Real Analysis II

12 Fourier Series

12.0 Power Series Revisited

We say f is **analytic on an open interval** (a,b) if for any $x_0 \in (a,b)$, there is a power series centered at x_0 that represents f in some open interval containing x_0. That is, there exists $a \le c < x_0 < d \le b$ such that for all $x \in (c,d)$

$$f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

Remark 1) If $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$, then

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

and hence f is infinitely differentiable. We write $f \in C^\infty(a,b)$

2) The **Taylor Series for** f **at** x_0 is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

3) The nth **Taylor Polynomial for** f **at** x_0 is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

4) The **remainder term of order** n is defined by $R_n(x) = f(x) - P_n(x)$.

5) **(Taylor's Formula)** If f is analytic in (a,b) and $x_0 \in (a,b)$, then there exists c_x between x and c such that

$$R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$$

6) **(Lagrange)** If $f \in C^\infty(a,b)$, then for any $x, x_0 \in (a,b)$,

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n)}(t)\,dt$$

7) If $f \in C^\infty(a,b)$ and if

$$f^{(n)}(x) \ge 0 \quad \text{for all } \ x \in (a,b),$$

then f is analytic on (a,b).

8) Suppose f and g are analytic on (a,b) and $x_0 \in (a,b)$. If $f(x) = g(x)$ for all $x \in (a,x_0)$, then there exists a $\delta > 0$ such that

$$f(x) = g(x) \quad \text{for all} \quad x \in (x_0-\delta, x_0+\delta).$$

9) **(Analytic Continuation)** Suppose that I and J are open interval, that f is analytic on I and g is analytic on J. If $a < b$ are points in $I \cap J$ and $f(x) = g(x)$ for all $x \in (a,b)$, then $f(x) = g(x)$ for all $x \in I \cap J$.

Examples a) Polynomials are analytic.

b) All convergent power series are analytic.

c) The exponential function e^x, the trig functions $\sin x$ and $\cos x$ are analytic on $(-\infty, \infty)$. The rational function $1/(1-x)$ is analytic on $(-1, 1)$. Furthermore, on the intervals of analyticity, we have

i) $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ ii) $\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$

iii) $\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!}$ iv) $\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$

d) **(Cauchy)** The function

$$f(x) = \begin{cases} e^{-2/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

belongs to $C^\infty(-\infty, \infty)$ but is not analytic on any open interval that contains $x = 0$. Find its Taylor series.

12.1 Definition of Fourier Series

Theorem 1 (Orthogonality Theorem)

(a) $\int_{-\pi}^{\pi} \cos(kx)\cos(nx)\,dx = 0 \quad (n \neq k; n, k = 0, 1, 2, 3, \cdots)$

(b) $\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \begin{cases} \pi, & n \geq 1 \\ 2\pi, & n = 0 \end{cases}$

(c) $\int_{-\pi}^{\pi} \sin(kx)\sin(nx)\,dx = 0, \quad (k \neq n; n, k = 1, 2, 3, \cdots)$

(d) $\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi, \quad (n = 1, 2, 3, \cdots)$

(e) $\int_{-\pi}^{\pi} \cos(kx)\sin(nx)\,dx = 0, \quad (n, k = 1, 2, 3, \cdots)$

Proof Use the trig identities

$$\cos(kx + nx) = \cos(kx)\cos(nx) - \sin(kx)\sin(nx)$$
$$\cos(kx - nx) = \cos(kx)\cos(nx) + \sin(kx)\sin(nx)$$

$$\int_{-\pi}^{\pi} \cos(ax) = \frac{1}{a}\sin(ax)\Big|_{-\pi}^{\pi} = 0, \quad (a = 1, 2, 3, \cdots)$$
$$\int_{-\pi}^{\pi} \sin(ax) = \frac{-1}{a}\cos(ax)\Big|_{-\pi}^{\pi} = 0, \quad (a = 1, 2, 3, \cdots)$$

Remark 1. Let f be a function defined on $[-\pi, \pi]$. Assume that, for each $x \in [-\pi, \pi]$, $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big[a_k \cos(kx) + b_k \sin(kx)\Big]$$

Then using Theorem 1, we have the following

$$\int_{-\pi}^{\pi} f(x)\, dx = \pi a_0$$

For $n \geq 1$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx &= \frac{a_0}{2}\int_{-\pi}^{\pi} \cos(nx)\, dx \;+\; \sum_{k=1}^{\infty}\Big[a_k \int_{-\pi}^{\pi} \cos(nx)\cos(kx)\, dx \;+\; b_k \int_{-\pi}^{\pi} \cos(nx)\sin(kx)\, dx\Big] \\ &= a_n \int_{-\pi}^{\pi} \cos^2(nx)\, dx = \pi a_n\end{aligned}$$

Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx.$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx.$$

Definition If f is Riemann integrable over $[-\pi, \pi]$, then the **Fourier Series** of f is the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \Big[a_k \cos(kx) + b_k \sin(kx)\Big],$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\, dx \qquad (k = 0, 1, 2, 3, \cdots)$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\, dx, \qquad (k = 1, 2, 3, \cdots.)$$

The a_k and b_k are called the **Fourier Coefficients** of f. For the purpose of clarity, we will write $a_k(f)$ instead of a_k and $b_k(f)$ instead of b_k .

