Mathematica® for Calculus I and II

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Chapter 1. Introduction

1.1. Getting Started

Welcome to *Mathematica*! To make your getting started with this computer algebra system as easiest as possible, we recommend that you read this introductory chapter very carefully. First of all, a student must have a computer account on the Rowan network in order to access the *Mathematica* software package. Computer accounts can be obtained from the Office of Instructional Technology.

First-time users of *Mathematica* 4.0: you must install certain *Mathematica* fonts before using the software. After logging into your account, go inside the Math Classes network directory by clicking on the [Start] icon and following the trail of icons: Network Applications/Class Applications/Math Classes. Look for the icon Install *Mathematica* Fonts and click on it. Then reboot your computer. You may now go back to the Math Classes directory and click on the icon *Mathematica* 3.0 to run the front end of the program where commands can be entered. *Mathematica* will automatically create a new notebook. Just start typing and simultaneously press the SHIFT+ENTER keys whenever you want *Mathematica* to evaluate an expression; this will launch the kernel application, which does all the computations. This is a one-time procedure and may take a few seconds, so please be patient.

There is an on-line help menu within *Mathematica*; please feel free to use it if you have questions about the program. There is also the search command given by the question mark **?**. For example, evaluating **?Plot** will provide all *Mathematica* commands containing the expression Plot.

1.2. *Mathematica*'s Conventions for Inputting Commands

1.2.1. Naming

Built-in *Mathematica* commands, functions, constants, and other expressions begin with capital letters, and are (for the most part) one or more full-length English words. Furthermore, *Mathematica* is case sensitive; a common cause of error is the failure to capitalize command names. For example, **Plot**, **Integrate**, and **FindRoot** are valid functions names. **Sin**, **Exp**, **Det**, **GCD**, and **Max** are some of the standard mathematical abbreviations that are exceptions to the full-length English word(s) rule.

User-defined functions and variables can be any mixture of upper and lower case letters and numbers. However, a name cannot begin with a number. By convention, user-defined functions begin with a lower-case letter, but this is not required. For example, **f**, **g1**, **myPlot**, **r12**, **sOLution**, **Method1** are permissible functions names.

1.2.2. Parenthesis, Brackets and Braces

Mathematica interprets various types of brackets differently. Using an incorrect type of bracket is another source of error. *Mathematica*'s bracketing conventions are:

1) Parentheses, ( ), are used only for grouping expressions. For example, (x - y)^2, 1/(a + b), (x^3 - y)/(x + 3 y^2) demonstrate proper use of parenthesis. Users should realize that *Mathematica* understands f(2) as f multiplied with 2 and not as the function f evaluated at 2.

2) Square brackets, [ ], are used for function arguments. For example, Sqrt[346], Sin[Pi], and Simplify[(x^3 - y^3)/(x - y)] are valid uses of square brackets. Therefore, to evaluate a function f at 2, we can type f[2].

3) Braces or curly brackets, { }, are used for lists, ranges and iterators. In all cases, elements are separated by commas. Here are some typical uses of braces:

{1, 4, 9, 16, 25, 36} lists the square of the first six positive integers;

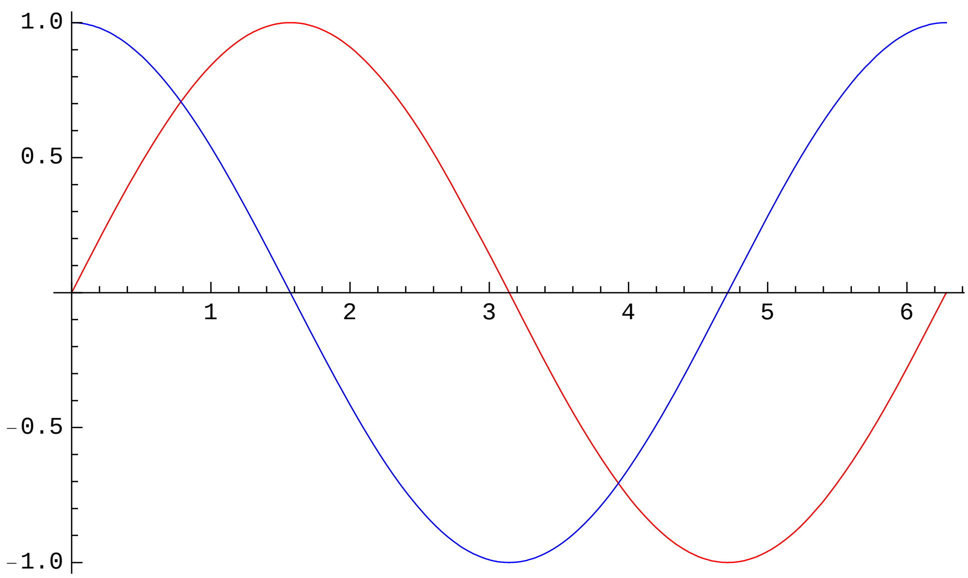
Plot[f[x],{x,-5,5}] specifies the range of values for x in plotting f;

Table[m^3,{m,1,100}] specifies the values of the iterator m in generating a table of elements.

