Real Analysis I: Hints for Problems of Chapter 3

Section 3.1

3. Assume that $f(x)$ is continuous. We want to show that $|f(x)|$ is also continuous. Let $\{x_n\}$ be a sequence that converges to x. Then by definition of continuity, $\{f(x_n)\}$ converges to $f(x)$. But then $\{|f(x_n)|\}$ converges to $|f(x)|$. Hence $|f(x)|$ is continuous.

4. Let $f(x) = \ln(\sin x)$. Then $x \in \text{Dom}(f)$ if and only if $\sin x > 0$ if and only if $x \in [2n\pi, (2n+1)\pi]$ where $n = 0,\ 1,\ 2,\ \cdots$.

5. Suppose f is continuous and $f(x) = 0$ for $x \in \mathbf{Q}$. We want to show $f(x) = 0$ for all x. Let $x \in \mathbf{R}$. If x is a rational number, then $f(x) = 0$ by assumption. If x is not a rational number, then for each $n \in \mathbf{N}$, we can choose a rational number x_n between x and $x + 1/n$. (Draw a number line to show this.) But then $|x_n - x| < 1/n$ and hence $\{x_n\}$ converges to x. By continuity of f we conclude that $\{f(x_n)\}$ converges to $f(x)$, that is, $\lim_{n\to\infty} f(x_n) = f(x)$. Since $f(x_n) = 0$ for all n, we see that the limit is also 0. Thus $f(x) = 0$.

6. If f is defined only on integers and a is an integer, then there is no sequence $\{x_n\}$ of integers converging to a for which $\{f(x_n)\}$ does not converge to $f(a)$. (In other words there is no sequence for which the definition of continuity fails)

7. Let $f(x) = 3x - 1$. Then $|(fx) - 2| \le \epsilon$ if and only if $|3x - 1 - 2| \le \epsilon$ if and only if $|3x - 3| \le \epsilon$ if and only if $|x - 1| \le \epsilon/3$. Thus we can choose $\delta \le \epsilon/3$.

8. Let $f(x) = x^2$.

a) $|x^2 - 1| \le \epsilon$ iff $|x+1||x-1| \le \epsilon$. If we choose x so that $|x - 1| \le 1$, then x is between 0 and 2 and hence $x + 1$ is between 1 and 3. Thus, for this choice of x, we have $|x^2 - 1| = |x+1||x-1| \le 3|x-1|$. To make $|x^2 - 1| \le \epsilon$, all we need to do is make $3|x - 1| \le \epsilon$ and for this we need $|x - 1| \le \epsilon/3$. Thus we may choose $\delta = \min\{1, \epsilon/3\}$.

b) To make $|x^2 - 4| \le \epsilon$, we note that $|x^2 - 4| = |x+2||x-2|$. As in (a) above, choose x so that $|x - 2| \le 1$. (Draw a number line for all such x.) Then $|x + 2| \le 5$. Therefore, $|x^2 - 4| = |x+2||x-2| \le 5|x-2|$. If $|x - 2| \le \epsilon/5$, then $|x^2 - 4| \le \epsilon$. Thus we choose $\delta = \min\{1,\ \epsilon/5\}$.

c) δ gets smaller.

9. Let $f(x) = 3x^3 - 2$. Let $\epsilon > 0$ be given. $|f(x) - 1| = |3x^3 - 3| \le \epsilon$ iff $3|x^3 - 1| \le \epsilon$. We now factor $x^3 - 1 = (x-1)(x^2+x+1)$ and estimate x^2+x+1 assuming $|x-1| \le 1$. Note that $|x-1| \le 1$ implies $0 \le x \le 2$ and hence $0 \le x^2 \le 4$. Thus, if $|x-1| \le 1$, then $1 \le x^2+x+1 \le 7$. Now $|f(x)-1| \le 3|x-1||x^2+x+1| \le 3{\cdot}7|x-1| \le \epsilon$ iff $|x - 1| \le \epsilon/21$. We can choose $\delta = \min\{1, \epsilon/21\}$.

