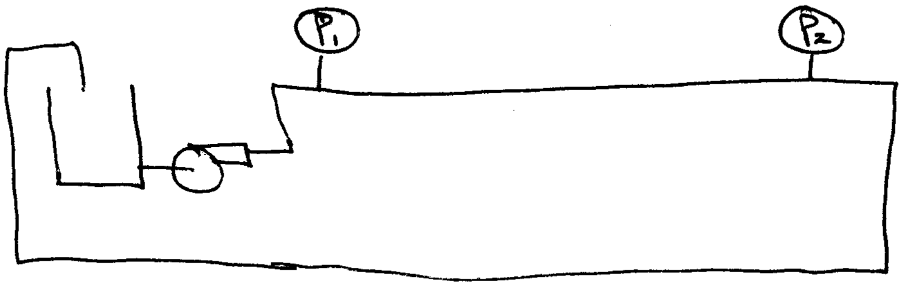
Fluid Mechanics

Look at Pipe experiment



<u>Relationships</u> (1) P_1 & P_2 ?

$\therefore P_2 > P_1 \qquad \Delta P = P_2 - P_1$

(2) flowrate vs ΔP

$\Delta P \propto v$ or v^2

(3) ΔL & D vs ΔP

$\Delta P \propto \frac{\Delta L}{D}$

fanning friction factor $\Delta P = 4 f_{fan} \rho \frac{\Delta L}{D} \frac{v^2}{2}$

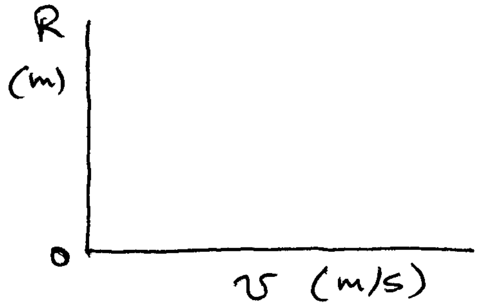
for turbulent flow & laminar if appropriate f is used.

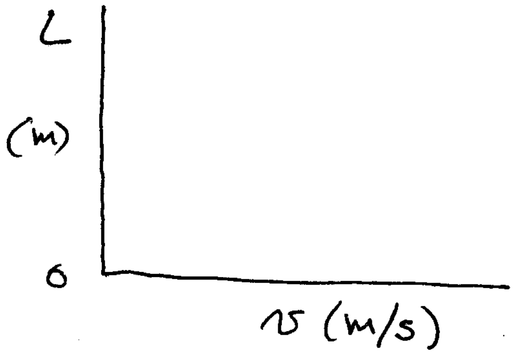
for laminar $f_{fan} = 16/Re$

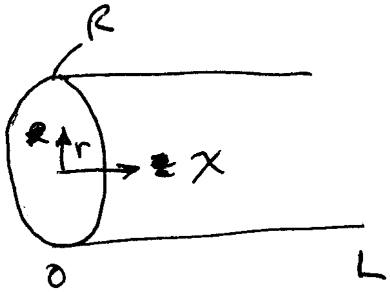
$$\Delta P = \frac{32 \mu v (\Delta L)}{D^2}$$

– this is the macroscopic approach
empirical

Now think about the flow of fluid in a pipe.







Momentum balance (mv) ← momentum

$$F_{net} = m a_{net}$$

tool to remember

$$\sum F = \rho \frac{dv}{dt} \Delta V$$

Review de Nevers Section 7 through 7.2

for flow systems

accumulation of momentum in a control volume = Σ Forces + rate of momentum in − rate of momentum out

$$\Delta V \frac{d}{dt}(\rho v) = \Sigma \text{ Forces} + (\dot{m} v)_{in} - (\dot{m} v)_{out}$$

volume $\frac{1}{s}$ $\left(\frac{\text{momentum}}{\text{volume}}\right)$

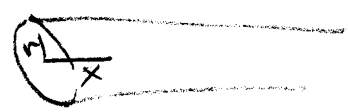
rate of momentum is called a flux of momentum