It is also common to write

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big[a_k \cos(kx) + b_k \sin(kx)\Big].$$

We shall also write $S_n f$ for the nth partial sum of the Fourier series of f. Thus,

$$S_0 f(x) = \frac{a_0}{2}$$

and for $n \geq 1$,

$$(S_n f)(x) = \frac{a_0}{2} \;+\; \sum_{k=1}^{n} \Big[a_k \cos(kx) \;+\; b_k \sin(kx)\Big]$$

Example Show that the Fourier series for $f(x) = x$ is

$$2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)$$

Since $x\cos(kx)$ is odd and $x\sin(kx)$ is even, we see that

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = 0, \qquad k = 0, 1, 2, 3, \cdots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(kx)\,dx \qquad k = 1, 2, 3, \cdots.$$

Integration by parts yields

$$b_k = \frac{2}{\pi}\left(-\frac{x\cos(kx)}{k}\bigg|_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos(kx)\,dx\right) = \frac{2(-1)^{k+1}}{k}.$$

Example Show that the Fourier series for $f(x) = |x|$ is

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos((2k-1)x)}{(2k-1)^2}$$

Since $|x|\cos(kx)$ is even and $|x|\sin(kx)$ is odd, we have

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = 0 \qquad k = 1, 2, 3, \cdots.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos(kx)\,dx \qquad k = 0, 1, 2, 3, \cdots.$$

If $k = 0$, then we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) = \pi$$

and if $k \geq 1$, integration by parts yields

$$a_k = \frac{2}{\pi k^2}\left(\cos(k\pi) - 1\right) = \begin{cases} 0 & \text{if } k \text{ is even} \\ -\frac{4}{\pi k^2} & \text{if } k \text{ is odd.} \end{cases}$$

12.2 Formulation of Convergence and Summability Problems

Convergence Question. Given a function f periodic on $\mathbf{R}$ and integrable on $[-\pi, \pi]$, does the Fourier series of f converge to f?

Uniqueness Question. If a trigonometric series converges to f, is the series the Fourier series of f?

Theorem 2. If the trigonometric series

$$\frac{a_0}{2} \ + \ \sum_{k=1}^{\infty} \Big[a_k \cos(kx) \ + \ b_k \sin(kx)\Big]$$

converges to f uniformly, then it is the Fourier series of f. That is,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, \qquad k = 0, 1, 2, 3, \cdots$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx, \qquad k = 1, 2, 3, \cdots.$$

Proof This follows from Remark 1 and the fact that the series converges uniformly.

Definition. 1) A **Dirichlet kernel of order** n is the function defined by

$$D_0(x) = \frac{1}{2}, \ \ D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx).$$

2) The **Fejer kernel of order** n is defined by

$$K_0(x) = \frac{1}{2}, \ \ K_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right)\cos(kx).$$

Lemma 1.

$$K_n(x) = \frac{D_0(x) + D_1(x) + \cdots + D_n(x)}{n+1}.$$

Proof. The formula is trivially true if $n = 0$. Suppose $n \geq 1$. Then

$$\begin{aligned} K_n(x) &= \frac{1}{n+1}\left(\frac{n+1}{2} + \sum_{k=1}^{n}(n-k+1)\cos(kx)\right) = \frac{1}{n+1}\left(\frac{1}{2} + \frac{n}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} 1\cdot\cos(kx)\right) \\ &= \frac{1}{n+1}\left(\frac{1}{2} + \sum_{j=1}^{n}\left[\frac{1}{2} + \sum_{k=1}^{j}\cos(kx)\right]\right) = \frac{D_0(x) + D_1(x) + D_2(x) + \cdots + D_n(x)}{n+1} \end{aligned}$$

Lemma 2. If $x \in \mathbf{R}$, $x \neq 2k\pi$ for $k \in \mathbf{I}$, then for each $n = 0, 1, 2, \cdots$,

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)} \qquad \text{and} \qquad K_n(x) = \frac{2}{n+1}\left(\frac{\sin\left(\frac{n+1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}\right)^2$$

Proof. For $n = 0$, the lemma is trivial.

Fix $n \geq 1$ and apply the sum-angle and telescoping to get

$$\begin{aligned} \left[D_n(x) - \frac{1}{2}\right] \sin\left(\frac{x}{2}\right) &= \sum_{k=1}^{n} \cos(kx) \sin\left(\frac{x}{2}\right) = \frac{1}{2} \sum_{k=1}^{n} \left[\sin\left(k + \frac{1}{2}\right) x - \sin\left(k - \frac{1}{2}\right) x\right] \\ &= \frac{1}{2}\left[\sin\left(n + \frac{1}{2}\right) x - \sin\left(\frac{x}{2}\right)\right] \end{aligned}$$

and hence

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right) x}{2 \sin\left(\frac{x}{2}\right)}$$

To prove the second formula we use the formula just proved and sum-angle formula we have

$$D_k(x) \sin^2\left(\frac{x}{2}\right) = \frac{1}{2} \sin\left(\frac{x}{2}\right) \sin\left(k + \frac{1}{2}\right) x = \frac{1}{4}\left[\cos(kx) - \cos(k+1)x\right]$$

By Lemma 1 and telescoping, we get

$$\begin{aligned} (n+1) K_n(x) \sin^2\left(\frac{x}{2}\right) &= \sum_{k=0}^{n} D_k(x) \sin^2\left(\frac{x}{2}\right) = \frac{1}{4} \sum_{k=0}^{n} \left[\cos(kx) - \cos(k+1)x\right] \\ &= \frac{1}{4}\left[1 - \cos(n+1)x\right] = \frac{1}{2} \sin^2\left(\frac{x}{2}\right) \end{aligned}$$

and the second formula of the lemma follows by dividing.

