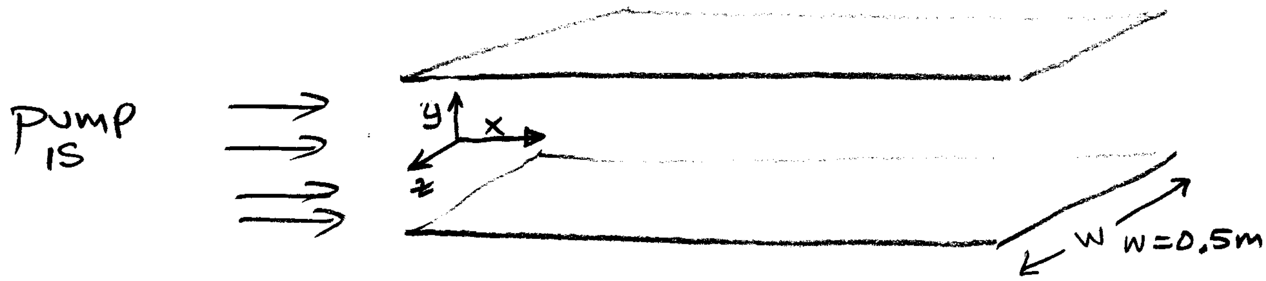
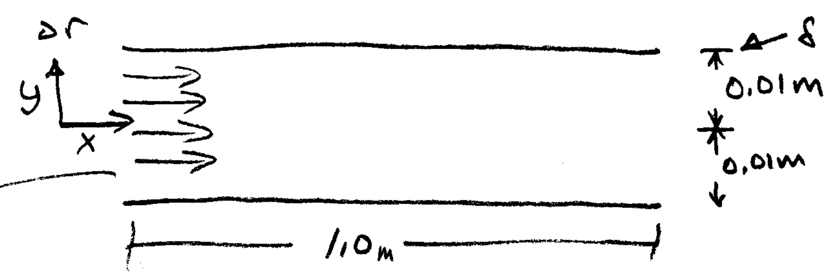
Flow between parallel Plates – Non-Newtonian fluid

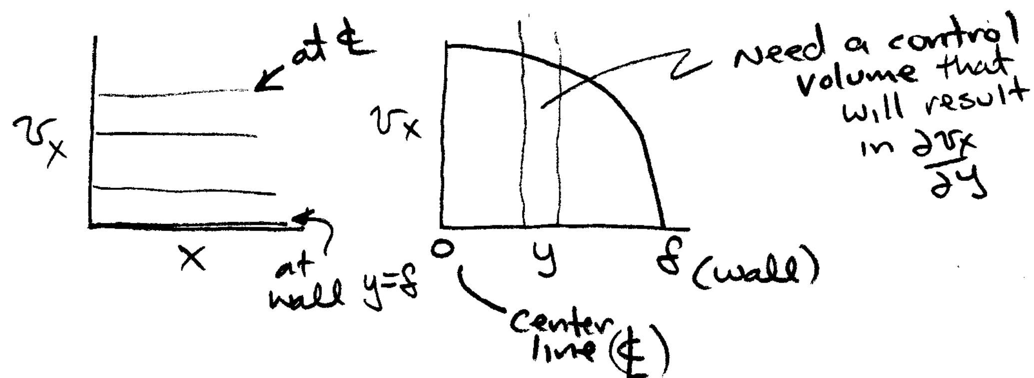


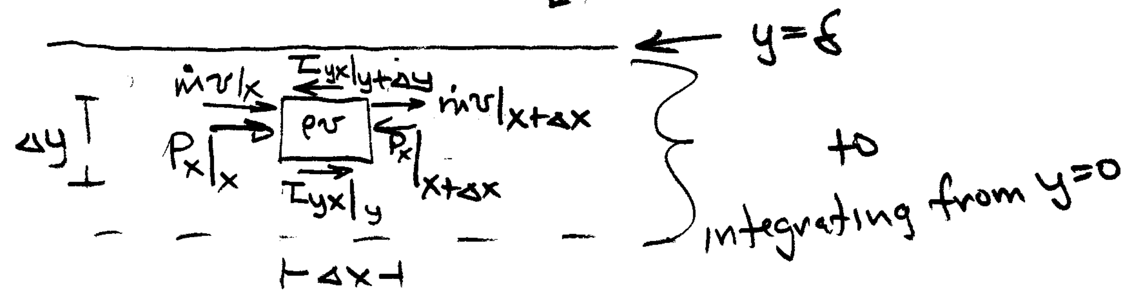


1) Assume flow is fully developed at x=0

velocity profile is constant with respect to x

2) Pump is pushing fluid





this is a constant so it can be outside the derivative

$$\Delta V \frac{\partial}{\partial t}(\rho v) = \dot{m}v|_x - \dot{m}v|_{x+\Delta x} + P_x \Delta y W|_x - P_x \Delta y W|_{x+\Delta x}$$