Compare this equation to de Nevers equation 7.14

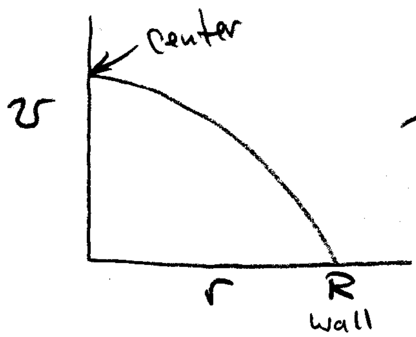
$$\cancel{m^3} \; \frac{1}{s} \; \frac{kg \; m/s}{\cancel{m^3}} \; [=] \; N \qquad \frac{kg}{s}\frac{m}{s}$$

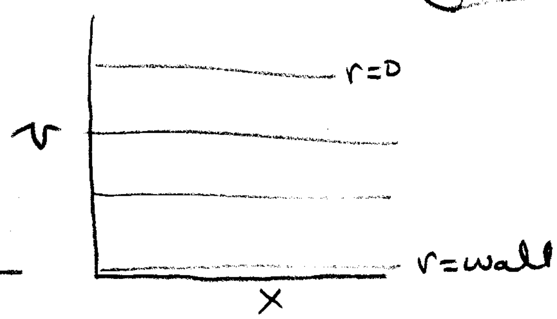
$$\underbrace{kg \frac{m}{s^2}} \qquad \underbrace{kg \frac{m}{s^2}} \qquad kg \frac{m}{s^2}$$

units must be the same



Re drew:





1). Equation describing this profile?

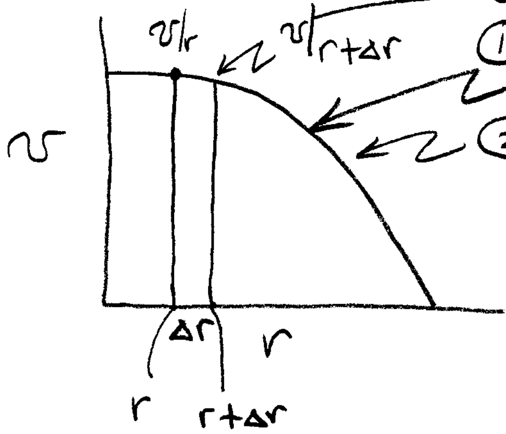
2) What is avg velocity ← used in Bernoulli eqn?

3) What is the shear stress on wall?

Define a control volume ΔV ← small volume

(think of a piece of paper rolled into a cylinder.)

evaluated at



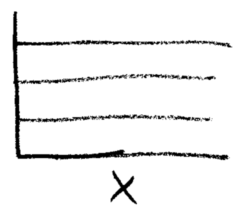
① We want to find $v = f(r)$

∴ v changes with r

② slope of line on curve

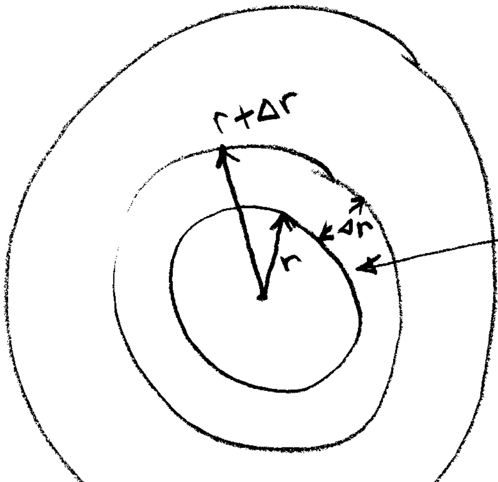
$$= \frac{dv}{dr} \simeq \frac{\Delta v}{\Delta r}$$

