Real Analysis I Some Examples

1. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: We will show that (i) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and (ii) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. To prove (i), let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. (Why?) Thus $x \in A$ and $x \in B$ or $x \in C$.(Why?) Hence $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. In other words, we have $x \in A \cap B$ or $x \in A \cap C$. Therefore, $x \in (A \cap B) \cup (A \cap C)$, that is $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
Please prove (ii) in a similar manner. Also show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. Prove or give a counter example.

a) $A - B = B - A$.

This is false. Here is one counter example. Let $A = \{1, 3, a\}$ and $B = \{1, 2, 3\}$. Then $A - B = \{a\}$ while $B - A = \{2\}$. (Can you find another counter example)

b) $(A \cup B) - C = A \cap (B - C)$.

This is false. Let A and B be as above and $C = \Phi$=The empty set. Then $(A \cup B) - C = A \cup B = \{1, 2, 3, a\}$ while $A \cap (B - C) = A \cap B = \{3\}$. Can you give different example in which C is nonempty.

c) $(A \cup B) - A = B$.

This is also false. For a counter example let A and B be as in (a) above. Explain why the statement is false.

d) If $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

This is true and here is why. Assume $A \subset C$ and $B \subset C$. Let $x \in (A \cup B)$. Then $x \in A$ or $x \in B$. If $x \in A$ then $x \in C$.(Why?) If $x \notin A$ then $x \in B$. But then $x \in C$. Thus in either cases, we see that $x \in C$. Therefore, whenever $x \in (A \cup B)$, then $x \in C$ and hence $A \cup B \subset C$.

e) $A \subset B$ if and only if $A \cup B = B$.

This is true. Note that there are two things we must show:(i) Assuming $A \subset B$, we must show $A \cup B = B$ and (ii) assuming $A \cup B = B$, we must show $A \subset B$.
Let us prove (i). Assume $A \subset B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then since $A \subset B$, we must $x \in B$. If $x \notin A$, then by the definition of union, $x \in B$. In either cases, we have $x \in B$. Thus $A \cup B \subset B$. On the other hand, if $x \in B$, then $x \in A \cup B$. (This is true whether A is a subset of B or not!) Hence $B \subset A \cup B$. Consequently, we have $A \cup B = B$.

To prove (ii), assume $A \cup B = B$. We want to show $A \subset B$. We will use proof by contradiction. (You should give direct proof!) If $A \not\subset B$, then there an element $x \in A$ but $x \notin B$. Since $x \in A$, we must have $x \in A \cup B$. Thus this element x belongs to $A \cup B$ but does not belong to B. Therefore, $A \cup B$ and B cannot be equal. This contradicts the assumption.

3. Prove that $S = \{1/4, 1/8, 1/12, 1/16, \cdots\}$ is countable by exhibiting a bijection from S on to $\mathbf{N}$.

Solution: Define $f : \mathbf{N} \to \mathbf{S}$ by $f(n) = 1/4n$. By definition of S f is onto. To show that f is one-to-one, suppose $f(n) = f(m)$. Then $1/4n = 1/4m$ and hence $m = n$. Thus f is one-to-one. Therefore f is both one-to-one and onto. By definition of countability, we see that S is countable.

4. Use the Principle of Mathematical Induction to show that $\sum_{k=1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}$ for all $n \in \mathbf{N}$.

Proof: *Step 1.* Let $n = 1$. Then the left hand side is $\sum_{k=1}^{1}(2k-1)^2 = 1^2 = 1$ and the right hand side is $(4n^3 - n)/3 = (41^2 - 1)/3 = 1$. Thus the statement is true for $n = 1$.
Step 2. Assume $\sum_{k=1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}$ is true for some $n \geq 1$. We need to show that $\sum_{k=1}^{n+1}(2k-1)^2 = \frac{4(n+1)^3-(n+1)}{3}$. We begin with the left hand side and use the assumption: (Make sure that you fill in the reason(s) for each equality.)

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1)^2 &= \sum_{k=1}^{n}(2k-1)^2 + (2(n+1)-1)^2 \qquad \text{(Why?)}\\ &= \frac{4n^3-n}{3} + (2n+1)^2 \qquad \text{(Why?)}\\ &= \frac{4n^3-n}{3} + 4n^2 + 4n + 1 = \frac{4n^3+12n^2+11n+3}{3} \qquad \text{(Why?)}\end{aligned}$$

On the other hand,

$$\frac{4(n+1)^3-(n+1)}{3} = \frac{4(n^3+3n^2+3n+1)-n-1}{3} = \frac{4(n^3+12n^2+11n+3}{3}.$$

Therefore, $\sum_{k=1}^{n+1}(2k-1)^2 = \frac{4(n+1)^3-(n+1)}{3}$ is true whenever $\sum_{k=1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}$. By the Principle of Mathematical Induction, the formula holds for all positive integers n.

5. Assume that there is a rational number between any two given real numbers. Use this assumption to prove that any nonempty open interval (a, b) contains an infinite number of rational numbers.

Proof: By assumption there is a rational number, call it y, between a and b. Note that $a < y$. But then a and y are real numbers and the assumption we are making guarantees the existence of a rational number, say x_0, between a and y. Now x_0 and y are rational numbers between a and b. Then the midpoint of x_0 and y, call it x_1 is a rational point between a and b. Clearly $x_0 \neq x_1$. Let x_2 be the midpoint of x_0 and x_2, x_3 be the midpoint of x_1 and x_2, and so on. Thus x_n is the midpoint of x_{n-1} and x_{n-2} for each $n > 2$. The set $\{x_1, x_2, x_3, \cdots\}$ is an infinite set of rational numbers between a and b. (Can you rewrite this proof to make it shorter?)

6. If a, b, c, d are positive real numbers such that $a/b < c/d$, then show that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{b}{d}.$$

Proof: Since b and d are positive, we can multiply $a/b < c/d$ to obtain $ad - bc < 0$ and hence $bc - ad > 0$. (Can you think of which properties we have used? But then we have

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc-ad}{b(b+d)} > 0. \quad \text{(Here we used algebra!)}$$

Thus $\frac{a}{b} < \frac{a+c}{b+d}$. Prove the other inequality.