SOLUTION II

Another method makes use of the result obtained previously in Eq. 10.2-13. Although T_0 is not known in the present problem, we can nonetheless use the result. From Eqs. 10.1-2 and 10.2-16 we can get the temperature difference

$$T_0 - T_{air} = \frac{\pi R^2 L S_e}{h(2\pi R L)} = \frac{S_e R}{2h} \tag{10.2-24}$$

Substraction of Eq. 10.2-24 from Eq. 10.2-13 enables us to eliminate the unknown T_0 and gives Eq. 10.2-23.

§10.3   HEAT CONDUCTION WITH A NUCLEAR HEAT SOURCE

Transport phenomena 2nd ed. Bird, Stewart & light foot 2002.

We consider a spherical nuclear fuel element as shown in Fig. 10.3-1. It consists of a sphere of fissionable material with radius $R^{(F)}$, surrounded by a spherical shell of aluminum "cladding" with outer radius $R^{(C)}$. Inside the fuel element, fission fragments are produced that have very high kinetic energies. Collisions between these fragments and the atoms of the fissionable material provide the major source of thermal energy in the reactor. Such a volume source of thermal energy resulting from nuclear fission we call S_n (cal/cm$^3 \cdot$ s). This source will not be uniform throughout the sphere of fissionable material; it will be the smallest at the center of the sphere. For the purpose of this problem, we assume that the source can be approximated by a simple parabolic function

$$S_n = S_{n0}\left[1 + b\left(\frac{r}{R^{(F)}}\right)^2\right] \tag{10.3-1}$$

Here S_{n0} is the volume rate of heat production at the center of the sphere, and b is a dimensionless positive constant.

We select as the system a spherical shell of thickness Δr within the sphere of fissionable material. Since the system is not in motion, the energy balance will consist only of heat conduction terms and a source term. The various contributions to the energy balance are:

Rate of heat in by conduction at r

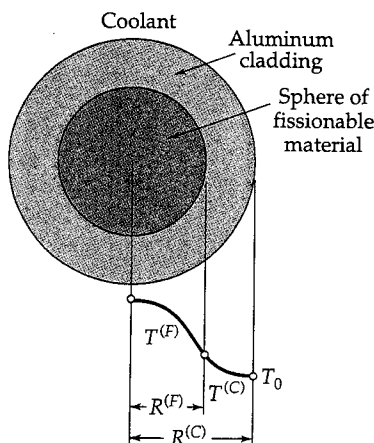$$q_r^{(F)}|_r \cdot 4\pi r^2 = (4\pi r^2 q_r^{(F)})|_r \tag{10.3-2}$$

Fig. 10.3-1. A spherical nuclear fuel assembly, showing the temperature distribution within the system.

Rate of heat out
by conduction
at $r + \Delta r$
$$q_r^{(F)}|_{r+\Delta r} \cdot 4\pi(r + \Delta r)^2 = (4\pi r^2 q_r^{(F)})|_{r+\Delta r} \qquad (10.3\text{-}3)$$

Rate of thermal
energy produced
by nuclear fission
$$S_n \cdot 4\pi r^2 \, \Delta r \qquad (10.3\text{-}4)$$

Substitution of these terms into the energy balance of Eq. 10.1-1 gives, after dividing by $4\pi \, \Delta r$ and taking the limit as $\Delta r \to 0$

$$\lim_{\Delta r \to 0} \frac{(r^2 q_r^{(F)})|_{r+\Delta r} - (r^2 q_r^{(F)})|_r}{\Delta r} = S_n r^2 \qquad (10.3\text{-}5)$$

Taking the limit and introducing the expression in Eq. 10.3-1 leads to

$$\frac{d}{dr}(r^2 q_r^{(F)}) = S_{n0}\left[1 + b\left(\frac{r}{R^{(F)}}\right)^2\right]r^2 \qquad (10.3\text{-}6)$$

The differential equation for the heat flux $q_r^{(C)}$ in the cladding is of the same form as Eq. 10.3-6, except that there is no significant source term:

$$\frac{d}{dr}(r^2 q_r^{(C)}) = 0 \qquad (10.3\text{-}7)$$

Integration of these two equations gives

$$q_r^{(F)} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}}\frac{r^3}{5}\right) + \frac{C_1^{(F)}}{r^2} \qquad (10.3\text{-}8)$$

$$q_r^{(C)} = + \frac{C_1^{(C)}}{r^2} \qquad (10.3\text{-}9)$$

in which $C_1^{(F)}$ and $C_1^{(C)}$ are integration constants. These are evaluated by means of the boundary conditions:

B.C. 1:   at $r = 0$, $q_r^{(F)}$ is not infinite   (10.3-10)

B.C. 2:   at $r = R^{(F)}$, $q_r^{(F)} = q_r^{(C)}$ (10.3-11)

Evaluation of the constants then leads to

$$\boxed{q_r^{(F)} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}}\frac{r^3}{5}\right)} \qquad (10.3\text{-}12)$$

$$\boxed{q_r^{(C)} = S_{n0}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r^2}} \qquad (10.3\text{-}13)$$

These are the heat flux distributions in the fissionable sphere and in the spherical-shell cladding.