$$+ \tau_{yx} \Delta x W|_y - \tau_{yx} \Delta x W|_{y+\Delta y}$$

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divide by ΔV & rearrange & take limits
$\dot{m}v = \rho v v \Delta A$

$$\lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{\partial}{\partial t}(\rho v) = -\left(\frac{\rho v v \Delta y w|_x - \rho v v \Delta y w|_{x+\Delta x}}{\Delta y w\,(x+\Delta x - x)}\right) - \left(\frac{P_x \Delta y w|_{x+\Delta x} - P_x \Delta y w|_x}{\Delta y w\,(x+\Delta x - x)}\right)$$

$$-\left(\frac{\tau_{yx}\Delta x w|_{y+\Delta y} - \tau_{yx}\Delta x w|_y}{\Delta x w\,(\underbrace{y+\Delta y - y}_{\text{this matches}})}\right)$$

$$\frac{\partial}{\partial t}(\rho v) = -\frac{\partial(\rho v v)}{\partial x}\frac{\cancel{\Delta y w}}{\cancel{\Delta y w}} - \frac{\partial P_x}{\partial x}\frac{\cancel{\Delta y w}}{\cancel{\Delta y w}} - \frac{\partial(\tau_{yx}\cancel{\Delta x w})}{\cancel{\Delta x w}\,\partial y}$$

constant with respect to y

Based on a mass balance for a filled pipeline of constant ID.

$v|_x = v|_{x+\Delta x} \quad \partial v/\partial x = 0$

Steady state $\frac{\partial}{\partial t}(\rho v) = 0$

$$\frac{\partial \tau_{yx}}{\partial y} = -\frac{\partial P_x}{\partial x}$$

these are the same steps of deriving a momentum balance as for a Newtonian fluid

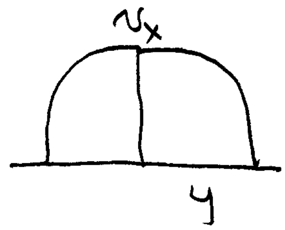
$$\int \partial \tau_{yx} = -\int \frac{\partial P_x}{\partial x}\,\partial y$$

$$\tau_{yx} = -\frac{\partial P_x}{\partial x} y + c_1 \qquad \text{at } y=0 \quad \tau_{yx}=0$$

or

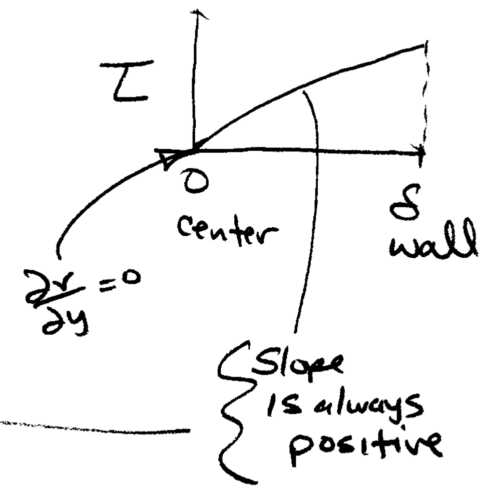
$\frac{\partial v}{\partial y} = 0$ = Symmetry condition

$0 = 0 + c_1$
$\therefore c_1 = 0$



New steps

$$\tau_{yx} = -\mu \left| \frac{\partial v_x}{\partial y} \right|^{n-1} \frac{\partial v_x}{\partial y}$$



