1. Equation of motion in rectangular coordinates. For Newtonian fluids for constant ρ and μ for the x component, y component, and z component we obtain, respectively,

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) - \frac{\partial p}{\partial x} + \rho g_x \qquad \text{(C.15)} \quad \textbf{(3.7-36)}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) - \frac{\partial p}{\partial y} + \rho g_y \qquad \text{(C.16)} \quad \textbf{(3.7-37)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - \frac{\partial p}{\partial z} + \rho g_z \qquad \text{(C.17)} \quad \textbf{(3.7-38)}$$

Combining the three equations for the three components, we obtain

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{v} \qquad \textbf{(3.7-39)}$$

2. Equation of motion in cylindrical coordinates. These equations are as follows for Newtonian fluids for constant ρ and μ for the r, θ, and z components, respectively.

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \qquad \text{(C.18)} \quad \textbf{(3.7-40)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \qquad \text{(C.19)} \quad \textbf{(3.7-41)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z}$$
$$+ \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad \text{(C.20)} \quad \textbf{(3.7-42)}$$

3. Equation of motion in spherical coordinates. The equations for Newtonian fluids are given below for constant ρ and μ for the r, θ, and ϕ components, respectively.

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left(\nabla^2 v_r - \frac{2}{r^2} v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2} v_\theta \cot\theta - \frac{2}{r^2 \sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_r \qquad \textbf{(3.7-43)}$$

Using the relations from Eq. (3.6-26) with Eq. (3.6-20), the equation of continuity in cylindrical coordinates is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta}+\frac{\partial(\rho v_z)}{\partial z}=0 \tag{3.6-27}$$

C.13

2. *Equation of motion in cylindrical coordinates.* These equations are as follows for Newtonian fluids for constant ρ and μ for the r, θ, and z components, respectively.

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}$$
$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho g_r \tag{3.7-40}$$

C.18

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}$$
$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]+\rho g_\theta \tag{3.7-41}$$

C.19

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z \tag{3.7-42}$$

C.20

2. *Shear-stress components for Newtonian fluids in cylindrical coordinates*

$$\tau_{rr}=-\mu\left[2\frac{\partial v_r}{\partial r}-\frac{2}{3}(\nabla\cdot\mathbf{v})\right] \tag{3.7-21}$$

$$\tau_{\theta\theta}=-\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)-\frac{2}{3}(\nabla\cdot\mathbf{v})\right] \tag{3.7-22}$$

$$\tau_{zz}=-\mu\left[2\left(\frac{\partial v_z}{\partial z}-\frac{2}{3}(\nabla\cdot\mathbf{v})\right]\right. \tag{3.7-23}$$

$$\tau_{r\theta}=\tau_{\theta r}=-\mu\left[r\frac{\partial(v_\theta/r)}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right] \tag{3.7-24}$$

$$\tau_{\theta z}=\tau_{z\theta}=-\mu\left[\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial\theta}\right] \tag{3.7-25}$$

$$\tau_{zr}=\tau_{rz}=-\mu\left[\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right] \tag{3.7-26}$$

Corrected

$$(\nabla\cdot\mathbf{v})=\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z} \tag{3.7-27}$$

Boundary conditions - summary

1). No slip at a solid/liquid interface

D >> molecular dimensions

Both Newtonian & non-Newtonian

Polymer melts (except under extreme processing conditions)

2) fluid - fluid interface (air/water or fluid1/fluid2)

v & τ must be continuous - eg. (must be equal)

τ — tangential

(Normal stress may have a surface tension term required)

3) finiteness of v or τ

symmetry resulting from geometry

Procedure for solutions of flow Problems

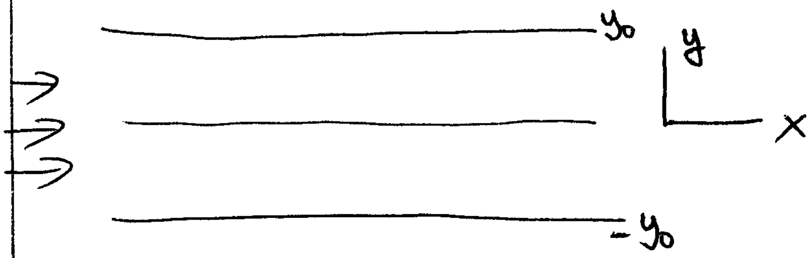
1). Examine fluid flow within given geometry. Determine the relation between v and coordinates (eg v_z vs r, θ, z). The result of this step is a hypothesis which must be checked at each subsequent step