NOTE: one can write comments on any input line. The comments should be enclosed between (* and *). For example,

```
In[1]:= Plot[{Sin[x], Cos[x]}, {x, 0, 2 Pi},
  PlotStyle -> {RGBColor[0.996109, 0, 0], RGBColor[0, 0, 0.996109]}]
 (* this commands plots the graph of two functions in different colors. *)
```

Out[1]=



1.2.3. Equal Signs

Here are *Mathematica*'s rules regarding the use of equal signs:

1) A single equal sign (=) assigns a value to a variable. Thus, entering **q = 3** means that q will be assigned the value 3. If we then evaluate **10 + q^3**, *Mathematica* will return 37. As another example, suppose the expression **y = x^3 - x + 1** is entered. If we then assign a value for x, say x=2, then in any input containing y afterwards, *Mathematica* will use this value of x to calculate y, which would be 7 in our case.

2) A colon-equal sign (:=) creates a delayed statement for an expression and is used to define a function. For example, typing **f[x_]:= x^3 - x + 1** tells *Mathematica* to reevaluate the function f for each new value of x. We will say more about defining functions in section 1.3 of this notebook.

3) A double-equal sign (= =) is a test of equality between two expressions. If we had previously set **x= - 5** , then evaluating x = = -5 returns True, whereas evaluating **x = = 5** returns False. Another common usage of = = is to solve equations, such as in **Solve[x^3-x+1= = 0, x]** (see section 1.5 below).

1.2.4. Referring to Previous Results

Mathematica saves all input and ouput in a session. A previous output can be referred to by using the percent sign **%** . A single **%** refers to *Mathematica*'s last output, **%%** refers to the next-to-last ouput, and so forth. **%k** refers to the output line numbered k. For example, **%45** refers to output line number 45.

NOTE: CTRL+L reproduces the last input and CTRL+SHIFT+L reproduces the last output.

■ 1.3. Basic Calculator Operations

Mathematica uses the standard symbols **+**, **-**, *****, **/**, **^**, **!** for addition, subtraction, multiplication, division, raising exponents, and factorials, respectively. Multiplication can also be performed by leaving a blank space between factors. Powers can also be entered by pressing CTRL+6 and fractions can be entered by pressing CTRL+/.

To convert numerical values to decimal form, use the command **N[expr]** or **N[expr,n]**. In most cases, **N[expr]** returns six digits or expr and may be in the form $n.\text{abcde} * 10^m$ (scientific notation), whereas **N[expr,n]** attempts to return n digits of expr. NOTE: *Mathematica* can perform calculations to arbitrary precision and handle numbers that are arbitrarily large or small. Here are some examples:

In[2]:=
```
Pi
```
Out[2]= π

In[3]:=
```
N[Pi]
```
Out[3]=
```
3.14159
```

In[4]:=
```
N[Pi, 200]
```
Out[4]=
```
3.1415926535897932384626433832795028841971693993751058209749445923078164062 \
86208998628034825342117067982148086513282306647093844609550582231725359408 \
128481117450284102701938521105559644622948954930382 0
```

In[5]:= 6^{5^4}

Out[5]=
```
2 210 708 544 304 025 665 789 890 545 869 282 983 189 550 730 342 026 817 054 484 706 923 \
451 925 215 263 872 221 875 601 412 877 526 055 033 568 150 952 983 731 997 599 172 762 \
855 409 042 386 638 455 130 114 567 918 179 610 415 056 135 043 685 865 981 465 821 197 \
678 998 054 981 600 364 232 459 680 450 883 986 513 397 952 866 100 532 961 319 277 446 \
513 221 836 325 497 685 382 494 082 501 890 188 075 860 096 650 899 943 982 604 939 901 \
346 570 765 022 869 199 395 889 789 728 382 946 141 484 842 179 531 904 056 612 897 175 \
359 078 633 987 736 867 003 878 781 857 613 656 893 578 474 392 372 463 398 376 238 316 \
805 554 810 164 724 551 909 376
```

In[6]:=
```
1 / 300 !
```
Out[6]=
```
1 /
 306 057 512 216 440 636 035 370 461 297 268 629 388 588 804 173 576 999 416 776 741 259 ⋱
  476 533 176 716 867 465 515 291 422 477 573 349 939 147 888 701 726 368 864 263 907 ⋱
  759 003 154 226 842 927 906 974 559 841 225 476 930 271 954 604 008 012 215 776 252 ⋱
  176 854 255 965 356 903 506 788 725 264 321 896 264 299 365 204 576 448 830 388 909 ⋱
  753 943 489 625 436 053 225 980 776 521 270 822 437 639 449 120 128 678 675 368 305 ⋱
  712 293 681 943 649 956 460 498 166 450 227 716 500 185 176 546 469 340 112 226 034 ⋱
  729 724 066 333 258 583 506 870 150 169 794 168 850 353 752 137 554 910 289 126 407 ⋱
  157 154 830 282 284 937 952 636 580 145 235 233 156 936 482 233 436 799 254 594 095 ⋱
  276 820 608 062 232 812 387 383 880 817 049 600 000 000 000 000 000 000 000 000 000 ⋱
  000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
```

In[7]:=
```
N[%] (* This command returns a decimal answer of the last output *)
```
Out[7]= $3.267359761105326 \times 10^{-615}$

Example 1.1. How close is $e^{\sqrt{163}\,\pi}$ to being an integer?