So we do not need the | | sign

$$\tau_{yx} = \mu \left(-\frac{\partial v_x}{\partial y} \right)^n$$

$$\mu \left(\frac{\partial v_x}{\partial y} \right)^n = -\frac{dP_x}{dx} y$$

$$-\frac{\partial v_x}{\partial y} = \left(\frac{1}{\mu} \right)^{1/n} \left(-\frac{dP_x}{dx} \right)^{1/n} y^{1/n}$$

$$\int \partial v_x = -\left(\frac{1}{\mu}\left(-\frac{dP_x}{dx} \right) \right)^{1/n} \int y^{1/n} dy$$

$$v_x = -\left(\frac{1}{\mu}\left(-\frac{dP_x}{dx} \right) \right)^{1/n} \left[\frac{y^{1/n+1}}{1 + 1/n} \right] + c_2$$

$$v_x = -\left(\frac{1}{\mu}\left(-\frac{dP_x}{dx} \right) \right)^{1/n} \left[\frac{y^{(1+n)/n}}{\frac{(1+n)}{n}} \right] + c_2$$

$v_x = 0$ at $y = \delta$

$$\left(\frac{1}{\mu} \frac{dP}{dx} \right)^{1/n} \delta^{(1+n)/n} + c_2 \qquad c_2 = +\delta^{(1+n)/n} \left(\frac{1}{\mu}\left(-\frac{dP}{dx} \right) \right)^{1/n} \frac{n}{1+n}$$

$$v_x = \left(\frac{1}{\mu}\left(-\frac{dP}{dx}\right)\right)^{1/n}\left(\frac{n}{1+n}\right)\left[\delta^{(1+n)/n} - y^{(1+n)/n}\right]$$

or

$$v_x = \left[\frac{1}{\mu}\left(-\frac{dP}{dx}\right)\right]^{1/n}\left(\frac{n}{1+n}\right)\delta^{\frac{1+n}{n}}\left[1 - \left(\frac{y}{\delta}\right)^{\frac{1+n}{n}}\right]$$

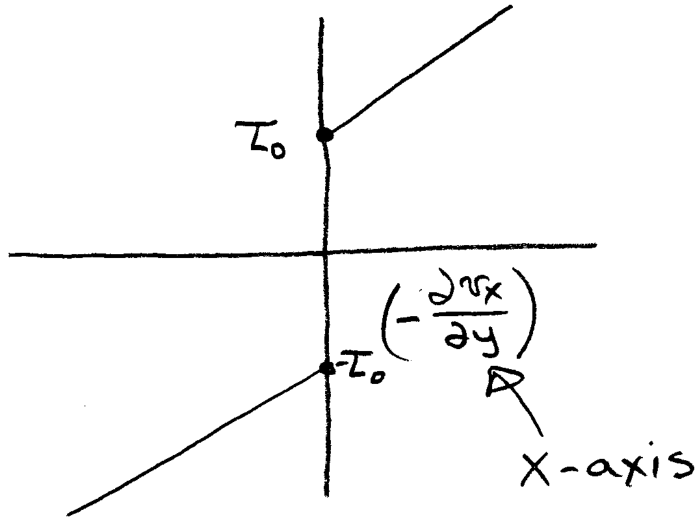
Check at $y=\delta$ $v_x=0$

$y=0$ $v_x = v_x|_{max}$

Parallel Flow – Bingham Plastic

What is a Bingham Plastic?

Experiments give



observation
at $\tau \le \tau_o$
the fluid flows
as a plug
at $\tau > \tau_o$
the fluid has,
a plug flow region
and a "liquid like"
or Newtonian Region

for $|\tau_{yx}| \le \tau_o$ $\quad \frac{\partial v_x}{\partial y} = 0$

for $|\tau_{yx}| > \tau_o$

I and $\tau_{yx} > 0$ $\quad \tau_{yx} = \tau_o - \mu \frac{\partial v_x}{\partial y}$

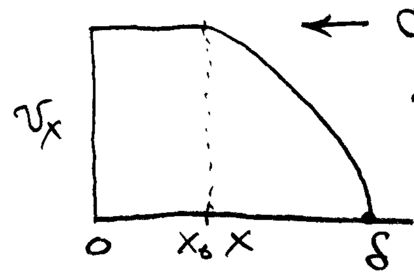
<u>OR</u>

II and $\tau_{yx} < 0$ $\quad \tau_{yx} = -\tau_o + \mu \frac{\partial v_x}{\partial y}$

first look at $|\tau_{yx}| \le \tau_o \quad \therefore \frac{\partial v_x}{\partial y} = 0$

this gives $\int \partial v_x = \int 0 \quad v = \text{constant}$

Now plot velocity profile



← Constant $= v$
let $v = v_o$
at
$0 < x \le x_o$

The stress profile is independent of fluid type: from previous derivation from Newtonian fluid