Look at



← v does not change with x

Look at cylinder



then the control volume

$\Delta V = (\text{Area})(\text{length})$

Δx

$\text{Area} = \pi(r+\Delta r)^2 - \pi r^2$

$= \pi(r^2 + 2r\Delta r + \Delta r^2 - r^2)$

if $r = R = 0.1\,m$

and $\Delta r = 0.0001\,m = 0.1\,mm$

Area $= \pi($ find each term

$$\text{Area} = \pi\,(\overset{r^2}{\cancel{0.01}m^2} + \overset{2r\Delta r}{0.00002m^2} + \overset{\Delta r^2}{1\times10^{-8}m^2} - \cancel{0.01}m^2)$$

$$\pi\,(2\times10^{-5} + 1\times10^{-8})\,m^2 = (0.0000200 1\,m^2)\,\pi$$

in other words $\Delta r^2 << 2r\Delta r\pi$

$\text{Area} \cong 2r\Delta r$

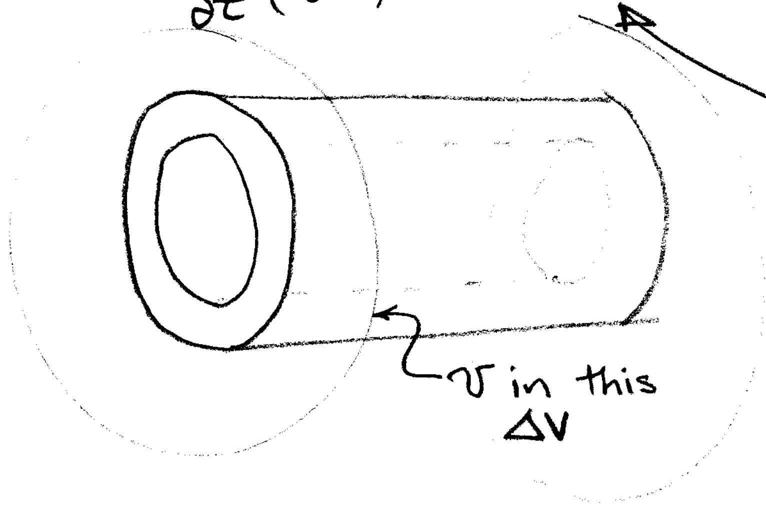
for $\Delta r \rightarrow$ really small then $\Delta A \equiv 2r\Delta r$

$$\Delta V = (\text{Area})\,\Delta x = \underbrace{2\pi r}_{\text{circumference}}\,\Delta r\,\Delta x$$

Δr: thickness — think paper

$$2r\Delta r\Delta x\,\frac{\partial}{\partial t}(\rho v) = \Sigma\text{Forces} + \dot{m}v\big|_{in} - \dot{m}v\big|_{out}$$

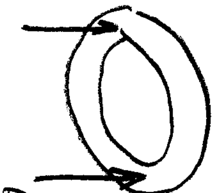
what are the sum of the forces?



v in this ΔV

Is there a pushing force (from a pump?)

$F = PA$



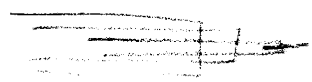
$P_x A$

what is this? $\underbrace{2\pi r}_{\text{circumference}}\ \Delta r$ — thickness

Pressure acts $\perp$ to area

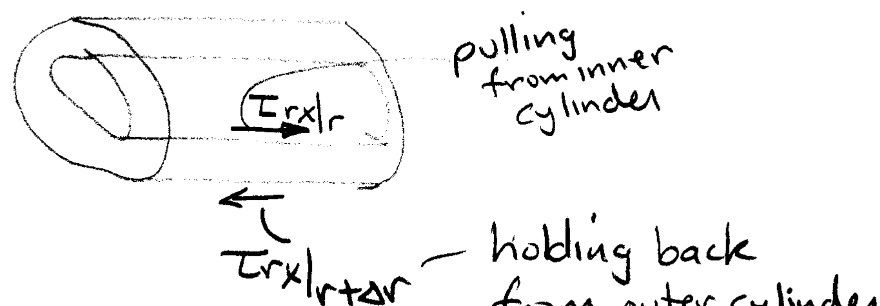
Next force?

Gravity? Assume horizontal and not needed.



What happens when you pull the paper out?

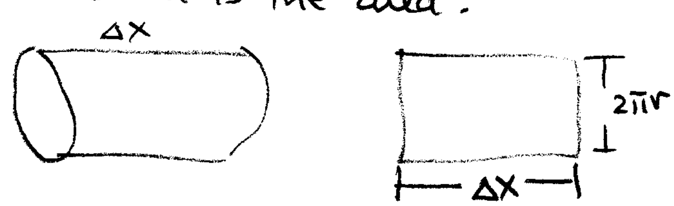
Is one layer of paper rubbing on another layer?



$$\text{Stress} = \frac{\text{Force}}{\text{Area}} = \frac{N}{m^2} = \frac{kg\, m}{s^2\, m^2}$$

Force = (Stress) Area

what is the area?