In[8]:=
```
E^(Pi * Sqrt[163])
```
Out[8]= $e^{\sqrt{163}\,\pi}$

In[9]:=
```
N[%, 40]
```
Out[9]= $2.625374126407687439999999999992500725972 \times 10^{17}$

We can rewrite this output by moving the decimal place 17 places to the right. This shows that $e^{\sqrt{163}\,\pi}$ is very close to being an integer. We use the command **Mod[n,m]**, which returns the remainder of n when divided by m, to obtain the fractional part of $e^{\sqrt{163}\,\pi}$:

In[10]:=
```
Mod[%, 1]
```
Out[10]= 0.99999999999992500725972

In[11]:=
```
1 - %
```
Out[11]= $7.499274028 \times 10^{-13}$

■ 1.4. Functions

There are two different ways to define functions in *Mathematica*, depending on how they are to be used. Consider the following example:

Example 1.2. Enter the function $f(x) = \frac{x^2+x+2}{x+1}$ into *Mathematica*.

Method 1: Simply enter

In[12]:=
```
f = (x^2 + x + 2) / (x + 1)
```
Out[12]= $\frac{2 + x + x^2}{1 + x}$

To evaluate $f(x)$ at $x = 10$, we use the substitution command slash-period **/.** as follows:

In[13]:=
```
f /. x -> 10
```
Out[13]= $\frac{112}{11}$

Method 2: An alternative way to define $f(x)$ by using the argument x is to write

In[14]:=
```
Clear[f] (* This clears the argument f *)
f[x_] := (x^2 + x + 2) / (x + 1)
```

Typing **f[10]** now tells *Mathematica* to evaluate f at $x = 10$. More generally, the command **f[{a, b, c, ...}]** evaluates $f(x)$ for a list of values of x:

In[16]:=
```
f[10]
f[{-3, -2, -1, 0, 1, 2, 3}]
```
Out[16]= $\frac{112}{11}$

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Out[17]= $\left\{-4, -4, \text{ComplexInfinity}, 2, 2, \frac{8}{3}, \frac{7}{2}\right\}$

Here, *Mathematica* is warning us that it has encountered the undefined expression $\frac{1}{0}$ in evaluating $f(-1)$ by returning the message ComplexInfinity.

Warning: Recall that *Mathematica* reads f(x) as f multiplied with x.

Method 3: If there is no need to attach a label to the expression $\frac{x^2+x+2}{x+1}$, then we can directly enter this expression into *Mathematica*:

In[18]:= $\frac{x^2+x+2}{x+1}$ /. x -> 10

Out[18]= $\frac{112}{11}$

In[19]:= $\frac{x^2+x+2}{x+1}$ /. x -> {-3, -2, -1, 0, 1, 2, 3}

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Out[19]= $\left\{-4, -4, \text{ComplexInfinity}, 2, 2, \frac{8}{3}, \frac{7}{2}\right\}$

Piecewise functions can be defined using the command **If[*p*, else, *q*]**.

Example 1.3. Enter the piece-wise function $f(x) = \begin{cases} \tan\frac{\pi x}{4}, & \text{if } |x| < 1; \\ x, & \text{if } |x| \geq 1, \end{cases}$ into *Mathematica*.

In[20]:=
```
f[x_] := If[Abs[x] < 1, Tan[Pi * x / 4], x]
```

1.5. Palettes

Mathematica allows us to enter commonly used mathematical expressions and commands from 6 different palettes. Palettes are calculator pads containing buttons that can be clicked on to insert the desired expression or command into a command line. These palettes can be found under the File/Palettes menu. By default, the Basic Inputs Palette should appear when the *Mathematica* application is opened. In fact, you can create your own palette.

Palette Instructions:

1. Move the cursor to the location where an expression will be inserted.

2. Click on the corresponding palette button to insert the expression.

3. Press the TAB key to move between placeholders (i.e. the square boxes) inside the expression.

4. Press CTRL+Spacebar to move outside of a placeholder or an expression.

Example 1.4. Enter $\sqrt{\frac{3}{\pi^4}}$ into a notebook.

Here is one set of instructions for entering this expression:

a) Open the BasicInput palette (it should have already opened on your screen by default).

b) Click on the palette button $\sqrt{\square}$.

c) Click on $\frac{\square}{\square}$.

d) Enter the number 3 into the highlighted top placeholder.

e) Press the TAB key to move the cursor to the bottom placeholder.

f) Click on $\square^{\square}$.

g) To insert π into the base position, click on the palette button for π.

h) Press the TAB key to move the cursor to the superscript placeholder.

i) Enter the number 4.