Definition. A series $\sum_{k=0}^{\infty} a_k$ with partial sums $s_n = \sum_{k=0}^{n} a_k$ is said to be **Cesaro summable** to a finite number L if and only if

$$\sigma_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}$$

converges to L. Cesaro summable is also called $(C, 1)$ **summable** and we write

$$\sum_{k=0}^{\infty} a_k = L \qquad (C, 1).$$

(Sections 2.11, and 3.9 of the text have more on this.)

Example The series

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

is divergent, since

$$s_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

However,

$$\sigma_n = \begin{cases} \frac{n+2}{2(n+1)} & \text{if } n \text{ is even} \\ \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$\lim_{n \to \infty} \sigma_n = \frac{1}{2}, \quad \text{and so} \quad \sum_{k=0}^{\infty} (-1)^k = \frac{1}{2} \quad (C, 1).$$

Definition. The **Cesaro means** of a Fourier series of f is denoted by $\sigma_n f$ and is given by

$$(\sigma_n f)(x) = \frac{(S_0 f)(x) + (S_1 f)(x) + \cdots + (S_n f)(x)}{n+1},$$

where $S_k f$ is the k partial sum of the Fourier series of f.

Lemma 3. If f is periodic on $\mathbf{R}$ and integrable on $[-\pi, \pi]$, then for all $x \in \mathbf{R}$ and $n = 0, 1, 2, 3 \cdots$, we have

$$(\sigma_n f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) K_n(t)\, dt.$$

Proof For simplicity, let us write a_k for $a_k(f)$ and b_k for $b_k(f)$. For each j, we have

$$\begin{aligned} a_j \cos(jx) + b_j \sin(jx) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \cos(ju) \cos(jx)\, du + \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \sin(ju) \sin(jx)\, du \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \Big[\cos(ju) \cos(jx) + \sin(ju) \sin(jx) \Big]\, du \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \cos(j(u-x))\, du. \end{aligned}$$

Summing over $j = 1, 2, \cdots, k$ and adding $a_0/2$, we have

$$\begin{aligned} (S_k f)(x) &= \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos(jx) + b_j \sin(jx) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \left(\frac{1}{2} + \sum_{j=1}^{k} \cos(j(x-u)) \right) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) D_k(x-u)\, du \end{aligned}$$

We now use the fact that f and D_k are periodic and make change of variables $t = x - u$ to obtain

$$S_k f(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) D_k(t)\, dt.$$

Using Lemma 2 we have

$$\begin{aligned} (\sigma_n f)(x) &= \frac{1}{n+1} \sum_{k=0}^{n} (S_k f)(x) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) D_k(t)\, dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) K_n(t)\, dt. \end{aligned}$$

Lemma 4. For $n = 0, 1, 2, 3, \cdots$, we have

(i) $K_n(t) \geq 0, \quad$ for all $\quad t \in \mathbf{R}$,

(ii) $\frac{1}{\pi} \int_{-\pi}^{\pi} K_n(x)\, dx = 1$,

(iii) $\lim_{n\to\infty} \int_{\delta}^{\pi} |K_n(x)|\, dx = 0$ for any $0 < \delta < \pi$.

Proof. (i) follows from

$$K_n(x) = \frac{2}{n+1} \left(\frac{\sin\left(\frac{n+1}{2}\right) x}{2 \sin\left(\frac{x}{2}\right)} \right)^2$$

To prove (ii), note that

$$\int_{-\pi}^{\pi} K_(x)\,dx = \int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\cos(kx)\right)dx = \pi.$$

To prove (iii), note that if $0<\delta<t<\pi$, then $\sin(\delta/2)<\sin(t/2)$ and using Lemma 2 we get

$$\int_{\delta}^{\pi}|K_n(x)\,dx \;\le\; \frac{2}{n+1}\int_{\delta}^{\pi}\left(\frac{\sin\left(\frac{n+1}{2}\right)x}{2\sin\left(\frac{\delta}{2}\right)}\right)^2 dt \;\le\; \frac{\pi}{2(n+1)}\frac{1}{\sin^2\left(\frac{\delta}{2}\right)}$$

and then take limit as $n\to\infty$.

Theorem 3. (Fejer) Suppose f is periodic on $\mathbf{R}$ and integrable on $[-\pi,\pi]$.

1) If

$$L=\lim_{h\to 0}\frac{f(x_0+h)+f(x_0-h)}{2}$$

exists for some $x_0\in\mathbf{R}$, then $\lim_{n\to\infty}(\sigma_n f)(x_0)=L$.