Area $= 2\pi r \Delta x$

$$2\pi r \Delta r \Delta x \frac{\partial}{\partial t}(\rho v) = P_x 2\pi r \Delta r\Big|_x - P_x 2\pi r \Delta r\Big|_{x+\Delta x}$$

$$\tau_{rx} 2\pi r \Delta x\Big|_r - \tau_{rx} 2\pi r \Delta x\Big|_{r+\Delta r} + (\text{More})$$

these r's are not equal

this r is $r|_{r+\Delta r} = \underline{\underline{r+\Delta r}}$

$$2\pi r \Delta r \Delta x \frac{\partial}{\partial t}(\rho v) = P_x 2\pi r \Delta r\Big|_x - P_x 2\pi r \Delta r\Big|_{x+\Delta x}$$

$$\tau_{rx} 2\pi r \Delta x\Big|_r - \tau_{rx} 2\pi r \Delta x\Big|_{r+\Delta r}$$

$$\underbrace{\dot{m} v}\Big|_x - \dot{m} v\Big|_{x+\Delta x}$$

we can rewrite this as $\dot{m} = (\rho v \Delta A)$

$$(\rho v \Delta A\, v\Big|_x - (\rho v \Delta A\, v\Big|_{x+\Delta x}$$

what is ΔA? $\Delta A = 2\pi r \Delta r$

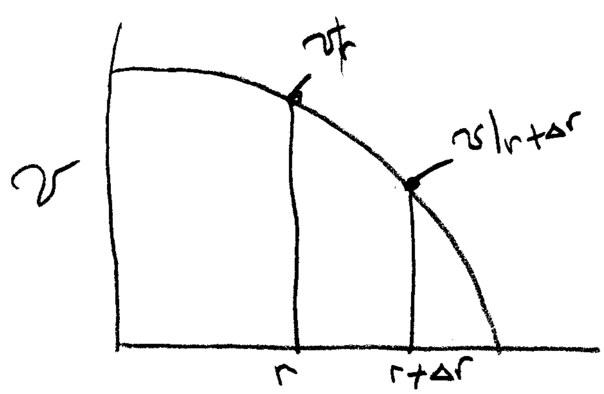
Divide by $\Delta r \Delta x (2\pi r)$

$$\frac{\partial}{\partial t}(\rho v) = \frac{P_x 2\pi r \Delta r|_x - P_x 2\pi r \Delta r|_{x+\Delta x} + \tau_{rx} 2\pi r \Delta x|_r}{2\pi r \Delta r \Delta x}$$

see Next page for P

$$- \frac{\tau_{rx} 2\pi r \Delta x|_{r+\Delta r} + \rho v v 2\pi r \Delta r|_x - \rho v v 2\pi r \Delta r|_{x+\Delta x}}{2\pi r \Delta r \Delta x}$$

now take limit as $\Delta r \to 0$ & $\Delta x \to 0$

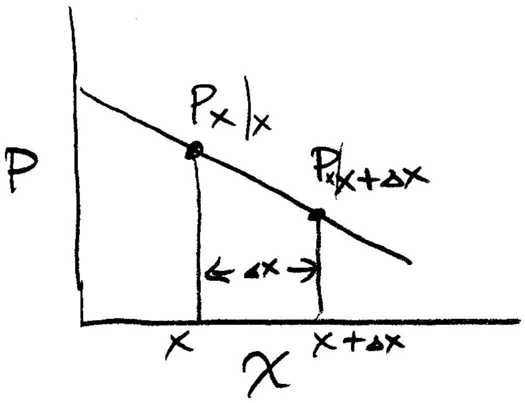


$$\frac{\overset{\text{final}}{v|_{r+\Delta r}} - \overset{\text{initial}}{v_r}}{(r+\Delta r) - r} = \underline{\underline{\text{slope}}}$$

