Real Analysis I: Hints for Problems of Chapter 2

Section 2.1

2. For a given $\epsilon > 0$, verify that the value of N in parenthesis is the appropriate choice. (You should try to obtain this bound for N)

(a) $\lim_{n\to\infty}\left(1+\frac{10}{\sqrt{n}}\right)=1$ $\left(N\geq\frac{100}{\epsilon^2}\right)$ (b) $\lim_{n\to\infty}\left(1+\frac{1}{\sqrt[3]{n}}\right)=1$ $\left(N\geq\frac{1}{\epsilon^3}\right)$

(c) $\lim_{n\to\infty}(3+2^{-n})=3$ $\left(N\geq-\frac{\ln(\epsilon)}{\ln(2)}\right)$ (d) $\lim_{n\to\infty}\left(\sqrt{\frac{n+1}{n}}\right)=1$ $\left(N\geq\frac{1}{\epsilon^3}\right)$

3. For a given $\epsilon > 0$, verify that the value of N in parenthesis is the appropriate choice. (You should try to obtain this bound for N)

(a) $\lim_{n\to\infty}\left(5-\frac{2}{\ln n}\right)=5$ $\left(N\geq e^{2/\epsilon}\right)$ (b) $\lim_{n\to\infty}\left(\frac{3n+1}{n+2}\right)=3$ $\left(N\geq\frac{5}{\epsilon}-2\right)$

(c) $\lim_{n\to\infty}\left(\frac{n^2+6}{2n^2-2}\right)=\frac{1}{2}$ $\left(N\geq\sqrt{\frac{5}{\epsilon}+1}\right)$ (d) $\lim_{n\to\infty}\left(\frac{2^n}{n!}\right)=1$ $\left(N\geq 2+\frac{\ln(\epsilon)1}{\ln(2)-\ln(3)}\right)$

Note that for (d) we have used the fact that if $n > 2$, then

$$\frac{2^n}{n!}=\frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdots\frac{2}{n}\leq 2\left(\frac{2}{3}\right)^{n-2}.$$

4. For a given $M > 0$, verify that the value of N in parenthesis is the appropriate choice. (You should try to obtain this bound for N)

(a) $\lim_{n\to\infty}2^n=\infty$ $\left(N\geq\frac{\ln M}{\ln 2}\right)$ (b) $\lim_{n\to\infty}(-n^2)=-\infty$ $\left(N\geq\sqrt{M}\right)$

(c) $\lim_{n\to\infty}\sqrt{\ln n}=\infty$ $\left(N\geq e^{M^2}\right)$

5. Suppose $\lim_{n\to\infty}a_n=a$ and also $lim_{n\to\infty}a_n=b$, where $a\neq b$. Let $\epsilon=\frac{|a-b|}{2}$. The $\epsilon>0$ and hence there exist positive integers N_1 and N_2 such that $|a_n-a|\leq\frac{\epsilon}{3}$ for all $n\geq N_1$ and $|a_n-b|\leq\frac{\epsilon}{3}$ for all $n\geq N_2$. Let $N=\max\{N_1,N_2\}$ and choose $n>N$. Then

$$\epsilon=|a-b|=|a-a_n+a_n-b|\leq|a-a_n|+|a_n-b|\leq\frac{\epsilon}{3}+\frac{\epsilon}{3}=\frac{2}{3}\epsilon.$$

(Verify each of the above inequality.) But this implies that $1\leq 2/3$, which is absurd. Therefore $a=b$ and hence a sequences cannot have more than one limit.

6. Suppose $\lim_{n\to\infty}a_n=a$ and $b<a$. Let $\epsilon=\frac{a-b}{2}$. Then $\epsilon>0$ and so there exist a positive integer N such that $|a_n-a|\leq\epsilon$ for all $n\geq N$. For all $n\geq N$, we then conclude that $-\epsilon\leq a_n-a\leq\epsilon$. The first inequality implies that for all $n\geq N$, $a_n\geq a-\epsilon=a-(a-b)/2=(a+b)/2>b$. Hence there are infinitely many n for which $a_n>b$.

7. Suppose $\lim_{n\to\infty}a_n=a$ and $a_n\geq b$ for all n. If possible assume $a<b$. Let $\epsilon=\frac{b-a}{2}$. Then $\epsilon>0$ and so there exist a positive integer N such that $|a_n-a|\leq\epsilon$ for all $n\geq N$. For all $n\geq N$, we then conclude that $-\epsilon\leq a_n-a\leq\epsilon$. The second inequality implies that for all $n\geq N$, $a_n\leq a+\epsilon=a+(b-a)/2=(a+b)/2<b$. Hence there are infinitely many n for which $a_n<b$. But this contradict the assumption that $a_n\geq b$ for *all* n. Therefore $a\geq b$.

