Real Analysis I: Exercises for Chapter Two.

1. Use the definition of the limit to establish the following limits.

a) $\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right) = 0.$

b) $\lim_{n\to\infty}\left(\frac{n^2-3}{2n^2+1}\right) = \frac{1}{2}.$

c) $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+10}\right) = 0.$

d) $\lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0.$

2. Show that a) $\lim_{n\to\infty}\left(\frac{n^2}{n!}\right) = 0.$ b) $\lim_{n\to\infty}\left(\frac{2^n}{n!}\right) = 0.$

3. If $\{b_n\}$ is a bounded sequence and $\lim_{n\to\infty} a_n = 0$, then show that $\lim_{n\to\infty}(a_n b_n) = 0$. Explain why the product rule for limits cannot be used.

4. Let $a_n = \sqrt{n+1} - \sqrt{n}$. Show that $\{a_n\}$ $\{\sqrt{n}a_n\}$ both converge and find their limits.

5. Suppose $0 < a < b$. (a) Determine $\lim_{n\to\infty}\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$. b) $\lim_{n\to\infty}(a^n+b^n)^{1/n} = b$.

6. Let $\{a_\}$ be a sequence of positive numbers such that $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = r$ exists. If $r < 1$, then show that $\lim_{n\to\infty} a_n = 0$. (Hint: Choose b so that $r < b < 1$. [Give an example of such a number b]. Let $\epsilon = b - r$. Then argue that there exists a number N so that $a_n < a_N b^{n-N}$ for all $n > N$. Show that $\lim_{n\to\infty} a_N b^{n-N} = 0$ and use the Squeeze Theorem.)

7. Let $0 < a < 1$ and $b > 1$. Use the result of Problem 6 to find the following limits.

a) $\lim_{n\to\infty}(a^n)$.

b) $\lim_{n\to\infty}\left(\frac{b^n}{2^n}\right)$.

c) $\lim_{n\to\infty}\left(\frac{n}{b^n}\right)$.

d) $\lim_{n\to\infty}\left(\frac{2^{3n}}{3^{2n}}\right)$.

8. Let $a_1 = 4$ and $a_{n+1} = \frac{1}{2}a_n + 2$. Prove that $\{a_n\}$ is a bounded and monotone. Find the limit.

9. Let $a_1 = \sqrt{p}$, where $p > 0$. Define $a_{n+1} = \sqrt{p + a_n}$. Prove that $|a_n| \le 1 + 2\sqrt{p}$. Also show that $\{a_n\}$ is monotone. Find the limit.

10. Give an example of an unbounded sequence that has a convergent subsequence.

11. Show that the following sequences are divergent. a) $\{1 - (-1)^n + 1/n\}$. b) $\{\cos\left(\frac{n\pi}{4}\right)\}$.

12. Let $S = [0, \infty)$. Find three lower bounds of S. Show that $\inf S = 0$ and that S has no upper bound.

13. Let $T = (0, \infty)$. Does $\inf T$ exist? Does T have an upper bound? Does $\sup T$ exist? Prove your statements.

14. Let $S = \left\{\frac{n}{n+1} \mid n \text{ is a positive integer}\right\}$. Show in detail that $\inf S = 1/2$ and that $\sup S = 1$.

15. Suppose a is a lower bound of a set S. If there is a sequence $\{s_n\}$ of elements of S that converges to a, then show that a is the greatest lower bound of S.

16. Let $a_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ Does $\lim_{n\to\infty} a_n$ exist? Prove your assertion.

17. Let $a_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$ Does $\lim_{n\to\infty} a_n$ exist? Prove your assertion.

18. Let $a_n \geq 0$ for all n and suppose $\lim_{n\to\infty} a_n = a$. Use the definition of limit to prove that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

19. Give example of two sequences that do not converge but for which

(a) the sum converges. (b) the product converges. (c) the quotient converges.

(Note: The sequences do not have to be the same sequences for all cases.)

20. Give an example of sequences $\{a_n\}$ and $\{b_n\}$ for which $\{a_n\}$ converges, $\{b_n\}$ is bounded but $\{a_n b_n\}$ does not converge.

21. Let $a_1 = 1$ and for each $n \geq 1$, let $x_{n+1} = \frac{1}{4}(2x_n + 3)$. Show that $\{x_n\}$ is convergent and find its limit. (Hint: Show that $\{x_n\}$ is bounded and increasing.)

22. Let $s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$.

a) Show that $\{s_n\}$ is not Cauchy. b) Show that $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$.

23. Suppose $|a_{n+2} - a_{n+1}| \leq r|a_{n+1} - a_n|$ where $0 < r < 1$. Show that $\{a_n\}$ is a Cauchy sequence. (Hint: Show that $|a_{n+1} - a_n| \leq r^{n-1}|a_2 - a_1|$.)

24. For each of the following sets find $\inf S$ and $\sup S$. Make sure to justify your answers.

(a) $S = \{x \in \mathbf{R} \mid x^2 < 2\}$ (b) $S = \{x \in \mathbf{R} \mid x > 0 \text{ and } x^2 > 2\}$

(c) $S = \{x \in \mathbf{R} \mid x^2 > 2\}$ (d) $S = \{n - \frac{1}{n} \mid n \in \mathbf{N}\}$

25. If A and B are nonempty subsets of $\mathbf{R}$ such that $A \subseteq B$, then show that $\inf B \leq \inf A \leq \sup A \leq \sup B$.

26. Show that a nonempty subset finite set contains both its maximal and minimal elements.