as $\Delta r \to 0$

$$\text{slope} = \underline{\underline{\frac{\partial v}{\partial r}}}$$

what does pressure look like



$$\frac{P_x|_{x+\Delta x} - P_x|_x}{x+\Delta x - x} = \text{slope}$$

if $\Delta x \to 0$ then slope $= \dfrac{\partial P_x}{\partial x}$

$$\frac{(P_x 2\pi r \Delta r)|_x - (2\pi r \Delta r P_x)|_{x+\Delta x}}{(2\pi r \Delta r)\,\Delta x}$$

these look similar

Rearrange

$$\lim_{\Delta x \to 0} \frac{-\left(P_x 2\pi r \Delta r|_{x+\Delta x} - P_x 2\pi r \Delta r|_x \right)}{(2\pi r \Delta r)\left[x + \Delta x - x \right]}$$

Add x & subtract x

as $\Delta x \to 0$ $\quad -\dfrac{\partial (P_x 2\pi r \Delta r)}{\partial x}$

negative sign

partial derivative with respect to x.

$$\frac{\rho v v\, 2\pi r \Delta r|_x - \rho v v\, 2\pi r \Delta r|_{x+\Delta x}}{(2\pi r \Delta r)\Delta x}$$

Again

$$\lim_{\Delta x \to 0} - \frac{\left(\rho v v\, 2\pi r \Delta r|_{x+\Delta x} - \rho v v\, 2\pi r \Delta r|_x\right)}{(2\pi r \Delta r)(x + \Delta x - x)}$$

$$-\frac{\partial(\rho v v\, 2\pi r \Delta r)}{2\pi r \Delta r\, \partial x}$$

stress term is an r term

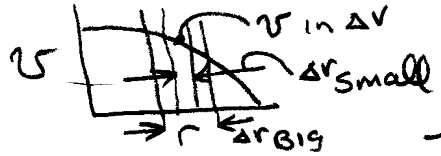
$$\frac{(\tau_{rx}\, 2\pi r \Delta x)|_r - (\tau_{rx}\, 2\pi r \Delta x)|_{r+\Delta r}}{2\pi r \Delta r \Delta x}$$

$$\lim_{\Delta r \to 0} - \frac{\left(\tau_{rx}\, 2\pi r \Delta x|_{r+\Delta r} - (\tau_{r}\, 2\pi r \Delta x)|_{r+\Delta r}\right.}{2\pi r \Delta x\,(r + \Delta r - r)}$$

$$-\frac{\partial(2\pi r \Delta x\, \tau_{rx})}{2\pi r \Delta x\, \partial r}$$

$$\frac{\partial}{\partial t}(\rho v) = -\frac{\partial(2\pi r \Delta r P_x)}{2\pi r \Delta r\, \partial x} - \frac{\partial(\rho v v\, 2\pi r \Delta r)}{2\pi r \Delta r\, \partial x} - \frac{\partial(2\pi r \Delta x\, \tau_{rx})}{2\pi r \Delta x\, \partial r}$$

does $\Delta x \to 0$, $\Delta r \to 0$ affect this?



v did not change

$$-\frac{\partial\,(2\pi r\Delta r P_x)}{2\pi r\Delta r\,\partial x} \Rightarrow -\frac{\overbrace{2\pi r\cancel{\Delta r}}^{\text{Constants in } X}}{\cancel{2\pi} r\Delta r}\frac{\partial P_x}{\partial x}$$

partial derivative with respect to x

hold r constant and take derivative with respect to x

$$-\frac{\partial\,(\rho v v\, 2\pi r\Delta r)}{2\pi r\Delta r\,\partial x} \Rightarrow -\left(\frac{2\pi \cancel{r\Delta r}}{2\pi \cancel{r\Delta r}}\right)\overbrace{\frac{\partial\,(\rho v v)}{\partial x}}^{\text{constant in } x}$$

$$-\frac{\partial\,(\tau_{rx}\, 2\pi r\Delta x)}{2\pi r\Delta x\,\partial r} = -\frac{\overbrace{\cancel{2\pi\Delta x}}^{\text{constant with } r}}{\cancel{2\pi} r\cancel{\Delta x}}\frac{\partial\,(\tau_{rx} r)}{\partial r}$$

