Real Analysis I: Problems for Chapter 3.

1. Prove or find a counterexample to the following statements. In all cases we assume that f and g are functions defined on the indicated intervals.

(a) If f is bounded on $[a, b]$, then f is continuous on $[a, b]$.
(b) If f is continuous on$[a, b]$, then f is bounded on $[a, b]$.
(c) If f is continuous on(a, b), then f is bounded on (a, b).
(d) If $(f(x))^2$ is continuous on(a, b), then f is continuous on (a, b).
(e) If f and g are not continuous on(a, b), then fg is not continuous on (a, b).
(f) If f and g are not continuous on(a, b), then $f + g$ is not continuous on (a, b).
(g) If f and g are not continuous on(a, b), then $f \circ g$ is not continuous on (a, b).
(h) If $f + g$ and g are continuous on(a, b), then f is continuous on (a, b).
(i) If fg and g are continuous on(a, b), then f is continuous on (a, b).
(j) If $|f(x)|$ is continuous on(a, b), then f is continuous on (a, b).

2. Determine whether or not the given functions are uniformly continuous.

(a) $f(x) = \frac{1}{x}$ with $x \in (0, 1)$
(b) $f(x) = x^3$ with $x \in [0, 2)$.
(c) $f(x) = \frac{x}{x+4}$ with $x \in [0, 2)$.

3. Let $f(x) = (x^2 + x - 6)/(x - 2)$. Can f be defined at $x = 2$ so that f is continuous everywhere?

4. Let $f(x)=\begin{cases} 2x, & \text{if } x \in \mathbf{Q} \\ x+3, & \text{if } x \notin \mathbf{Q} \end{cases}$
Find all points at which f is continuous.

5. Let f be a continuous function on $\mathbf{R}$ and let $S = \{x \in \mathbf{R} \mid f(x) = 0\}$. If $\{x_n\}$ is a sequence of points in S and if $x_n \to x$, then show that $x \in S$.

6. Let f and g be continuous functions on $\mathbf{R}$ and let $S = \{x \in \mathbf{R} \mid f(x) \geq g(x)\}$. If $\{x_n\}$ is a sequence of points in S and if $x_n \to x$, then show that $x \in S$.

7. Let f and g be continuous functions on $\mathbf{R}$. If $f(x) = g(x)$ for all rational numbers x, is it true that $f(x) = g(x)$ for all real numbers x? Explain.

8. Use Intermediate Value Theorem to show that the polynomial $p(x) = x^4 + 7x^3 - 9$ has two real root. Then use your calculator to find the roots.

9. Let $f(x) = x^2$ with $x \in [0, \infty)$. Find a positive number ϵ and two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty}(x_n - y_n) = 0$ but $|f(x_n) - f(y_n)| \geq \epsilon$. Then conclude that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$.

10. Let $f(x) = \sqrt{x}$ for $x \in [0, \infty)$. Show that f is uniformly continuous but f is not Lipschitz continuous on $[0, \infty)$.

11. For each of the following functions, find $L_P(f)$ and $U_P(f)$, where $P = \{x_0,\ x_1,\ x_2,\ \cdots,\ x_N\}$ is the partition of $[a, b]$ such that each subinterval has length equal to $h = \frac{b-a}{N}$.

a) $f(x) = x^2 + 2x + 1$ on $[0, 1]$ with $N = 4$, $N = 8$, and N is any integer.
b) $f(x) = x^2 - 2x + 1$ on $[0, 2]$ with $N = 4$, $N = 8$, and N is any integer.

12. Let $f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 2, & \text{if } 1 < x \le 2. \end{cases}$

a) Find a partition P of $[0, 2]$ such that $U_P(f) - L_P(f) \le \frac{1}{10}$.
b) For any $\epsilon > 0$, show that there exists a partition P such that $U_P(f) - L_P(f) \le \epsilon$.
c) Use the definition to find $\int_0^2 f(x)\,dx$.

13. Use upper and lower sums to evaluate the following integrals.

a) $\int_0^1 3x^2\,dx$ b) $\int_0^2 (x^2 - 2x + 1)\,dx$

14. Suppose $f(x) \ge 0$ for all $x \in [a, b]$. If f is bounded, show that $L_P(f) \ge 0$ for all partitions P of $[a, b]$.

15. Prove or give a counter example: If $f(x) \le g(x) \le h(x)$ for all x and if f and h are Riemann integrable on $[a, b]$, then so is g.

16. Give an example of a function with domain $[0, 1]$, such that

a) f is bounded but not Riemann integrable.
b) f is Riemann integrable but not monotone.
c) f is Riemann integrable but not continuous.

17. Suppose f is bounded on $[a, b]$ and $f(x) = 0$ for all x except at finitely many values in $[a, b]$, prove that $\int_a^b f(x)\,dx = 0$.

18. If f is Riemann integrable on $[a, b]$, then prove that $|f|$ is also Riemann integrable on $[a, b]$. Is the converse of this statement true? Prove or give a counter example.

19. (Refer to problems #3 and #7 page 102.) Define

$$T_N = \frac{h}{2}\Big(f(x_0)\ +\ 2f(x_1)\ +\ 2f(x_2)\ +\ \cdots\ +\ 2f(x_{N-1})\ +\ f(x_N)\Big)$$

and for even N

$$S_N = \frac{h}{6}\sum_{i=1}^{N}\Big(f(x_{i-1})\ +\ 4f(\bar{x}_i)\ +\ f(x_i)\Big),$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ for each i.

a) For $f(x) = x^2$ on $[0, 1]$, find M_N, T_N, and S_N for $N = 4$ and $N = 8$.
b) For $f(x) = 1/x$ on $[1, 2]$, find M_N, T_N, and S_N for $N = 4$ and $N = 8$.

20. DO problem #9 on page 103.