2) If f is continuous on some closed interval $[a,b]$, then $\sigma_n f\to f$ uniformly on $[a,b]$

Proof. Since f is periodic, we may assume that $x_0\in[-\pi,\pi]$. Fix $n\ge 1$. By Lemmas 2 and 3 and change of variables, we have

$$\begin{aligned}(\sigma_n f)(x_0)-L &= \frac{1}{\pi}\int_{-\pi}^{\pi}K_n(t)[f(x_0-t)-L]dt = \frac{2}{\pi}\int_0^{\pi}K_n(t)\left[\frac{f(x_0+t)+f(x_0-t)}{2}-L\right]dt\\ &= \frac{2}{\pi}\int_0^{\pi}K_n(t)F(x_0,t)dt\end{aligned}$$

where

$$F(x_0,t)=\frac{f(x_0+t)+f(x_0-t)}{2}-L.$$

Let $\epsilon>0$. By definition of L, we can choose $\delta>0$ with $\delta<\pi$ such that if $|t|<\delta$, then $|F(x_0,t)|<\epsilon/3$. Using Lemma 3, we get

$$\left|\frac{2}{\pi}\int_0^{\delta}K_n(t)F(x_0,t)dt\right|\le\frac{2\epsilon}{3\pi}\int_0^{\delta}|K_n(t)|\,dt<\frac{2\epsilon}{3}.$$

Let $M=\sup_{-\pi\le x\le\pi}|f(x)|$. Then $|F(x_0,t)|\le M$. Using the third equation of Lemma 3, we can choose N_1 such that for all $n\ge N_1$,

$$\left|\int_{\delta}^{\pi}K_n(t)\,dt\right|<\frac{\epsilon}{3M}.$$

Thus, we have

$$\left|\frac{2}{\pi}\int_{\delta}^{\pi}K_n(t)F(x_0,t)dt\right|\le M\int_0^{\delta}|K_n(t)|\,dt<\frac{\epsilon}{3}.$$

Therefore for $n\ge N_1$, we have

$$|(\sigma_n f)(x_0)-L| \;\le\; \frac{2}{\pi}\int_0^{\delta}|K_n(t)F(x_0,t)|+\frac{2}{\pi}\int_{\delta}^{\pi}|K_n(t)F(x_0,t)| \;\le\; \frac{2\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.$$

We proved the very definition of (i).

To prove (ii), we note that if f is continuous on $[-\pi, \pi]$, then it is uniformly continuous on $[-\pi, \pi]$. The above inequalities are valid if we replace x_0 by any $x \in [-\pi, \pi]$. (You should cary out the details.)

Corollary 1. If f is continuous and periodic on $\mathbf{R}$, then

$$\sigma_n f \to f \qquad \text{uniformly on} \quad \mathbf{R}$$

Proof. Since f is periodic, we may assume that f is continuous on $-\pi, \pi]$ and apply Fejer's Theorem.

Corollary 2. (Completeness) If f is continuous and periodic on $\mathbf{R}$, and if $a_k(f) = 0$ and $b_k(f) = 0$ for all $k = 0, 1, 2, 3, \cdots$, then $f(x) = 0$ for all $x \in \mathbf{R}$

Proof. From the assumption we have $\sigma_n f(x) = 0$ for all x. By Corollary 1, we have $f(x) = \lim_{n\to\infty}(\sigma_n f)(x) = 0$.

Corollary 3. If f is continuous and periodic on $\mathbf{R}$, then there is a sequence of trigonometric polynomials $T_1, T_2, \cdots$ such that

$$T_n \to f \qquad \text{uniformly on} \quad \mathbf{R}$$

Proof $S_n f$ is a trig polynomial implies $\sigma_n f$ is a trig polynomial. Take T_n to be $\sigma_n f$ and apply Fejer's Theorem.

Theorem 4.(Weierstrass Approximation Theorem) Let f be continuous on a closed and bounded interval $[a, b]$. Given $\epsilon > 0$, there exists a polynomial

$$P(x) = \sum_{k=0}^{n} p_k x^k,$$

where $p_k \in \mathbf{R}$ such that for all $x \in [a, b]$,

$$|f(x) - P(x)| < \epsilon.$$

12.3 Growth of Fourier Coefficients

Lemma 5. If f is integrable on $[-\pi, \pi]$, then for $n = 0,\ 1,\ 2,\ 3,\ \cdots,$

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(S_n f)(x)\,dx &= \frac{|a_0(f)|^2}{2} + \sum_{k=1}^{n}\Big(|a_k(f)|^2 + |b_k(f)|^2\Big) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} |(S_n f)(x)|^2\,dx\end{aligned}$$

Theorem 5. (Bessel's Inequality) If f is Riemann integrable on $[-\pi, \pi]$, then

$$\sum_{k=1}^{\infty} |a_k(f)|^2 \qquad \text{and} \qquad \sum_{k=1}^{\infty} |b_k(f)|^2$$

are both convergent. Moreover,

$$\frac{|a_0(f)|^2}{2} + \sum_{k=1}^{n}\Big(|a_k(f)|^2 + |b_k(f)|^2\Big) \le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