8. For a_n and b_n given below, verify that all conditions hold. Also find your own.
(a) $a_n=\frac{1}{n^2}$ $b_n=n.$ (b) $a_n=\frac{1}{n}$ $b_n=n^2$ (c) $a_n=\frac{1}{n}$ $b_n=n+2$ (d) $a_n=\frac{(-1)^n}{n}$ $b_n=n$

Section 2.2

2. Note first that $\lim_{n\to\infty} a_n = 0$ iff $lim_{n\to\infty}|a_n| = 0$. Now verify the following inequality for each given sequence and apply the Squeezing Theorem.

(a) $|a_n| \le e^{-n}$ (b) $|a_n| \le \left|\sin\left(\frac{1}{n}\right)\right|$ (c) $|a_n| \le \frac{1}{\ln n}$

6. First show that if $\lim_{n\to\infty} a_n = a$ and k is a positive integer then $lim_{n\to\infty}(a_n)^k = a^k$. Then let $P(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ be any polynomial. Now use repeated application of the sum rule for limits, to show that

$$\lim_{n\to\infty} P(a_n) = \lim_{n\to\infty} b_k (a_n)^k + \lim_{n\to\infty} b_{k-1}(a_n)^{k-1} + \cdots + \lim_{n\to\infty} b_1 (a_n) + b_0 = P(a).$$

7. Suppose $\lim_{n\to\infty} a_n = \infty$ and $a_n \le b_n$. Let $M > 0$ be given. Then there exists N such that $a_n \ge M$ for all $n \ge N$. But then for all $n \ge N$ we have $b_n \ge a_n \ge M$ and hence $\lim_{n\to\infty} b_n = \infty$.

8. Note that

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Hence $0 \le a_n \le \frac{1}{2\sqrt{n}}$. Now use the squeeze theorem to show that $\lim_{n\to\infty} a_n = 0$.

Section 2.4

4. If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, the they are convergent. Hence $\{x_n - y_n\}$ also converges. Since all convergent sequences are Cauchy, we conclude that $\{x_n - y_n\}$ is Cauchy. (Can you give a direct proof without using a theorem?)

5. We argue the same way as in #4. If $\{a_n\}$ is Cauchy it is convergent and hence, by the product rule for limit of sequences, $\{a_n^2\}$ converges. Thus it is a Cauchy sequence. The converse is not true. Here is an example you should verify: $a_n = (-1)^n$.

6. Suppose $\lim_{n\to\infty} b_n = 0$ and $|a_m - a_n| \le b_m$ for all $m \ge n$. We need to show $\{a_n\}$ is Cauchy. Let $\epsilon > 0$ be given. Then there exists N such that $|b_n| \le \epsilon$ for all $n \ge N$. But then for any $m \ge n \ge N$, we have $|a_m - a_n| \le b_n \le \epsilon$. Hence $\{a_n\}$ is Cauchy.

6. Suppose $|a_{n+1} - a_n| \le 2^{-n}$ for all n. Note then that(Explain each step)

$$|a_{n+2} - a_n| = |a_{n+2} - a_{n+1} + a_{n+1} - a_n| \le |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n| \le 2^{-(n+1)} + 2^{-n} = \left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^n$$

Repeating this yields

$$|a_m - a_n| \le \left(\frac{1}{2}\right)^{m-1} + \left(\frac{1}{2}\right)^{m-2} + \cdot + \left(\frac{1}{2}\right)^n.$$

For example if we repeat this we get

$$|a_{10} - a_6| \le |a_{10} - a_9| + |a_9 - a_8| + |a_8 - a_7| + |a_7 - a_6| \le \left(\frac{1}{2}\right)^9 + \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^6.$$

But then we have (Explain each step.)

$$\left(\frac{1}{2}\right)^{m-1} + \left(\frac{1}{2}\right)^{m-2} + \cdot + \left(\frac{1}{2}\right)^n = \sum_{k=n}^{m-1} \left(\frac{1}{2}\right)^k \le \sum_{k=n}^{\infty} \left(\frac{1}{2}\right)^k = \frac{\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = \left(\frac{1}{2}\right)^{n-1}.$$

With $b_n = \left(\frac{1}{2}\right)^{n-1}$,we get $|a_m - a_n| \le b_m$ for all $m \ge n$. Now apply the #6.