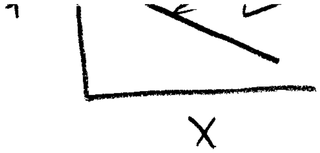
r is <u>not</u> Constant with respect to r!!!

$$= -\frac{1}{r}\frac{\partial\,(\tau_{rx} r)}{\partial r}$$

$$\frac{\partial\,(\rho v)}{\partial t} = -\frac{\partial P_x}{\partial x} - \frac{\partial\,(\rho v v)}{\partial x} - \frac{1}{r}\frac{\partial\,(\tau_{rx} r)}{\partial r}$$

We can solve problems in cylindrical coordinates with this equation

- Polymath
- Comsol
- Analytical (By hand)



P vs X is linear see mv-7

$$-\frac{dP_x}{dx} = \text{constant}$$

$$0 = -\frac{dP_x}{dx} - \cancel{\frac{\partial(\rho v v)}{\partial x}} - \frac{1}{r}\frac{\partial}{\partial r}(\tau_{rx} r)$$

this is v in x direction

0

does this change with x

from mass balance $v_1 \rho_1 A_1 = v_2 \rho_2 A_2$

$v_1 = v_2$

$\frac{\partial v}{\partial x} = 0$

$$0 = -\frac{\partial P_x}{dx} - \frac{1}{r}\frac{\partial}{\partial r}(\tau_{rx} r)$$

$$-\int r \frac{dP_x}{dx}\,dr = \int \partial(\tau_{rx} r) \Rightarrow -\frac{r^2}{2}\frac{dP_x}{dx} + c_1 = (\tau_{rx} r)$$

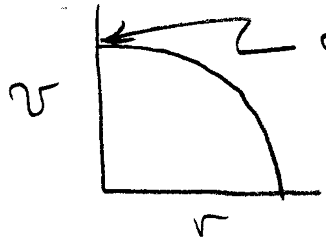
what is τ_{rx}?

Constitutive equation — found from experiments

Newtonian Fluids (water)

$$\tau_{rx} = -\mu \frac{\partial v_x}{\partial r}$$

Before pluggin in what is τ_{rx} at $r=0$



$\frac{\partial v}{\partial r} = 0$ so $\tau_{rx} = 0$ at $r=0$

divide by r

$$-\frac{1}{r}\frac{r^2}{2}\frac{dP_x}{dx}\bigg|_{r=0} + C_1 = \tau_{rx}\Big|_{r=0} = 0$$

$$0$$

$$C_1 = 0$$

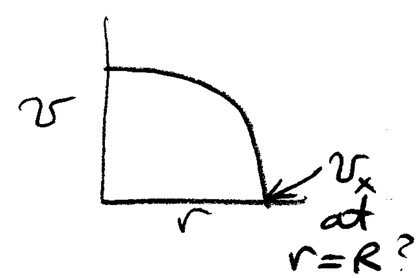
$$\tau_{rx} = -r\frac{dP_x}{dx}$$

for newtonian fluids

$$-\mu\frac{dv_x}{dr} = -\frac{r}{2}\frac{dP_x}{dx}$$

$$\cancel{-}\mu\int dv_x = \cancel{-}\frac{dP_x}{dx}\int r\,dr$$

$$\mu v_x = \frac{dP_x}{dx}\frac{r^2}{4} + C_2$$



what is v_x at $r=R$

$$\mu v_x = 0 = \frac{dP_x}{dx}\frac{R^2}{4} + C_2 \quad \therefore \quad C_2 = -\frac{dP_x}{dx}\frac{R^2}{4}$$

$$v_x = +\frac{dP_x}{dx}\frac{r^2}{4\mu} + \left(-\frac{dP_x}{dx}\frac{R^2}{2\mu}\right)$$

$$= -\frac{1}{4\mu}\frac{dP_x}{dx}\left(R^2 - r^2\right)$$

$$= -\frac{R^2}{4\mu}\frac{dP_x}{dx}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

check at $r=0$ $v_x = \text{const}$ $\frac{dv_x}{dr} = 0$

$r=R$ $v_x = 0$