Corollary (Riemann - Lebesgue Lemma) If f is Riemann integrable on $[\pi, \pi]$, then

$$\lim_{k\to\infty} a_k(f) = \lim_{k\to} b_k(f) = 0.$$

Lemma 6. If f is Riemann integrable on $[-\pi, \pi]$ and

$$T_n = \frac{c_0}{2} + \sum_{k=1}^{n}\Big[c_k\cos(kx) + d_k\sin(kx)\Big]$$

is any trigonometric polynomial of degree n, then

$$\int_{-\pi}^{\pi} |f(x) - (S_n f)(x)|^2\,dx \le \int_{-\pi}^{\pi} |f(x) - T_n(x)|^2\,dx$$

Theorem 6. (Parseval's Identity) If f is periodic and continuous on $\mathbf{R}$, then

$$\frac{|a_0(f)|^2}{2} + \sum_{k=1}^{n}\Big(|a_k(f)|^2 + |b_k(f)|^2\Big) = \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

Theorem 7. (Riemann - Lebesgue Lemma) If $f^{(j)}$ exists and is Riemann integrable on $[\pi, \pi]$ and if $f^{(l)}$ is periodic for $1 \le l < j$, then

$$\lim_{k\to\infty} k^j a_k(f) = \lim_{k\to\infty} k^j b_k(f) = 0.$$

12.4 A Digression: Functions of Bounded Variation

Definition Let $\phi : [a, b] \to \mathbf{R}$ be a function and let $P = \{x_0, x_1, \cdots x_n\}$ be a partition of $[a, b]$. Define

$$V(\phi, P) = \sum_{k=1}^{n} |\phi(x_k) - \phi(x_{k-1})| .$$

The **total variation of** ϕ **on** $[a, b]$ is defined by

$$Var(\phi) = \sup\{V(\phi, P)|P \text{ is a partition of } [a, b]\}$$

A function ϕ is said to be **of bounded variation** if $V(\phi) < \infty$.

Lemma 7. If $\phi \in C^1[a, b]$, then ϕ is of bounded variation on $[a, b]$.

Proof: Let $P = \{x_0, x_1, \cdots x_n\}$ be a partition of $[a, b]$. Since ϕ' is continuous on $[a, b]$, by Extreme Value Thereom, there exits M such that

$$|\phi'(x)| \leq M \quad \text{for all} \quad x \in [a, b].$$

On the other hand, by Mean Value Theorem, there exists $c_k \in [x_{k-1}, x_k]$ such that

$$\phi(x_k) - \phi(x_{k-1}) = \phi'(c_k)(x_k - x_{k-1}).$$

Adding these, using the previous inequality, and telescoping, we see that

$$V(\phi, P) = \sum_{k=1}^{n} |\phi(x_k) - \phi(x_{k-1})| \leq M(b - a).$$

Taking the sup over all partitions P we see that

$$Var(\phi) \leq M(b - a).$$

Example Let $\phi(x) = x^2 \sin(1/x)$. Show that

a) ϕ is of bounded variation on $[0, 1]$.

b) $\phi \notin C^1[0, 1]$

Solution. a) Consider a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[0, 1]$.

Choose n a large positive integer so that the values of x_k that are close to zero are contained in the partition $Q = \{0/n, 1/n, 1/n - 1, \cdots, 1\}$

$$\begin{aligned} Var(\phi, Q) &= \sum_{k=1}^{n} x_k^2 \sin(1/x_k) - x_{k-1}^2 \sin(1/x_{k-1}) \leq \sum_{k=1}^{n} \left(x_k^2 + x_{k-1}^2\right) \\ &\leq 2\sum_{j=1}^{n} \frac{1}{k^2} \leq 2 + 2\sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 4 - \frac{2}{n} \leq 4. \end{aligned}$$

Thus $V(\phi, P) \leq V(\phi, Q) < 2$ and taking the sup we see that $Var(\phi) < \infty$.

b) But note that for $x \neq 0$,

$$\phi'(x) = 2x \sin(1/x) - \cos(1/x)$$

and hence

$$\lim_{x \to 0} \phi'(x)$$

does not exist while

$$\phi'(0) = \lim_{h \to \infty} \frac{h^2 \sin(1/h)}{h} = \lim_{h \to \infty} h \sin(1/h) = 0.$$

Therefore $\phi \notin C^1[0,1]$.

Example Let $\phi(x) = x^2 \sin(1/x^2)$. Show that ϕ is not of bounded variation on $[0,1]$.

Lemma 8 If ϕ is monotone on $[a,b]$, then ϕ is of bounded variation on $[a,b]$.

Proof: Suppose ϕ is increasing and let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$. Then

$$\sum_{k=1}^{n} |\phi(x_k) - \phi(x_{k-1})| = \sum_{k=1}^{n} (\phi(x_k) - \phi(x_{k-1})) = \phi(x_n) - \phi(x_0) = \phi(b) - \phi(a)$$

Since $M = \phi(b) - \phi(a)$ is finite, we see that the sup over all partitions P is also finite. Hence ϕ is of bounded variation.

Lemma 9. If ϕ is of bounded variation on $[a,b]$, then ϕ is bounded on $[a,b]$.