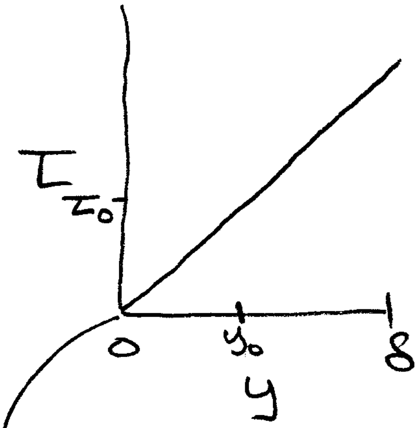
$$\frac{\partial(\rho v)}{\partial t} = -\frac{\partial P}{\partial x} - \frac{\partial \tau_{yx}}{\partial y}$$

$\cancel{\frac{\partial(\rho v)}{\partial t}} \to 0$ S.S.

$$\frac{\partial \tau_{yx}}{\partial y} = -\frac{dP}{dx}$$

$$\int \partial \tau_{yx} = -\int \frac{dP}{dx}\, dy$$

$$\tau_{yx} = \left(-\frac{dP}{dx}\right) y + C_1$$



at $y=0$ $\tau_{yx} \cong 0$! there is ~~no~~ only a very small stress at gas-liquid interface

$$0 = 0 + C_1 \quad \therefore \; C_1 = 0$$

which is the same as before

For flow between parallel plates there is an axis of symmetry between the 2 plates so the stress will be zero at this centerline.

Now look at region $y_0 \le y \le \delta$

$$\tau_{yx} = \tau_0 - \mu \frac{\partial v}{\partial y} = \left(-\frac{dP}{dx}\right) y$$

$$-\mu \frac{\partial v}{\partial y} = \left(-\frac{dP}{dx}\right) y - \tau_0$$

$$-\mu \int dv = \int \left(-\frac{dP}{dx}\right) y\, dy - \int \tau_0\, dy$$

$$-\mu v = \left(-\frac{dP}{dx}\right)\frac{y^2}{2} - \tau_0 y + C_2$$

$$\mu v_x = \tau_0 y - \left(-\frac{dP}{dx}\right) y^2/2 + c_2$$

at $y=\delta$ $v_x = 0$

this is in region of interest

$y_0 \le y \le \delta$

$$\mu(0) = \tau_0 \delta - \left(-\frac{dP}{dx}\right)\frac{\delta^2}{2} + c_2$$

$$c_2 = -\tau_0 \delta + \left(-\frac{dP}{dx}\right)\delta^2/2$$

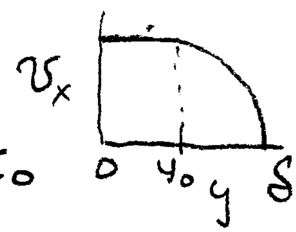
$$v_x = \frac{1}{\mu}\left[\tau_0 (y-\delta) + \left(-\frac{dP}{dx}\right)\left[\frac{\delta^2}{2} - \frac{y^2}{2}\right]\right]$$

$$v_x = \frac{\delta^2}{2\mu}\left(-\frac{dP}{dx}\right)\left(1-\left(\frac{y}{\delta}\right)^2\right) - \frac{\tau_0}{\mu}\delta\left(1 - y/\delta\right)$$

again for $y_0 \le y \le \delta$

at $y = y_0$ $\underbrace{v_x\Big|_{y\le y_0}}_{\text{inner}} = \underbrace{v_x\Big|_{y\ge y_0}}_{\text{outer}}$

Inner plug flow region $0 \le y \le y_0$



$$\tau_{yx}\Big|_{y=y_0} = \left(-\frac{dP_x}{dx}\right) y\Big|_{y_0} = -\frac{dP_x}{dx} y_0 = \tau_0$$