■ 1.6. Solving Equations

Mathematica has a host of built-in commands to help the user solve equations and manipulate expressions. The command **Solve[lhs == rhs, var]** solves the equation $\text{lhs} = \text{rhs}$ (i.e left hand side expression equals right hand side expression) for the variable var. For example, typing **Solve[x^2 − 1 == 0, x]** will solve the quadratic equation $x^2 - 1 = 0$ for x. A system of m equations in n unknowns can be solved with the command **Solve[{lhs$_1$ == rhs$_1$, lhs$_2$ == rhs$_2$, ..., lhs$_m$ == rhs$_m$}, {x_1, x_2, ..., x_n}]**. Numerical approximations of solutions can also be obtained through the command **NSolve[lhs == rhs, var]**.

There are commands to algebraically manipulate expressions: **Expand**, **Factor**, **Together**, **Apart**, **Cancel**, **Simplify**, **FullSimplify**, **TrigExpand**, **TrigFactor**, **TrigReduce**, **ExpToTrig**, **PowerExpand**, and **ComplexExpand**. These commands can also be entered from the AlgebraicManipulation palette; highlight the expression to be manipulated and click on the button corresponding to the command to be inserted.

■ Exercise 1.7

1. Evaluate the following expressions:

a) 103.41+20*76 b) $\frac{5^2+\pi}{1+\pi}$ c) $\frac{1}{1+\frac{1}{1+\frac{1}{4!}}}$ d) $\frac{2.06*10^9}{0.99*10^{-8}}$

e) What is the remainder of 1998 divided by 13?

2. Enter the following functions into *Mathematica*:

a) $f(x) = 2x^3 - 6x^2 + x - 5$ b) $g(x) = \frac{x^2-1}{x^2+1}$ c) $h(x) = \left|\sqrt{x} - 3\right|$

3. Evaluate the following functions using *Mathematica*:

a) $f(x) = 1001 + x^4$ at $x = 25$ b) $1 + \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x}$ at $x = \pi$

4. Enter the following six expressions into a notebook.

a) $\sqrt[3]{80}$ b) $\frac{\sqrt[5]{1024}}{2^{-3}}$ c) $\sqrt[3]{\sqrt{125}}$

d) $\sqrt{\sqrt[3]{10a^7 b}}$ e) $\left(\frac{x^{-3}y^4}{5}\right)^{-3}$ f) $\left(\frac{3m^{\frac{1}{6}}n^{\frac{1}{3}}}{4n^{-\frac{2}{3}}}\right)^2$.

5. Expand each of the following expressions:

a) $(x+1)(x-1)$ b) $(x+y-2)(2x-3)$

6. Factor each of the following expressions:

a) $x^3 - 2x^2 - 3x$ b) $4x^{2/3} + 8x^{1/3} + 3.6$ c) $6 + 2x - 3x^3 - x^4$

7. Simplify the following expressions using both of the commands **Simplify** and **FullSimplify**:

a) $\frac{x^2+4x-12}{3x-6}$ b) $\frac{\left(\frac{2}{x}-3\right)}{1-\frac{1}{x-1}}$ c) $(x(1-2x))^{-3/2} + (1-2x)^{-1/2}$

8. Perform the indicated operations:

a) $-\frac{1}{x} + \frac{2}{x^2+1} + \frac{1}{x^3+x}$ b) $\left(\frac{5}{y} - \frac{6}{2y+1}\right) \div \left(\frac{5}{y} + 4\right)$

9. Solve the following equations for x:

a) $x^2 - x + 1 = 0$ b) $x(1-2x)^{-3/2} + (1-2x)^{-1/2} = 0$

Chapter 2. Graphs of Functions

■ 2.1 Basic Plot

Plot[f**, {**x**,** a**,** b**}]** plots the graphs of f as a function of x with x ranging in value from a to b.

Plot[{ f_1**,** f_2**, ...,** f_N**}, {**x**,** a**,** b**}]** plots the graphs of $f_1, f_2, \ldots, f_N$ on the same set of axes.

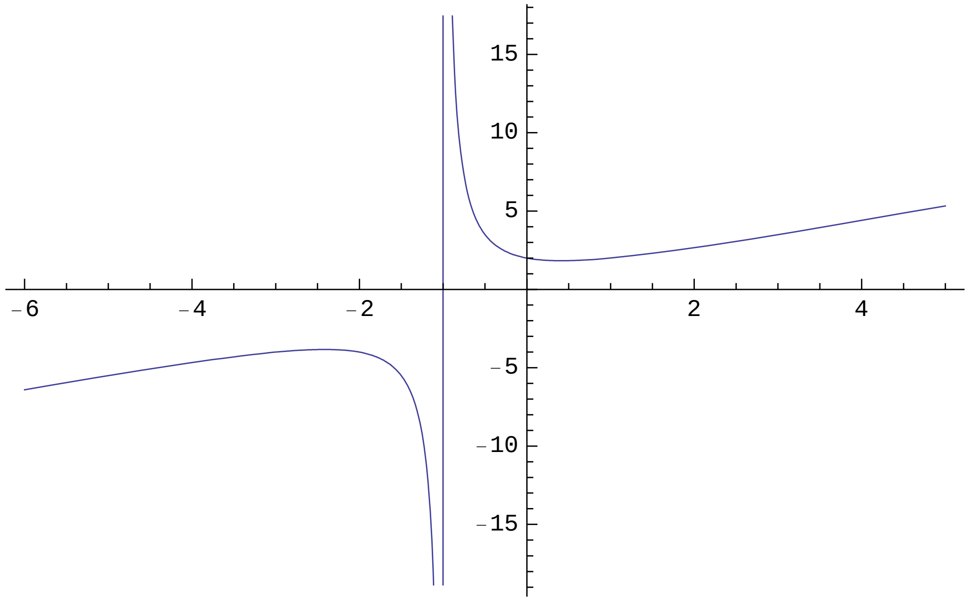
Plot[f**, {**x**,** a**,** b**}, PlotRange –> {**c**,** d**}]** plots the graph of f with a prescribed range of y-values set between c and d.