Proof: For any $x \in [a,b]$, we have

$$|\phi(x) - \phi(a)| \leq |\phi(x) - \phi(a)| + |\phi(b) - \phi(x)| \leq Var(\phi).$$

Thus

$$|\phi(x) \leq |\phi(x) - \phi(a)| + |\phi(a)| \leq Var(\phi) + |\phi(a)|$$

and hence ϕ is bounded.

Example The function

$$\phi(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is bounded (by 1).

But it is not of bounded variation.For if

$$x_j = \begin{cases} 0, & x = 0 \\ \frac{2}{(n-j)\pi}, & 0 < j < n-1. \end{cases}$$

then, as $n \to \infty$,

$$\sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1})| = 2n \to \infty.$$

Thus ϕ is not of bounded variation on $[0, 2/\pi]$.

Theorem 8. If ϕ and ψ are functions of bounded variation on $[a,b]$, then so are $\phi \pm \psi$, and $\phi \cdot \psi$. If there exists $\epsilon_0 > 0$ such that $\psi(x) \geq \epsilon_0$, then ϕ/ψ is also of bounded variation.

Proof: Let $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a,b]$. Then

$$\begin{aligned}\sum_{k=1}^{n} |(\phi(x_j) \pm \psi(x_j)) - (\phi(x_{j-1}) \pm \psi(x_{j-1}))| &\leq \sum_{k=1}^{n} |\phi(x_j) - \phi(x_{j-1})| + \sum_{k=1}^{n} |\psi(x_j) - \psi(x_{j-1})| \\ &\leq Var(\phi) + Var(\psi)\end{aligned}$$

Therefore,

$$Var(\phi \pm \psi) \leq Var(\phi) + Var(\psi).$$

By Lemma 8, there are constants M_1 and M_2 such that

$$|\phi(x)| \leq M_1 \quad \text{and} \quad |\psi(x)| \leq M_2 \quad \text{for all} \quad x \in [a,b].$$

But then

$$\begin{aligned}&\sum_{j=1}^{n} |\phi(x_j)\psi(x_j) - \phi(x_{j-1})\psi(x_{j-1})| \\ = &\sum_{k=1}^{n} |\phi(x_j)\psi(x_j)) - \phi(x_{j-1})\psi(x_j) + \phi(x_{j-1})\psi(x_j) - \phi(x_{j-1})\psi(x_{j-1})| \\ \leq &\; M_2 \sum_{k=1}^{n} |\phi(x_j) - \phi(x_{j-1})| + M_1 \sum_{k=1}^{n} |\psi(x_j) - \psi(x_{j-1})| \\ \leq &\; M_2 Var(\phi) + M_1 Var(\psi)\end{aligned}$$

Therefore,

$$Var(\phi\psi) \leq M_2 Var(\phi) + M_1 Var(\psi).$$

To prove the ϕ/ψ is also of bounded variation, we write

$$\begin{aligned}\sum_{j=1}^{n} \left| \frac{\phi(x_j)}{\psi(x_j} - \frac{\phi(x_{j-1})}{\psi(x_{j-1})} \right| &= \sum_{j=1}^{n} \left| \frac{\phi(x_j)\psi(x_j) - \phi(x_{j-1})\psi(x_{j-1})}{\psi(x_j)\psi(x_{j-1})} \right| \\ &\leq \frac{1}{\epsilon^2} \left(M_2 \sum_{k=1}^{n} |\phi(x_j) - \phi(x_{j-1})| + M_1 \sum_{k=1}^{n} |\psi(x_j) - \psi(x_{j-1})| \right)\end{aligned}$$

Therefore,

$$Var\left(\frac{\phi}{\psi}\right) \leq \frac{M_2}{\epsilon_0^2} Var(\phi) + \frac{M_1}{\epsilon_0^2} Var(\psi).$$

Definition. Let ϕ be of bounded variation on $[a,b]$. The **total variation of** ϕ is the function defined by

$$\Phi(x) = \sup\{\sum_{k=1}^{n} |\phi(x_j) - \phi(x_{j-1})|\}$$

where the sup is over all partitions $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,x]$.

Theorem 9. Let ϕ be of bounded variation and Φ be its total variation. Then

(i) $|\phi(y) - \phi(x)| \leq \Phi(y) - \Phi(x)$ for all $a \leq x \leq y \leq b$
(ii) Φ and $\Phi - \phi$ are increasing on $[a, b]$
(iii) $Var(\phi) \leq Var(\Phi)$.

Proof: (i) Let $x < y$ and let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, x]$. Then $Q = \{x_0, x_1, \cdots, x_n, y\}$is a partition of $[a, y]$. By definition of Φ we have

$$\begin{aligned} & \sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1})| \\ \leq & \sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1})| + |\phi(y) - \phi(x) \\ \leq & \Phi(y) \end{aligned}$$

Taking the sup over all such partitions P of $[a, x]$ we see that

$$\Phi(x) \leq \Phi(x) - |\phi(y) - \phi(x) \leq \Phi(y)$$

and (i) follows.

(ii) Since Φ is defined as the sup, it is clearly increasing.