2). Apply step one assumptions into continuity equation (C.E.)
- in some cases additional information is obtained

3) Substitute 1 & 2 results into momentum or Navier Stokes equations, as appropriate and solve.

a. integrate
b. substitute B.C.'s
c. check answer.

flow between 2 plates



assume: ① far from edges

② Pressure driven

③ S.S.

① Continuity

$$0 = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad v_y = v_z = 0$$

$$\frac{\partial v_x}{\partial x} = 0 \quad \text{so} \quad v_x \text{ is constant in } x\text{-direction}$$

② Navier Stokes for x-direction

$$\rho\left(\underbrace{\cancel{\frac{\partial v_x}{\partial t}}}_{\text{S.S.} \to 0} + v_x \underbrace{\cancel{\frac{\partial v_x}{\partial x}}}_{\text{continuity}} + \underbrace{\cancel{v_y}}_{0}\frac{\partial v_x}{\partial y} + \underbrace{\cancel{v_z}}_{0}\frac{\partial v_x}{\partial z}\right) = \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right) - \frac{\partial P_x}{\partial x} + \rho g_x$$

pressure

Since $\frac{\partial v_x}{\partial x} = 0$ $\quad \frac{\partial v_x}{\partial z} = 0$ so

~~constant~~

$$0 = \mu\left(\frac{\partial^2 v_x}{\partial y^2}\right) - \frac{dP_x}{dx} + \rho \cancel{g_x}$$

$g_x = 0$ for this problem horizontal

for vertical $\mathcal{P} = p + \rho g y$

if vertical

$$\frac{\partial P_x}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

P is not a function of z

if $2y_0$ is small the $P \neq f(y)$

then $\frac{dP_x}{dx}$ = constant

B.C. symmetry $\frac{\partial v_x}{\partial y} = 0$ at $y = 0$ (center)

$v_x = 0$ at $y = y_0$ no slip

$$v_x = \frac{1}{2\mu} \frac{dP}{dx} \left(y^2 - y_0^2 \right)$$

Using the relations from Eq. (3.6-26) with Eq. (3.6-20), the equation of continuity in cylindrical coordinates is

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta}+\frac{\partial(\rho v_z)}{\partial z}=0 \qquad \textbf{(3.6-27)} \qquad \boxed{\text{C.13}}$$

2. *Equation of motion in cylindrical coordinates.* These equations are as follows for Newtonian fluids for constant ρ and μ for the r, θ, and z components, respectively.

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}$$

$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho g_r \qquad \textbf{(3.7-40)} \qquad \boxed{\text{C.18}}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]+\rho g_\theta \qquad \textbf{(3.7-41)} \qquad \boxed{\text{C.19}}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z \qquad \textbf{(3.7-42)} \qquad \boxed{\text{C.20}}$$

2. *Shear-stress components for Newtonian fluids in cylindrical coordinates*

$$\tau_{rr}=-\mu\left[2\frac{\partial v_r}{\partial r}-\frac{2}{3}(\nabla\cdot\mathbf{v})\right] \qquad \textbf{(3.7-21)}$$

$$\tau_{\theta\theta}=-\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)-\frac{2}{3}(\nabla\cdot\mathbf{v})\right] \qquad \textbf{(3.7-22)}$$

$$\tau_{zz}=-\mu\left[2\left(\frac{\partial v_z}{\partial z}-\frac{2}{3}(\nabla\cdot\mathbf{v})\right]\right. \qquad \textbf{(3.7-23)}$$

$$\tau_{r\theta}=\tau_{\theta r}=-\mu\left[r\frac{\partial(v_\theta/r)}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \qquad \textbf{(3.7-24)}$$