Into these distributions we now substitute Fourier's law of heat conduction (Eq. B.2-7):

$$-k^{(F)}\frac{dT^{(F)}}{dr} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}}\frac{r^3}{5}\right) \qquad (10.3\text{-}14)$$

$$-k^{(C)}\frac{dT^{(C)}}{dr} = S_{n0}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r^2} \qquad (10.3\text{-}15)$$

These equations may be integrated for constant $k^{(F)}$ and $k^{(C)}$ to give

$$T^{(F)} = -\frac{S_{n0}}{k^{(F)}}\left(\frac{r^2}{6} + \frac{b}{R^{(F)2}}\frac{r^4}{20}\right) + C_2^{(F)} \tag{10.3-16}$$

$$T^{(C)} = +\frac{S_{n0}}{k^{(C)}}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r} + C_2^{(C)} \tag{10.3-17}$$

The integration constants can be determined from the boundary conditions

B.C. 3:   at $r = R^{(F)}$, $T^{(F)} = T^{(C)}$ (10.3-18)

B.C. 4:   at $r = R^{(C)}$, $T^{(C)} = T_0$ (10.3-19)

where T_0 is the known temperature at the outside of the cladding. The final expressions for the temperature profiles are

$$T^{(F)} = \frac{S_{n0}R^{(F)2}}{6k^{(F)}}\left\{\left[1 - \left(\frac{r}{R^{(F)}}\right)^2\right] + \frac{3}{10}b\left[1 - \left(\frac{r}{R^{(F)}}\right)^4\right]\right\}$$
$$+ \frac{S_{n0}R^{(F)2}}{3k^{(C)}}\left(1 + \frac{3}{5}b\right)\left(1 - \frac{R^{(F)}}{R^{(C)}}\right) \tag{10.3-20}$$

$$T^{(C)} = \frac{S_{n0}R^{(F)2}}{3k^{(C)}}\left(1 + \frac{3}{5}b\right)\left(\frac{R^{(F)}}{r} - \frac{R^{(F)}}{R^{(C)}}\right) \tag{10.3-21}$$

To find the maximum temperature in the sphere of fissionable material, all we have to do is set r equal to zero in Eq. 10.3-20. This is a quantity one might well want to know when making estimates of thermal deterioration.

This problem has illustrated two points: (i) how to handle a position-dependent source term, and (ii) the application of the continuity of temperature and normal heat flux at the boundary between two solid materials.

§10.4 HEAT CONDUCTION WITH A VISCOUS HEAT SOURCE

Next we consider the flow of an incompressible Newtonian fluid between two coaxial cylinders as shown in Fig. 10.4-1. The surfaces of the inner and outer cylinders are maintained at $T = T_0$ and $T = T_b$, respectively. We can expect that T will be a function of r alone.
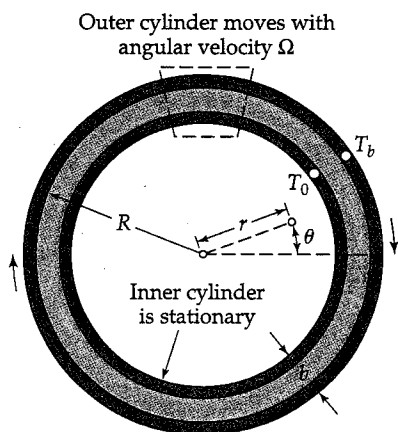

Fig. 10.4-1. Flow between cylinders with viscous heat generation. That part of the system enclosed within the dotted lines is shown in modified form in Fig. 10.4-2.

Now add the aluminum cladding

Equation:

"electrical wire" core
$$\frac{1}{r}\frac{\partial (q_r r)}{\partial r} = \dot{q}$$

insulation/conductor Aluminum
$$\frac{1}{r}\frac{\partial (q_r^{Al} r)}{\partial r} = 0$$

B.C.

WIRE
$$\begin{cases} r = 0 & q_r = \text{finite} \\ r = R_w & T^w = T^{Al} \\ & q_r^w = q_r^{Al} \quad (\text{same area}) \end{cases}$$

Al
$$\begin{cases} r = R_w \\ r = R_o & q_r^{Al} = h\left(T^{Al}\Big|_{r=R_o} - T_o\right) \end{cases}$$