NOTE: **Plot[** f**, {**x**,** a**,** b**}]** can also be entered from the menu File/Palette/BasicCalculations/Graphics (observe the other plot versions inside this palette).

Example 2.1. Plot the graph of $\frac{x^2+x+2}{x+1}$ along the interval $[-6, 5]$.

In[21]:=
```
Plot[(x^2 + x + 2) / (x + 1), {x, -6, 5}]
```

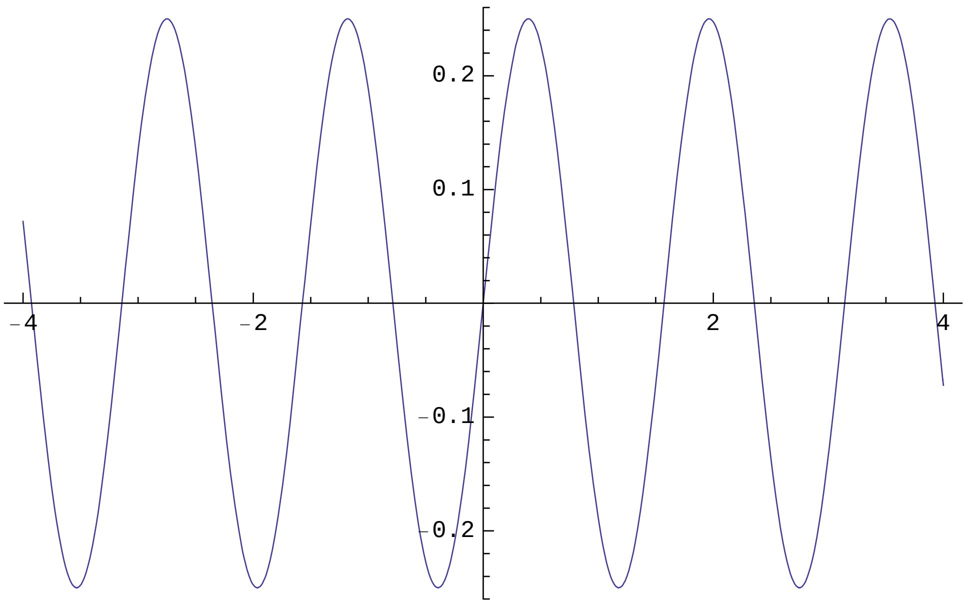
Out[21]=



Example 2.2. Plot the graph of $\frac{1}{4}\sin(4x)$ along the interval $[-4, 4]$.

In[22]:=
```
Plot[Sin[4 x] / 4, {x, -4, 4}]
```

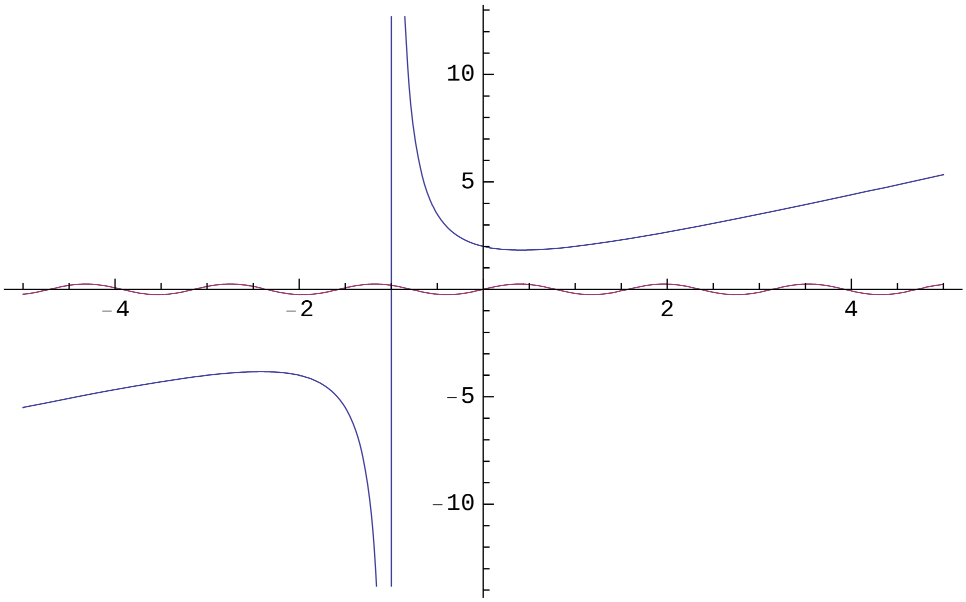
Out[22]=



Example 2.3. Plot the graphs of functions given in Examples 1 and 2 on the same set of axes.

