)]^2 = const. = [F(0)]^2 + (F'(0)]^2 = 0.$$

Thus $F(x) = 0$, which proves 14. Note also that the last equation implies

$$F'(x) = c(x + y) - c(x)c(y) + s(x)s(y) = 0$$

which gives 15.

Definiton. The functions $s(x)$ and $c(x)$ is called the sine and cosine functions of x, respectively, and we write

$$s(x) = \sin x \qquad c(x) = \cos x.$$

Thses are the basic trigonometic functions. The other four trigonometric functions are defined by

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Remark. For more on the trig functions and please read pages 233 and 234 of your text.