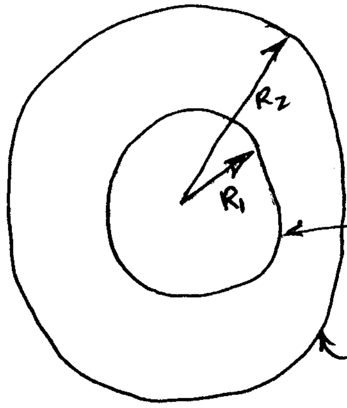
$$\tau_{\theta z}=\tau_{z\theta}=-\mu\left[\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right] \qquad \textbf{(3.7-25)}$$

$$\tau_{zr}=\tau_{rz}=-\mu\left[\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right] \qquad \textbf{(3.7-26)} \qquad \boxed{\text{Corrected}}$$

$$(\nabla\cdot\mathbf{v})=\frac{1}{r}\frac{\partial(r v_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z} \qquad \textbf{(3.7-27)}$$

Interesting Problems – rotational flows

Example 3.8.5



inner is fixed

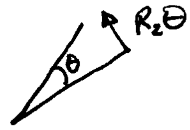
outer
Rotates with velocity ω

angular velocity
Rad/s

ρ = constant
μ = " "

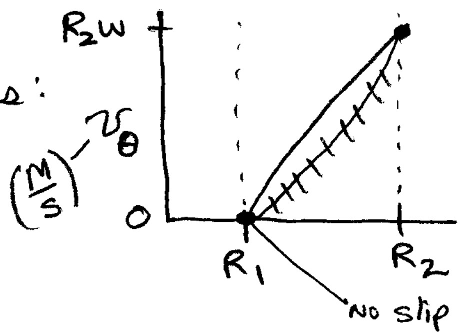
find velocity profile
Stress profile

distance $R_2\theta$ for angular velocity

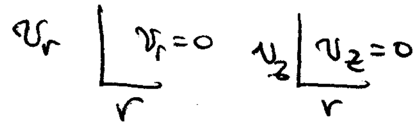


$v_\theta = R_2\omega$
$= R_2 \dfrac{d\theta}{dt}$

Sketch profiles:



This gives 2 B.C.'s



No pressure gradient

Equation of continuity (mass bal) — equation deNevers C.13 p622

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{SS} = \frac{1}{r}\underbrace{\cancel{\frac{\partial (\rho r v_r)}{\partial r}}}_{0} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \overbrace{\cancel{\frac{\partial (\rho v_z)}{\partial z}}}^{0} \qquad (3.6-27)$$

$$\frac{\partial}{\partial \theta}(\rho v_\theta) = 0 \qquad \text{or} \quad v_\theta \neq f(\theta)$$

cylindrical

r component: 3.7-40

SS; $v_r=0$; $v_r=0$; v_z & $v_r=0$

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r\cancel{\frac{\partial v_r}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta^2}{r} + v_z\cancel{\frac{\partial v_r}{\partial z}}\right) =$$

$$-\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cancel{\frac{\partial (r v_r)}{\partial r}}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right]$$

$v_r=0$; $v_r=0$; from continuity $=0$; $v_r=0$

$$+ \rho \cancel{g_r}$$

0

r component:

$$-\rho\frac{v_\theta^2}{r} = -\frac{\partial P}{\partial r} \qquad \text{Centrifugal force} \qquad 3.8\text{-}25$$

θ component 3.7-41

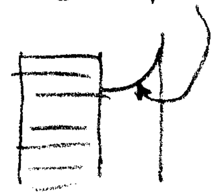
$$0 = \frac{d}{dr}\left(\frac{1}{r}\frac{d(r v_\theta)}{dr}\right) \qquad 3.8\text{-}26$$

z component 3.7-42

$$0 = -\frac{\partial P}{\partial z} + \rho g_z \qquad 3.8\text{-}27$$

gravity is g in z direction or opposite

Combining these 2 eqns gives shape of liq/gas interface BSL p94



$$\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} = \text{constant} = C_1$$