10. Let $f(x) = \sqrt{x}$. Let $\epsilon > 0$ be given. First let $c > 0$. We now assume that $|x - c| \le c/2$. (Draw a number a line to see the interval.) Then $c/2 \le x \le 3c/2$ and hence $\sqrt{c/2} \le \sqrt{x} \le \sqrt{3c/2}$. Adding $\sqrt{c}$ to all sides and taking the reciprocal, we get $\frac{1}{\sqrt{3c/2}+\sqrt{c}} \le \frac{1}{\sqrt{x}+\sqrt{c}} \le \frac{1}{\sqrt{c/2}+\sqrt{c}}$. We now use this in the hint given:

$$|\sqrt{x} - \sqrt{c}| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| \le \frac{|x - c|}{\sqrt{c/2} + \sqrt{c}}.$$

We can choose $\delta = \min\left\{c/2, \epsilon\left(\sqrt{c/2} + \sqrt{c}\right\}\right)$ (why?). For part(b), if $c = 0$, let $\delta = \epsilon^2$. (Verify that this works!)

Section 3.2

1. If $f(x)=x^3-4x+2$, then f is continuous on $[0,1]$ and $f(0)=2>0$ while $f(1)=-1<0$. Then by IVT, there exists a number $c\in[0,1]$ such that $f(c)=0$. Thus $f(x)$ hasa zero in the indicated interval.

2. Let $f(x)=ax^3+bx^2+cx+d$ be a cubic polynomial with $a\neq 0$. If $a>0$, then $\lim_{x\to\infty} f(x)=+\infty$ and $\lim_{x\to-\infty} f(x)=-\infty$. Thus there exist positive numbers M and N such that $f(-M)<0$ and $f(N)>0$. By IVT, there exists a number $y\in[-M,N]$ such that $f(y)=0$. What happens if $a<0$?

3. Assume that f is continuous on $[a,b]$ and $f(x)>0$ for all x. We want to show that there exists $\alpha>0$ such that $f(x)>\alpha$ for all x. Suppose such a number does not exist. Then for each positive integer n (taking $\alpha=1/n$), there must be a number x_n in $[a,b]$ such that $f(x_n)\leq 1/n$. But then $\{x_n\}$ is a sequence in $[a,b]$ and hence it is a bounded sequence. By Bolzano-Weierstrass Theorem, it has a convergent subsequence, $\{x_{n_k}\}$ that converges to a point x in $[a,b]$. (Why should x be in $[a,b]$? See Problems 2.1 #7) But then $\lim_{k\to\infty} f(x_{n_k})=f(x)$ by continuity of f. On the other hand $f(x_{n_k})\leq \frac{1}{n_k}$ and hence $\lim_{k\to} f(x_{n_k})=0$. This means that $f(x)=0$, which is a contradiction. (To what?)

5. Assume f and g are continuous on $[a,b]$ and $f(x)<g(x)$ for all x. We want to show that there exists $\alpha<1$ such that $f(x)\leq \alpha g(x)$ for all x. Suppose such a number does not exist. Then for each positive integer n (taking $\alpha=1-1/n$), there must be a number x_n in $[a,b]$ such that $f(x_n)\geq(1-1/n)g(x_n)$. But then $\{x_n\}$ is a sequence in $[a,b]$ and hence it is a bounded sequence. By Bolzano-Weierstrass Theorem, it has a convergent subsequence, $\{x_{n_k}\}$ that converges to a point x in $[a,b]$. But then $\lim_{k\to\infty} f(x_{n_k})=f(x)$ and $\lim_{k\to\infty} g(x_{n_k})=g(x)$ by continuity of f and g. On the other hand, $f(x_{n_k})\geq(1-1/n_k)g(x_{n_k})$ and hence $\lim_{k\to\infty} f(x_{n_k})\geq \lim_{k\to\infty}(1-1/n_k)g(x_{n_k})$. This means that $f(x)\geq g(x)$, which is a contradiction.

6. For (a), let $f(x)=x^3-3x+1$, $a=-2$, $b=2$, $c=0$, and $d=1$. Verify that $f(-2)<0<1<f(2)$. Draw the graph to see that S is not one interval but rather a union of three intervals.

For (b), we note that by IVT, there are two numbers x_0 and x_1 in $[a,b]$ such that $f(x_0)=c$ and $f(x_1)=d$. Now show that $S=[x_0,x_1]$. Here we must use the fact that the function is increasing.