In[23]:=
```
Plot[{(x^2 + x + 2) / (x + 1), Sin[4 x] / 4}, {x, -5, 5}]
```

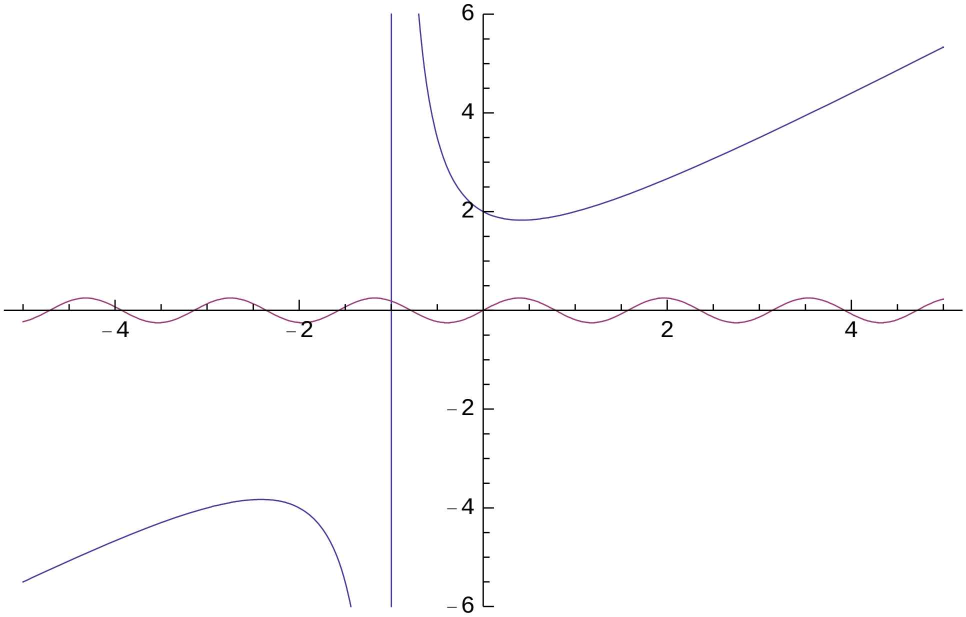
Out[23]=



Notice that the graph of $\frac{x^2+x+2}{x+1}$ is not visible in this output since the range (from -1 to 1) is too small. We can expand this range by using the **PlotRange** option:

In[24]:=
```
Plot[{(x^2 + x + 2) / (x + 1), Sin[4 x] / 4}, {x, -5, 5}, PlotRange -> {-6, 6}]
```