By part (i), we have

$$\phi(y) - \phi(x) \leq |\phi(y) - \phi(x)| \leq \Phi(y) - \Phi(x)$$

and hence $\Phi(x) - \phi(x) \leq \Phi(y) - \phi(y)$. Therefore, $\Phi - \phi$ is also increasing.

(iii) Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$. By part (i) and the definition of Φ, we have

$$\begin{aligned} \sum_{k=1}^{n} |\phi(x_k) - \phi(x_{k-1})| & \leq \sum)k = 1^n |\Phi(x_k) - \Phi(x_{k-1})| \\ & \leq Var(\Phi) \end{aligned}$$

Taking the sup over all such P we get (iii).

Corollary ϕ is of bounded variation on $[a, b]$ if and only if there exists increasing functions f and g on $[a, b]$ such that

$$\phi(x) = f(x) - g(x), \quad \text{for all} \quad x \in [a, b].$$

Proof: If ϕ is of bounded variation, let Φ be its total variation. Then by Theorem 9, the functions $f = \Phi$ and $g = \Phi - \phi$ are increasing and $\phi = f - g$.

Conversely, if f and g are monotone, then both are of bounded variation by Lemma 9. But then by Thereom 8 $\phi = f - g$ is of bounded variation.

Remarks. 1) If f is monotone on $[a,b]$, then the set points x in $[a,b]$ at which f is discontinuous is at *most countable.*
Thus if ϕ is of bounded variation on $[a,b]$, then it has at most a countable set of discontinuity on $[a,b]$.

2) If f is monotone, then for any $x_0 \in (a,b]$, the limit $\lim_{x\to x_0^-} f(x)$ exists. This limit is denoted by $f(x_0-)$.

Similarly, for any $x_0 \in [a,b)$, the limit $\lim_{x\to x_0^+} f(x)$ exists. This limit is denoted by $f(x_0+)$.
Thus if ϕ is of bounded variation on $[a,b]$, then the limits $\lim_{x\to x_0^+}\phi(x)$ and $\lim_{x\to x_0^+}\phi(x)$ both exist for all $x_0 \in (a,b)$.

3) Monotone functions are Riemann integrable over $[a,b]$.

Thus, if ϕ is of bounded variation on $[a,b]$, then ϕ is Riemann integrable. .

12.5 Convergence of Fourier Series

Lemma 10. If $\sum_{k=0}^{\infty} a_k$ converges to L, then it is Cesaro summable to L.

Proof: Let $\epsilon > 0$. Choose N_1 such that if $k \geq N_1$ then $|s_k - L| < \frac{\epsilon}{2}$. Use the Archimedean Property to choose $N_2 > N_1$ such that $\sum_{k=0}^{N_1} |s_k - L| < \frac{\epsilon N_2}{2}$. If $n > N_2$, then

$$\begin{aligned} |\sigma_n - L| &\leq \frac{1}{n+1}\sum_{k=0}^{N_1} |s_k - L| + \frac{1}{n+1}\sum_{k=N_1+1}^{n} |s_k - L| \\ &\leq \frac{\epsilon N_2}{2(n+1)} + \frac{\epsilon}{2}\left(\frac{n-N_2}{n+1}\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Theorem 10. (Tauberian Theorem) Let $a_k \geq 0$ and let $L \in \mathbf{R}$. If

$$\sum_{k=0}^{\infty} a_k = L \quad (C,1), \quad \text{then} \quad ;\sum_{k=0}^{\infty} a_k = L.$$

In other words, if a series of nonnegative terms is Cesaro summable to L, then it converges to L.

Proof: If we show the series converges, then by Lemma 8, we know that it must converge to L. Thus we need only show that $\sum_{k=0}^{\infty} a_k < \infty$. Suppose to the contrary that $\sum_{k=0}^{\infty} a_k = \infty$. Then given $M > 0$, there exists $n_0 > 1$ such that if $n \geq n_0$, then $s_n = \sum_{k=0}^{n} a_k \geq M$. Let $n \geq n_0$. Then

$$\begin{aligned} \sigma_n &= \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \\ &= \frac{s_0 + s_1 + \cdots + s_{n_0}}{n+1} + \frac{s_{n_0+1} + s_{n_0+2} + \cdots + s_n}{n+1} \\ &\geq 0 + \frac{n-n_0}{n+1}\cdot M \end{aligned}$$

If we take the limit as $n \to \infty$, we see that $L \geq M$ for all $M > 0$. This is a contradiction. (Take $M = L + 10$)

Corollary Let f be periodic on $\mathbf{R}$ and Riemann integrable on $[-\pi, \pi]$. If $a_k(f) = 0$ and $b_k(f) \geq 0$ for all $k \geq 1$, then

$$\sum_{k=1}^{\infty} \frac{b_k(f)}{k} \; < \; \infty.$$

Proof: Assume $a_0(f) = 0$. Otherwise take $g(x) = f(x) - a_0(f)$. Let

$$F(x) = \int_0^x f(t)dt.$$

Then F is continuous and periodic (note that $a_0(f) = 0$) on $\mathbf{R}$. Hence by Fejer's Theorem $(\sigma_n F)(0) \to F(0) = 0$ as $n \to \infty$. Integrating by parts we get

$$a_k(F) = \frac{b_k(f)}{k} \geq 0 \quad \text{and} \quad b_k(F) = \frac{a_k(f)}{k} = 0.$$

Hence

$$\sum_{k=1}^{\infty} \frac{b_k(f)}{k}$$

Cesaro summable. Since the terms are nonnegative, the corollary follows from Tauber's Theorem.