$$\int \partial (r v_\theta) = \int C_1 r \, dr$$

$$r v_\theta = \frac{C_1 r^2}{2} + C_2$$

Geankoplis

$$v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r} = C_1' r + \frac{C_2}{r}$$

for $r = R_1$ $\quad v_\theta = 0$

$r = R_2$ $\quad v_\theta = \omega R_2$

$$v_\theta = c_1 r + \frac{c_2}{r}$$

$$0 = C_1 R_1 + \frac{C_2}{R_1}$$

$$\omega R_2 = c_1 R_2 + \frac{C_2}{R_2} =$$

$$C_1 = -\frac{C_2}{R_1^2} = -\frac{C_2}{R_1^2}$$

$$\omega R_2 = -\frac{C_2}{R_1^2} R_2 + \frac{C_2}{R_2}$$

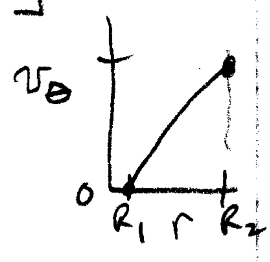
$$C_2 = \frac{\omega R_2}{-\frac{R_2}{R_1^2} + \frac{1}{R_2}}$$

$$C_1 = -\frac{\omega R_2}{R_1^2\left(-\frac{R_2}{R_1^2} + \frac{1}{R_2}\right)} = \frac{-\omega R_2}{-R_2 + \frac{R_1^2}{R_2}}$$

$$C_1 = \frac{-\omega}{\left(\frac{R_1}{R_2}\right)^2 - 1}$$

$$v_\theta =$$

$$\frac{v_\theta}{\omega R_2} = \frac{\cancel{\omega} R_1 R_2^{\cancel{2}}}{R_1^2 - R_2^2}\left[\frac{R_1}{r} - \frac{r}{R_1}\right]$$



$$v_\theta = \frac{\omega}{\frac{(R_1^2 - R_2^2)}{(R_1 R_2^2)}}\left[\frac{R_1}{r} - \frac{r}{R_1}\right]$$

3.8-29 !

check limits

$r = R_1$ $\quad (1-1)$ $\quad v_\theta = 0$

$r = R_2$

$$\left(\frac{\omega R_1 R_2^2}{R_1^2 - R_2^2}\right)\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right) = \omega\left[\frac{R_1^2 R_2}{R_1^2 - R_2^2} - \frac{R_2^2 R_2}{R_1^2 - R_2^2}\right] = \omega R_2 \left[\cancel{\frac{R_1^2 - R_2^2}{R_1^2 - R_2^2}}\right]^{1}$$

To find $\tau_{r\theta}$

$$\tau_{r\theta} = -\mu \left[r \frac{\partial (v_\theta / r)}{\partial r} + \frac{1}{r} \cancel{\frac{\partial v_r}{\partial \theta}} \right] \qquad \underline{3.7\text{-}24}$$

$v_r = 0$

$v_\theta = f(r)$ into this

$$\tau_{r\theta} = -2\mu\omega R_2^2 \left(\frac{1}{r^2}\right) \left[\frac{R_1^2/R_2^2}{1 - R_1^2/R_2^2} \right]$$

$R_1 = \frac{1}{2} R_2$

$\frac{R_1}{R_2} = 1/2$

at $r = R_2$ (outer) $\quad \tau \propto \dfrac{R_1^2/R_2^2}{1 - R_1^2/R_2^2} \quad \dfrac{1/4}{3/4} = \dfrac{1}{3}$

at $r = R_1$ (inner) $\quad \tau \propto \dfrac{1}{1 - R_1^2/R_2^2} = \dfrac{1}{1 - 1/4} = \dfrac{4}{3}$