Since τ_0 & $-\frac{dP_x}{dx}$ are determined from experiment then y_0 can be predicted from this equation

$$v_x\big|_{y_0} = +\frac{\delta^2}{2\mu}\left(-\frac{dP}{dx}\right)\left(1-\left(\frac{y_0}{\delta}\right)^2\right) - \frac{\tau_0}{\mu}\delta\left(1-\frac{y_0}{\delta}\right)$$

$$\tau_0 = -\frac{dP}{dx}y_0$$

$$v_x\big|_{y_0} = +\frac{\delta^2}{2\mu}\left(-\frac{dP}{dx}\right)\left(1-\left(\frac{y_0}{\delta}\right)^2\right) - \left(-\frac{dP}{dx}\right)y_0\frac{\delta}{\mu}\left(1-\frac{y_0}{\delta}\right)$$

$$v_x\big|_{y_0} = \left(-\frac{dP}{dx}\right)\frac{1}{\mu}\Bigg[\underbrace{+\frac{\delta^2}{2}\left(1-\left(\frac{y_0}{\delta}\right)^2\right) - y_0\delta + y_0^2}\Bigg]$$

$$= \left(-\frac{dP}{dx}\right)\frac{\delta^2}{2\mu}\left[\left(1-\left(\frac{y_0}{\delta}\right)^2\right) - \frac{2}{\delta}y_0 + \frac{2y_0^2}{\delta^2}\right]$$

$$1-\left(\frac{y_0}{\delta}\right)^2 - \frac{2y_0}{\delta} + 2\frac{y_0^2}{\delta^2}$$

$$\underbrace{1-\frac{2y_0}{\delta} + \frac{y_0^2}{\delta^2}}$$

$$\left(1-\frac{y_0}{\delta}\right)\left(1-\frac{y_0}{\delta}\right)$$

check: $1-\frac{y_0}{\delta}-\frac{y_0}{\delta}+\left(\frac{y_0}{\delta}\right)^2$

$$v_x\big|_{y_0} = v_0 = \frac{\delta^2}{2\mu}\left(-\frac{dP}{dx}\right)\left[1-\frac{y_0}{\delta}\right]^2$$

for $0 \le y \le y_0$ $\quad v_x = v_0 = \frac{\delta^2}{2\mu}\left(-\frac{dP}{dx}\right)\left[1-\frac{y_0}{\delta}\right]^2$

for $y_0 \le y \le \delta$ $\quad v_x = \frac{\delta^2}{2\mu}\left(-\frac{dP}{dx}\right)\left(1-\left(\frac{y}{\delta}\right)^2\right) - \frac{\delta}{\mu}\tau_0\left(1-\frac{y}{\delta}\right)$

check on $\tau_{yx} = \tau_o - \mu \frac{\partial v_x}{\partial y}$

at $0 \le y \le y_o$ $\quad v_x = v_o$ $\quad$ so $\quad \tau_{yx} = \tau_o - \mu \cancel{\frac{\partial}{\partial y} v_o}^{\,0}$

$$\tau_{yx} = \tau_o$$

$y_o \le y \le \delta$ $\quad \tau = \tau_o - \mu \frac{\partial}{\partial y}\left[v(y)\right]$

$$\frac{\partial}{\partial y} v(y) = \frac{\delta^2}{2\mu}\left(-\frac{dP}{dx}\right)\left[0 - \frac{2y}{\delta^2}\right] - \frac{\delta}{\mu}\tau_o\left(0 - \frac{1}{\delta}\right)$$

$$\mu\left[\frac{\partial v_x}{\partial y} = -\frac{1}{\mu}\left(-\frac{dP}{dx}\right) y + \frac{\tau_o}{\mu}\right]$$

$$\tau_{yx} = \tau_o - \left[-\left(-\frac{dP}{dx}\right) y + \tau_o\right]$$

$\tau_{yx} = +\left(-\frac{dP}{dx}\right) y$ $\quad$ for $\quad y_o \le y \le \delta$

this is the result from page BP-1

Again to find y_o:

from $\tau_o = -\frac{dP}{dx} y_o$ $\qquad y_o = \dfrac{\tau_o}{\left(-\frac{dP}{dx}\right)}$

τ_o is given in constitutive equation – Problem statement

$\frac{dP}{dx}$ from experiments

Power law Model in Comsol

$$\eta = m(\dot{\gamma})^{n-1}$$

and

$$\tau = \eta \frac{\partial v}{\partial r}$$ (for pipe flow)

$$\dot{\gamma} = \frac{\partial v}{\partial r}$$ Shear Rate

I gave this as

$$\tau = \underbrace{\mu_0 \left| \frac{\partial v}{\partial r} \right|^{n-1}}_{\eta \text{ in Comsol}} \frac{\partial v}{\partial r}$$

Cutlip & Shacham USE

$$\tau = k \left| \frac{\partial v}{\partial r} \right|^{n-1} \frac{\partial v}{\partial r}$$