Theorem 11. (Hardy) Let $E \subset \mathbf{R}$ and suppose the $\{f_k\}$ is a sequence of functions on E that satisfies

$$|kf_k(x)| \leq M$$

for all $x \in E$ and all $k \in \mathbf{N}$, and some $M > 0$. If $\sum_{k=0}^{\infty} f_k$ is uniformly Cesaro summable to a function f on E, then $\sum_{k=0}^{\infty} f_k$ converges uniformly to f on E.

Proof: Let $x \in E$ and assume, without loss of generality, that $M \geq 1$. For each $n = 0, 1, 2, \cdots$, set

$$s_n(x) = \sum_{k=0}^{n} f_k(x)$$

and

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + s_2(x) + \cdots + s_n(x)}{n+1}.$$

Consider (the delayed average) defined for $n, k \geq 0$ by

$$\sigma_{n,k}(x) = \frac{s_n(x) + s_{n+1}(x) + \cdots + s_{n+k}(x)}{k+1}.$$

Let $0 < \epsilon < 1$. For each n choose $k = k(n)$ such that

$$k + 1 \leq \frac{\epsilon n}{2M} < k + 2.$$

But then

$$\frac{n-1}{k+1} < \frac{n}{k+1} < \frac{2M}{\epsilon} < \infty.$$

Note also that

$$\begin{aligned} \sigma_{n,k}(x) - s_n(x) &= \frac{(s_n(x) - s_n(x)) + (s_{n+1}(x) - s_n(x)) + \cdots + (s_{n+k}(x) - s_n(x))}{k+1} \\ &= \sum_{j=n}^{k+n} \left(1 - \frac{j-n}{k+1}\right) f_j(x). \end{aligned}$$

By assumption $k|f_k(x)| \leq M$ and by choice of $k = k(n)$, we have

$$\begin{aligned} |\sigma_{n,k}(x) - s_n(x)| &\leq \sum_{j=n+1}^{n+k} |f_j(x)| \\ &\leq M \sum_{j=n+1}^{n+k} \frac{1}{j} < \frac{M(k+1)}{n+1} < \frac{\epsilon}{2} \end{aligned}$$

Since $\sigma_n \to f$ uniformly on E, we choose N so that for all $n \geq N$ and for all $x \in E$,

$$|\sigma_n(x) - f(x)| < \frac{\epsilon^2}{12M}.$$

Since

$$\sigma_{n,k} = \left(1 + \frac{n-1}{k+1}\right)\sigma_{n+k} - \left(\frac{n-1}{k+1}\right)\sigma_{n-1},$$

it follows that

$$\begin{aligned} |s_n(x) - f(x)| &\leq |s_n(x) - \sigma_{n,k}(x)| + |\sigma_{n,k}(x) - f(x)| \\ &\leq \frac{\epsilon}{2} + \left(1 + \frac{n-1}{k+1}\right)|\sigma_{n+k}(x) - f(x)| + \left(\frac{n-1}{k+1}\right)|\sigma_{n-1}(x) - f(x)| \\ &\leq \frac{\epsilon}{2} + \left(1 + \frac{n-1}{k+1}\right)\left(\frac{\epsilon^2}{12M}\right) + \frac{2M}{\epsilon}\left(\frac{\epsilon^2}{12M}\right) \\ &= \frac{\epsilon}{2} + \frac{\epsilon^2}{12M} + \frac{\epsilon}{3} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{12} + \frac{\epsilon}{3} < \epsilon. \end{aligned}$$

Therefore, $\sum_{k=0}^{\infty} f_k \to f$ uniformly on E.

Theorem 12. (Dirichlet - Jordan) If f is periodic on $\mathbf{R}$ and continuous on some closed interval $[a, b]$, then

$$S_n f \to f \qquad \text{uniformly on} \quad ; [a, b]$$

Remark For the uniqueness question posed earlier we have the following theorems, whose proofs can be found on pages 536 of William R. Wade's *An Introduction to Analysis* 3rd edition, published by Prentice Hall.

Theorem 13. (Cantor - Lebesgue Lemma) If

$$S = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big[a_k \cos(kx) + b_k \sin(kx) \Big]$$

is a trigonometric series that converges pointwise on some interval $[a, b]$, then

$$\lim_{k \to \infty} a_k = 0 \qquad \text{and} \qquad \lim_{k \to \infty} b_k = 0.$$

Theorem 14. (Cantor) Suppose

$$S = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big[a_k \cos(kx) + b_k \sin(kx) \Big]$$

is a trigonometric series that converges pointwise on $[-\pi, \pi]$ to a periodic continuous function f. Then S is the Fourier series of f, that is

$$a_k = a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx, \qquad k = 0, 1, 2, 3, \cdots$$

$$b_k = b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx, \qquad k = 1, 2, 3, \cdots.$$