7. Let f be Lipschitz continuous on S. We want to show that f is uniformly continuous on S. Let $\epsilon>0$ be given. Since f is Lipschitz continuous on S, there exists a constant $K>0$ such that $|f(x)-f(c)|\leq K|x-c|$ for ALL x and ALL c. If we choose $\delta=\epsilon/K$, we note that for all x and all c, if $|x-c|\leq\delta$, then $|f(x)-f(c)|\leq K|x-c|\leq K\delta=\epsilon$. Therefore, f is uniformly continuous on S.

9. Let $f(x)=1/x$. To show that $f(x)$ is not uniformly continuous on $(0,\infty)$, we need to find a positive number ϵ_0 and two sequences x_n and c_n such that $\lim_{n\to\infty}|x_n-c_n|=0$ but $|f(x_n)-f(c_n)|\geq\epsilon_0$. Let $\epsilon=1/2$, $x_n=\frac{1}{2n}$, and $c_n=\frac{n}{n^2+1}$. Then $\lim_{n\to\infty}|x_n-c_n|=0$ because $\lim_{n\to\infty}x_n=0$ and $\lim_{n\to\infty}c_n=0$. On the other hand, for $n>1$, we have

$$|f(x_n)-f(c_n)|=\left|\frac{1}{1/2n}-\frac{1}{n/(n^2+1)}\right|=2n-\frac{n^2+1}{n}=n-1/n\geq 1/2.$$

To show that $f(x)=1/x$ is uniformly continuous on $[\mu,\infty)$, where $\mu>0$, we note that if $K=1/\mu^2$, then for any x and any c in $[\mu,\infty)$, we have $x\geq\mu$ and $c\geq\mu$, and hence $1/x\leq 1/\mu$ and $1/c\leq 1/\mu$. Thus,

$$|f(x)-f(c)|=\left|\frac{1}{x}-\frac{1}{c}\right|=\frac{|x-c|}{xc}\leq\frac{|x-c|}{\mu^2}=K|x-c|.$$

Thus f is Lipschitz and we apply #7 above.

Section 3.3

1. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

Let $P = \{x_0,\ x_1,\ x_2,\ \cdots x_N\}$ be any partition of $[0,1]$. Then in any subinterval $[x_{i-1}, x_i]$, we can pick a rational number c_i and an irrational number d_i so that $M_i = f(d_i) = 1$ and $m_i = f(c_i) = 0$. Thus for any partition P, we have $L_P(f) = \sum_{i=1}^{N} m_i\,(x_i - x_{i-1}) = 0$, while

$$U_P(f) = \sum_{i=1}^{N} M_i\,(x_i - x_{i-1}) = \sum_{i=1}^{N} (x_i - x_{i-1}) = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_N - x_{N-1}) = x_N - x_0 = 1 - 0 = 1$$

. f is not Riemann integrable since $\inf\{U_P(f)\} = 1$ while $\sup\{L_P(f)\} = 0$.

2. Let $f(x) = \begin{cases} 1, & \text{if } x \neq \frac{1}{2} \\ 2, & \text{if } x = \frac{1}{2} \end{cases}$

Let $\epsilon > 0$ be given. Let $P = \{x_0,\ x_1,\ x_2,\ \cdots x_N\}$ be any partition of $[0,1]$ such that the length of the largest sub interval is less than ϵ. (It is a matter of choosing N so large that $\epsilon > 1/N$.) Suppose $1/2$ is in the jth subinterval $[x_{j-1}, x_j]$. On this subinterval $m_j = 1$ while $M_j = 2$. On all other subintervals we have $M_i = 1$ and $m_i = 1$ (why?). We now have

$$U_P(f) = \sum_{i=1}^{N} M_i\,(x_i - x_{i-1}) = \sum_{i \neq j} (x_i - x_{i-1}) + 2(x_j - x_{j-1})$$

and

$$L_P(f) = \sum_{i=1}^{N} m_i\,(x_i - x_{i-1}) = \sum_{i \neq j} (x_i - x_{i-1}) + (x_j - x_{j-1})$$

Subtracting the two equations we get

$$U_P(f) - L_P(f) = x_j - x_{j-1} < \epsilon.$$

Therefore, f is Riemann integrable by Lemma 3 (Page 89).