Temp $\neq$ heat flux must be continuous

substituting in

$$T^{Al} - T_o = \frac{\dot{q}R_w^2}{2k_{Al}}\left(\ln\frac{R_o}{r} + \frac{k_{Al}}{hR_o}\right)$$

$$T^{wire} - T_o = \frac{\dot{q}R_w^2}{4k_w}\left(1 - \left(\frac{r}{R_w}\right)^2 - \frac{2k_w}{k_{Al}}\ln\frac{R_w}{R_o} + 2\frac{k_w}{hR_o}\right)$$

$$\text{wire}: \begin{cases} q_r = \dfrac{C_1}{r} + \dot{q}\,{}^r\!/_2 \\[2mm] T = \dfrac{C_2}{-k_w} + 0 + \dot{q}\,\dfrac{r^2}{4}\dfrac{1}{-k_w} \end{cases}$$

again $\quad q_r = 0$ at $r=0 \quad C_1 = 0$

B.C. at $r=R_w \qquad T_w\Big|_{r=R_w} = T_c\Big|_{r=R_w} = T_{wc}$

$$T_{wc} = \dfrac{C_2}{-k_w} + 0 + \dot{q}\,\dfrac{R_w^2}{4}\dfrac{1}{-k_w}$$

$$C_2 = -T_{wc}\,k_w - \dot{q}\,\dfrac{R_w^2}{4}$$

$$T = \dfrac{-T_{wc}\,k}{-R_w} + \dot{q}\,\dfrac{R_w^2}{4}k_w + \dot{q}\,\dfrac{r^2}{-4k_w}$$

$$T = T_{wc} + \dot{q}\,\dfrac{R_w^2}{4k_w}\left(1 - \left(\dfrac{r}{R_w}\right)^2\right)$$

Now for cladding · Al

$$q_r = \dfrac{C_1}{r}$$

$$T = \dfrac{C_2}{-k_c} - \dfrac{C_1}{k_c}\ln r$$

at $r=R_w \quad T_c = T_w = T_{wc}$

$\qquad\qquad q_{cr} = q_{wr}$

$r=R_c \quad q = h\left(T\Big|_{r=R_c} - T_0\right)$

at $r=R_w$

$$q_r\Big|_w = q_r\Big|_{Al} = \dot{q}\,\dfrac{R_w}{2} = \dfrac{C_1}{R_w}$$

solve for $C_1 \qquad \dot{q}\,\dfrac{R_w^2}{2} = C_1$

$$T = \dfrac{C_2}{-k_c} - \dot{q}\,\dfrac{R_w^2}{2k_c}\ln r$$

$$\text{at } T\Big|_{r=R_c} = \text{flux}$$

$$-k_c \frac{dT}{dr} = -k_c\left[0 - \dot{q}\frac{R_w^2}{2k_c}\frac{1}{r}\right]$$

$$q\Big|_{r=R_c} = h\left(T\Big|_{r=R_c} - T_0\right) = \dot{q}\frac{R_w^2}{2}\frac{1}{R_c}$$

$$T\Big|_{r=R_c} = \dot{q}\frac{R_w^2}{2R_c}\frac{1}{h} + T_0 \qquad \frac{m^2\, sk}{s} \frac{w}{} \frac{J}{s\,m^3}$$

$$T\Big|_{r=R_c} = \dot{q}\frac{R_w^2}{2hR_c} + T_0 = \frac{C_2}{-k_c} - \dot{q}\frac{R_w^2}{2k_c}\ln R_c$$

$$C_2 = \dot{q}\frac{R_w^2}{2}\left[\frac{1}{hR_c} + \frac{\ln R_c}{R_c}\right](-k_c) - k_c T_0 = C_2$$

$$\frac{\cancel{s}}{s\,m^3}\frac{m^2}{\cancel{s}}\overset{\supset}{s\,m\,k}\ln R$$

$$T = \dot{q}\frac{R_w^2}{2}\left[\frac{1}{hR_c} + \frac{\ln R_c}{R_c}\right]\frac{(-k_c)}{-k_c} - \dot{q}\frac{R_w^2}{2k_c}\ln r - \frac{k_c T_0}{-k_c}$$

$$T = \dot{q}\frac{R_w^2}{2}\left[\frac{1}{hR_c} + \frac{\ln R_c}{k_c} - \frac{\ln r}{k_c}\right] + T_0$$

$$\boxed{T = \dot{q}\frac{R_w^2}{2k_c}\left[\frac{k_c}{hR_c} + \ln\frac{R_c}{r}\right] + T_0}$$
AR Cladding

$$\text{at } T\Big|_{r=R_w} = \dot{q}\frac{R_w^2}{2k_c}\left[\frac{k_c}{hR_c} + \ln\frac{R_c}{r}\right] + T_0 = T_{wc}$$

plug this into:

$$T = \frac{\dot{q}\,R_w^2}{2k_c}\left[\frac{k_c}{hR_c} + \ln\frac{R_c}{r}\right] + \dot{q}\,\frac{R_w^2}{4k_w}\left(1 - \left(\frac{r}{R_w}\right)^2\right)$$

$$= \frac{\dot{q}\,R_w^2}{2k_c}\left[\frac{k_c}{hR_c} + \ln\frac{R_c}{r} + \frac{k_c}{k_w}\left(1 - \left(\frac{r}{R_w}\right)^2\right)\right]$$

$$\frac{\dot{q}\,R_w^2}{2k_c}\,\frac{k_c}{k_w}\left[\frac{k_w}{hR_c} + \frac{k_w}{k_c}\ln\frac{R_c}{r} + \left(1 - \left(\frac{r}{R_w}\right)^2\right]$$