· Torque to move cylinder $= (\text{Force})\, R_2$

$$\text{Force} = \tau_{r\theta}\Big|_{r=R_2} (2\pi R_2 H)$$

Examine Shape of surface

$$-\rho \frac{v_\theta^2}{r} = -\frac{\partial P}{\partial r}$$

$$\frac{\partial P}{dz} = \rho g_z$$

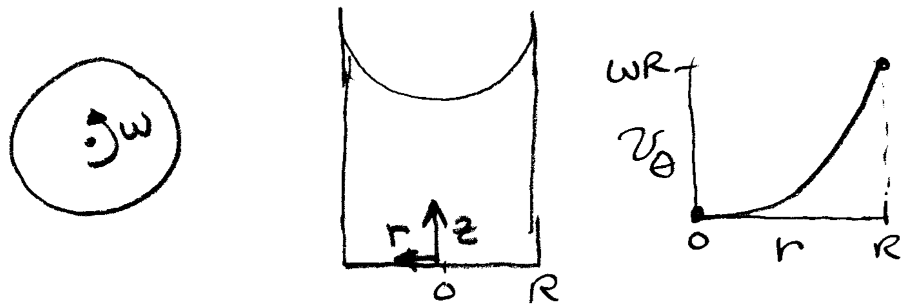
$$\left.\begin{aligned} \frac{\partial P}{\partial r} &= \frac{\rho}{r}\left[\frac{\omega R_1 R_2^2}{R_1^2 - R_2^2}\left[\frac{R_1}{r} - \frac{r}{R_1}\right]\right]^2 \\ &\text{and} \\ \frac{\partial P}{\partial z} &= -\rho g \end{aligned}\right\}$$

Pressure is a function of r & z

$$dP = \frac{\partial P}{\partial r} dr + \frac{\partial P}{\partial z} dz$$

plug in both equations and integrate for pressure profile

Rotating Liquid Cylinder



$v_z = 0 \quad v_r = 0$

Same equations result as in previous example

$$0 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right)$$

$$\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r} = c_1$$

$$\int \partial(r v_\theta) = \int c_1 r \, \partial r$$

$$r v_\theta = c_1 \frac{r^2}{2} + c_2$$

$$v_\theta = c_1 \frac{r}{2} + \frac{c_2}{r}$$

Evaluate constants

$r = 0 \quad v_\theta = ?$

$r = R \quad v_\theta = 0$

what happens at $r=0$ $\quad v_\theta = \infty$? to prevent this $c_2 = 0$

$$v_\theta = \omega R = \frac{c_1 R}{2} \qquad c_1 = 2\omega$$

$$v_\theta = \frac{2\omega r}{2} = \omega r$$

Now plug into $\rho \frac{v_\theta^2}{r} = \frac{dP}{dr}$ from r component centrifugal force

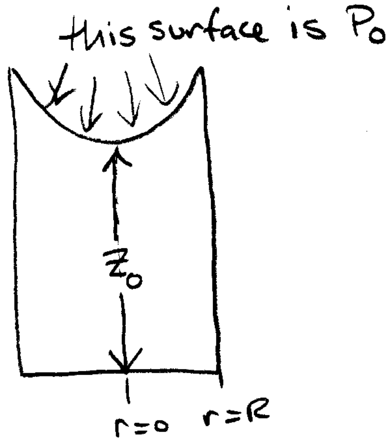
$$\frac{\partial P}{\partial r} = \frac{\rho}{r}(\omega r)^2 = \frac{\rho}{r}\omega^2 r^2 = \rho\omega^2 r$$

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial P}{\partial z}dz$$

$$\frac{dP}{dz} = -\rho g$$ since g is in opposite direction to z

$$dp = \rho\omega^2 r\,dr - \rho g\,dz$$

$$P = \rho\omega^2\frac{r^2}{2} - \rho g z + c_3$$



at $r=0$ & $z=z_o$ $P=P_o$

$$P_o = \rho\omega^2\frac{(0)^2}{2} - \rho g z_o + c_3$$

$$c_3 = P_o + \rho g z_o$$

$$P = \rho\omega^2\frac{r^2}{2} - \rho g z + \rho g z_o + P_o$$

$$(P - P_o) = \rho\omega^2\frac{r^2}{2} + \rho g(z_o - z)$$

to find shape of free surface $P = P_o$

$$z - z_o = \frac{\omega^2 r^2}{2g}$$ ← this is a parabola for $z > z_o$