3. Suppose f is Riemann Integrable and $f(x) \geq 0$ for all $x \in [a,b]$. (a) To show $\int_a^b f(x)\,dx \geq 0$, let $P = \{x_0,\ x_1,\ x_2,\ \cdots,\ x_N\}$. Then for each subinterval $[x_{i-1}, x_i]$ of this partition, we have $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \geq 0$. (Explain why M_i should be ≥ 0.) But then $U_P(f) \geq 0$ and hence $\inf_P\{U_P(f)\} \geq 0$. By definition $\int_a^b f(x)\,dx = \inf_P\{U_P(f)\}$ and hence $\int_a^b f(x)\,dx \geq 0$.

(b) Suppose $f(x) \geq 0$, f is continuous and $\int_a^b f(x)\,dx = 0$. We need to show $f(x) = 0$ for all x. Suppose not. Then there exists a number c in $[a,b]$ such that $f(c) > 0$. Since f is continuous, there exists a $delta > 0$ such that $f(x) > 0$ for all x in $[a,b]$ and $|x - c| \leq \delta$. We can choose smaller δ, if necessary, so that $c - \delta$ and $c + \delta$ are in $[a,b]$. Let $x_0 = a,\ x_1 = c - \delta,\ x_2 = c + \delta,\ x_3 = b$. Then $P = \{x_0, x_1, x_2, x_3\}$ is a partition of $[a,b]$.(Draw a number line and show this partition.) Since $f(x) \geq 0$ on the first and the third subintervals we have, $m_1 \geq 0$ and $m_3 \geq 0$. On the second subinterval $[c - \delta, c + \delta]$, the function is continuous. Hence it attains its maximum and minimum. On this interval (by the choice of δ) $f(x) > 0$ for all x. In particular $m_2 = f(d) = \inf\{f(x) \mid x \in [c - \delta, c + \delta]\} > 0$. Thus

$$L_P(f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2)\ \geq m_2(x_2 - x_1)\ > 0.$$

Since $\int_a^b f(x)\,dx$ is greater than or equal to any lower sum we see that $\int_a^b f(x)\,dx > 0$. This contradicts the assumption.

(Note: Once we assume that $f(c) > 0$ for some c, and $f(x) \geq 0$ for all x, then $\int_a^b f(x)\,dx$ is the area of a region under the graph and hence it must be positive. This is what we proved above, a seemingly trivial statement!!)

4. Let $f(x) = 3x$ on $[0, 1]$. and let $\epsilon >$ be given.

(a) Let $\delta = \epsilon/3$. Then for all x and for c in $[0, 1]$, $|x - c| \leq \delta$ implies $|f(x) - f(c)| \leq \epsilon$. Now choose a positive integer N large enough so that $1/N < \delta$ and let

$$P = \{0 = \frac{0}{N},\ \frac{1}{N},\ \frac{2}{N},\ \cdots,\ \frac{N}{N} = 1\}.$$

For each subinterval $[\frac{i-1}{N},\ \frac{i}{N}]$, we have $m_i = 3\frac{i-1}{N}$ and $M_i = 3\frac{i}{N}$. (Draw the graph of $f(x) = 3x$ and use $N = 8$ to see this or simply observe that the function is increasing.) But then

$$L_P(f) = \sum_{i=1}^{N} m_i(x_i - x_{i-1}) = 0 \cdot \frac{1}{N} + 3\frac{1}{N}\frac{1}{N} + 3\frac{2}{N}\frac{1}{N} + \cdot 3\frac{N-1}{N}\frac{1}{N}$$

and

$$U_P(f) = \sum_{i=1}^{N} M_i(x_i - x_{i-1}) = 3\frac{1}{N}\frac{1}{N} + 3\frac{2}{N}\frac{1}{N} + 3\frac{2}{N}\frac{1}{N} + \cdot 3\frac{N-1}{N}\frac{1}{N} + 3\frac{N}{N}\frac{1}{N}.$$

Hence $U_P(f) - L_P(f) = 3\frac{N}{N}\frac{1}{N} = 3/N < 3\delta = \epsilon$ as required.

(b) Let k be any integer and let $P = \{0 = \frac{0}{k},\ \frac{1}{k},\ \frac{2}{k},\ \cdots,\ \frac{k}{k} = 1\}$. Let $x_i^* = \frac{i}{k}$. Then x_i^* is in the subinterval $[\frac{i-1}{k},\ \frac{i}{k}]$. Note then that $x_i - x_{i-1} = 1/k$ and $f(x_i^*) = 3i/k$. We form the Riemann sum

$$s_k = \sum_{i=1}^{k} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{k} \frac{3i}{k}\frac{1}{k} = \frac{3}{k^2}\sum_{i=1}^{k} i = \frac{3}{k^2}\frac{k(k+1)}{2} = \frac{3(k+1)}{2k}.$$

Thus by Corollary 3.3.2 (page 91), we have

$$\int_0^1 f(x)dx = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{3(k+1)}{2k} = \frac{3}{2}.$$

5. Since f is continuous on $[a, b]$, by Corollary 3.3.2 $\int_a^b f(x)dx = \lim_{k\to\infty} s_k$, where $s_k = \sum_{i=1}^{k} f(x_i^*)(x_i - x_{i-1})$ is a Riemann sum. Any Riemann sum can be divided into parts where f is positive and f is negative. On the positive parts $f(x_i^*)(x_i - x_{i-1})$ is the area of the rectangle whose height is $f(x_i^*)$ and width is $(x_i - x_{i-1})$. On the negative parts $f(x_i^*)(x_i - x_{i-1})$ is negative one times the area of the rectangle whose height is $-f(x_i^*)$ and width is $(x_i - x_{i-1})$. Thus the Riemann sums are the sum of the areas of the rectangles above the x-axis minus the sum of the areas of the rectangles below. Passing to the limit, we conclude that $\int_a^b f(x)dx$ can be interpreted as the sum of the areas above the x-axis minus the areas below. The Theorems mentioned make sense because areas under graph of functions satisfy these properties. (You may want to draw graphs for each of the theorems and the corollary.)

7. Note that $f(x) = x^2$ is increasing on $[1, 2]$. Thus for any partition $P = \{x_0,\ x_1,\ x_2,\ \cdots, x_N\}$ of $[1, 2]$, $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. In other words, the lower sums are obtained by using the left endpoint of the subintervals while upper sums are obtained by using the right endpoints.

8. First observe that for any x and c in $[1, 3]$, we have $x \geq 1$ and $c \geq 1$. Hence

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x - c|}{xc} \leq |x - c|$$

. Next note that the function is decreasing and therefore for any partition $P = \{x_0,\ x_1,\ x_2,\ \cdots, x_N\}$ of $[1, 3]$, $m_i = f(x_i)$ and $M_i = f(x_{i-1})$. In other words, the lower sums are obtained by using the left endpoint of the subintervals while upper sums are obtained by using the right endpoints. Choose $N = 10^2$ and let the above partition be chosen so that$x_0 = 1, x_1 = 1 + 2/N,\ x_2 = 1 + 4/N,\ x_3 = 1 + 6/N$ and so on. (Note then that $x_N = 1 + 2n/N = 3$ as required!) Compute $L_P(f)$ and $U_P(f)$ and subtract.

9. Show that $e^x \leq \sqrt{1+x}e^x \leq \sqrt{2}e^x$ for all $x \in [0,1]$ and integrate.

13 First note if $x < c$, then $\int_a^c f(t)dt = \int_a^x f(t)dt \; + \; \int_x^c f(t)dt$. If f is continuous on $[a,b]$ then it is bounded: say $|f(t)| < M$ for all t in $[a,b]$. Let $\epsilon > 0$ be given and let $\delta = \epsilon/M$. The for all x and all c in $[a,b]$, is $|x-c| \leq \delta$ and $x < c$, then

$$|F(x) - F(c)| = \left| \int_a^x f(t)dt \; - \; \int_a^c f(t)dt \right| = \left| \int_x^c f(t)dt \right| \leq \int_x^c |f(t)|dt \leq M(c-x) \leq M\delta = \epsilon.$$

(Explain each eqality and inequality in the above argument.) Give the argument for the case if $x > c$.