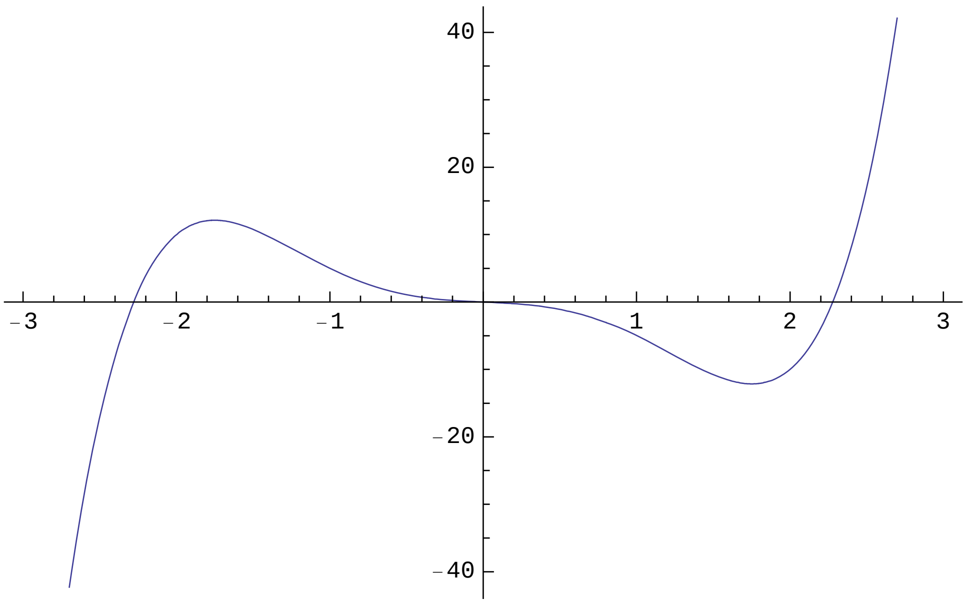
Out[24]=



Example 2.4. Draw the graph of $f(x) = x^5 - 5x^3 - x$

In[25]:=
```
Plot[x^5 - 5 x^3 - x, {x, -3, 3}]
```

Out[25]=



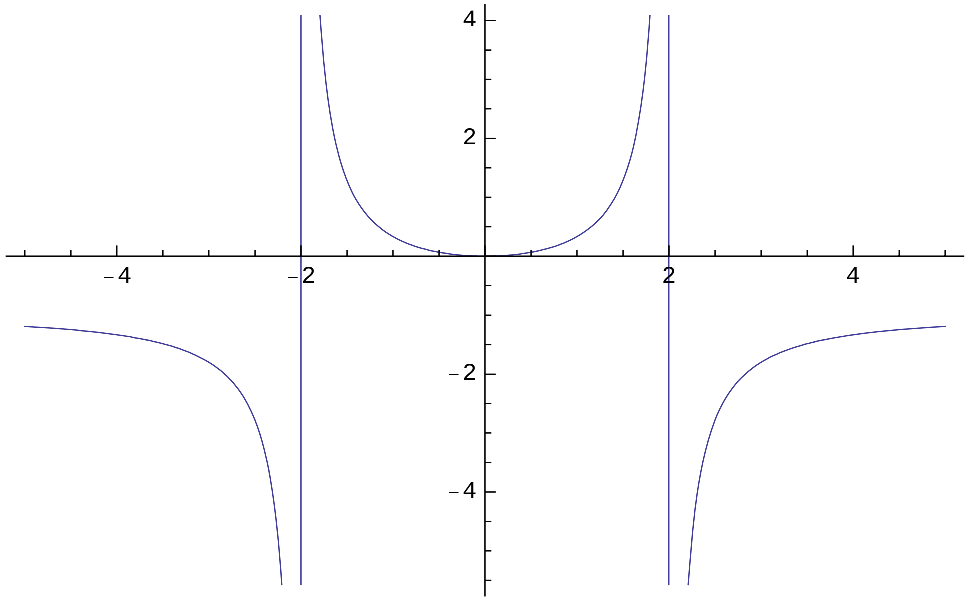
Note: Observe that *Mathematica* output of a **Plot** command is -Graphics-. To obtain the graph of the function without -Graphics-, we add a semicolon **(;)** at the end of the **Plot** command. We will do this for the next few examples.

Example 2.5. Draw the graph of $f(x) = \frac{x^2}{4 - x^2}$

In[26]:=

$$\text{Plot}\left[\frac{x^2}{4-x^2}, \{x, -5, 5\}\right]$$

Out[26]=



Notice that we have used the **BasicInput Palette** for the input. Here is a direct way of typing the input.

Plot[x^2 / (4-x^2) , {x, -5, 5}] ;

■ Exercise 2.1

1. Plot the graphs of the following functions on the specified interval:

a) $f(x) = x^2 + 1$ on $[-5, 5]$ b) $g(x) = \frac{1}{x-2}$ on $[0, 4]$ c) $h(x) = \frac{\sin x}{x}$ on $[-\pi, \pi]$

d) $f(x) = x^3 - 5x^2 + 10$ on $[-5, 5]$. e) $f(x) = \sqrt{32 - 2x^2}$ on $[-4, 4]$.

f) $f(x) = x + \frac{1}{x}$ for $[-10, 10]$.

2. Plot the graphs of $f(x) = 0.01\, x(x-3)(x+3)$ and $g(x) = 2x + 1$ together on the same set of axes and over the interval $[-20, 20]$. Use the **PlotRange** option to change the range so that their points of intersection are visible.

■ 2.2 Plot Options

In this section we will introduce various options of **Plot.** To begin with, the command **Options[Plot]** displays the following options:

In[27]:=
```
Options[Plot]
```

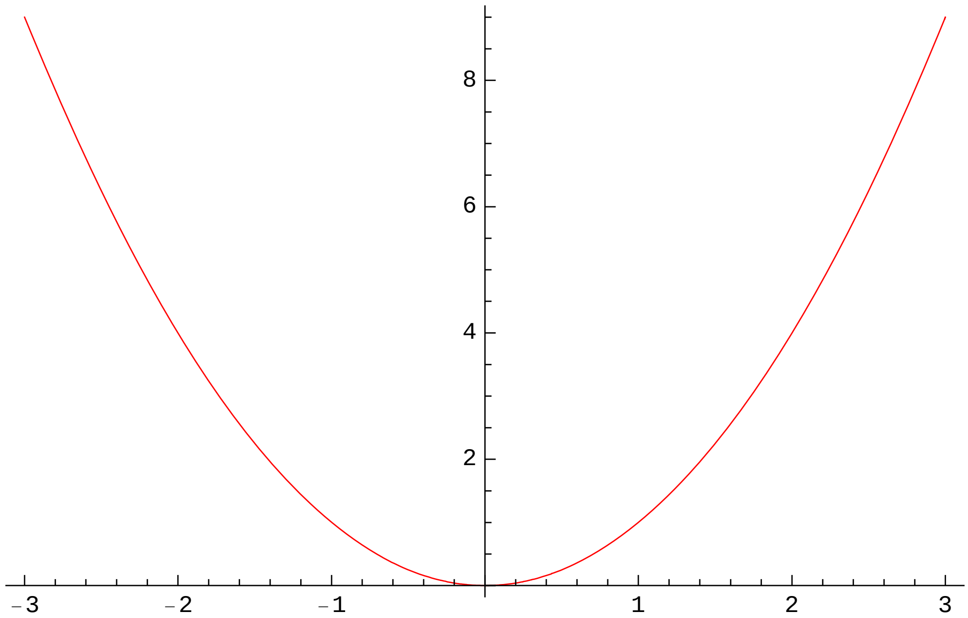
Out[27]=
```
{AlignmentPoint → Center, AspectRatio → 1/GoldenRatio, Axes → True,
 AxesLabel → None, AxesOrigin → Automatic, AxesStyle → {}, Background → None,
 BaselinePosition → Automatic, BaseStyle → {}, ClippingStyle → None,
 ColorFunction → Automatic, ColorFunctionScaling → True,
 ColorOutput → Automatic, ContentSelectable → Automatic,
 DisplayFunction :→ $DisplayFunction, Epilog → {}, Evaluated → Automatic,
 EvaluationMonitor → None, Exclusions → Automatic, ExclusionsStyle → None,
 Filling → None, FillingStyle → Automatic, FormatType :→ TraditionalForm,
 Frame → False, FrameLabel → None, FrameStyle → {}, FrameTicks → Automatic,
 FrameTicksStyle → {}, GridLines → None, GridLinesStyle → {},
 ImageMargins → 0., ImagePadding → All, ImageSize → Automatic, LabelStyle → {},
 MaxRecursion → Automatic, Mesh → None, MeshFunctions → {#1 &},
 MeshShading → None, MeshStyle → Automatic, Method → Automatic,
 PerformanceGoal :→ $PerformanceGoal, PlotLabel → None, PlotPoints → Automatic,
 PlotRange → {Full, Automatic}, PlotRangeClipping → True,
 PlotRangePadding → Automatic, PlotRegion → Automatic,
 PlotStyle → Automatic, PreserveImageOptions → Automatic, Prolog → {},
 RegionFunction → (True &), RotateLabel → True, Ticks → Automatic,
 TicksStyle → {}, WorkingPrecision → MachinePrecision}
```

2.2.1 PlotStyle

PlotStyle is an option for **Plot** that specifies the style of lines or points to be plotted. Among other things, one can use this option to specify a color of the graph and the thickness of the curve. **PlotStyle** should be followed by a minus sign(-) followed by greater than sign (>). We use **PlotStyle** for choosing a color of a plot. There are two ways of choosing colors. (1) Click on the **Input** menu and choose **Color Selector.** A box of various colors will be displayed. We click on the color of our choice and click **Ok** to complete the process. (2) Specify the color by typing **RGBColor[a, b, c]** , where **a, b,** and **c** are numbers between 0 and 1. Here is an example:

In[28]:= `Plot[x^2, {x, -3, 3}, PlotStyle → {RGBColor[1, 0, 0]}]`

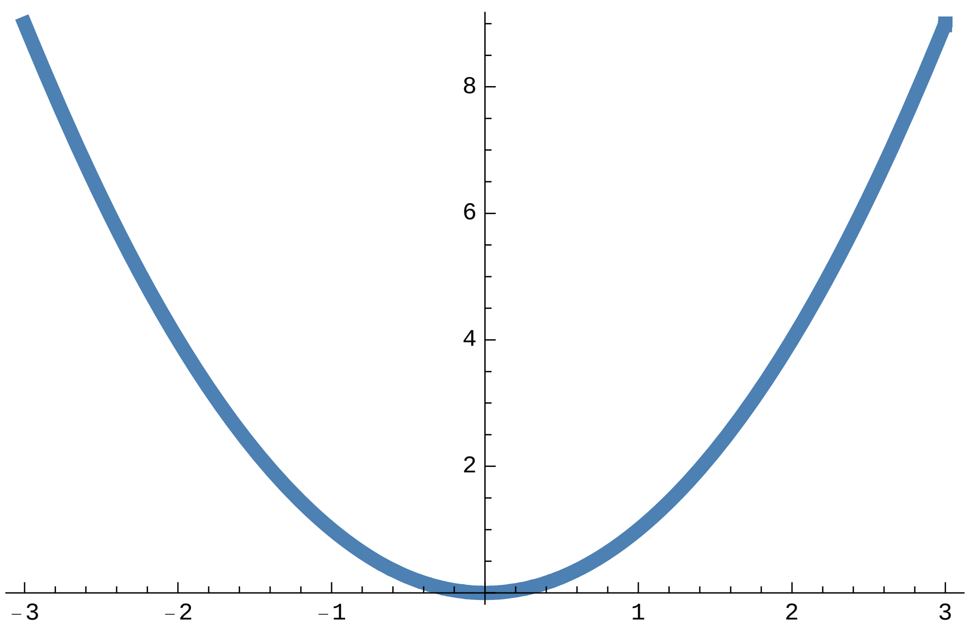
Out[28]=



The next example shows the use of **PlotStyle** with two *styles:* a color and thickness.

In[29]:= `Plot[x^2, {x, -3, 3},`
`PlotStyle → {RGBColor[0.3`, 0.500008`, 0.7`], Thickness[0.015`]}]`

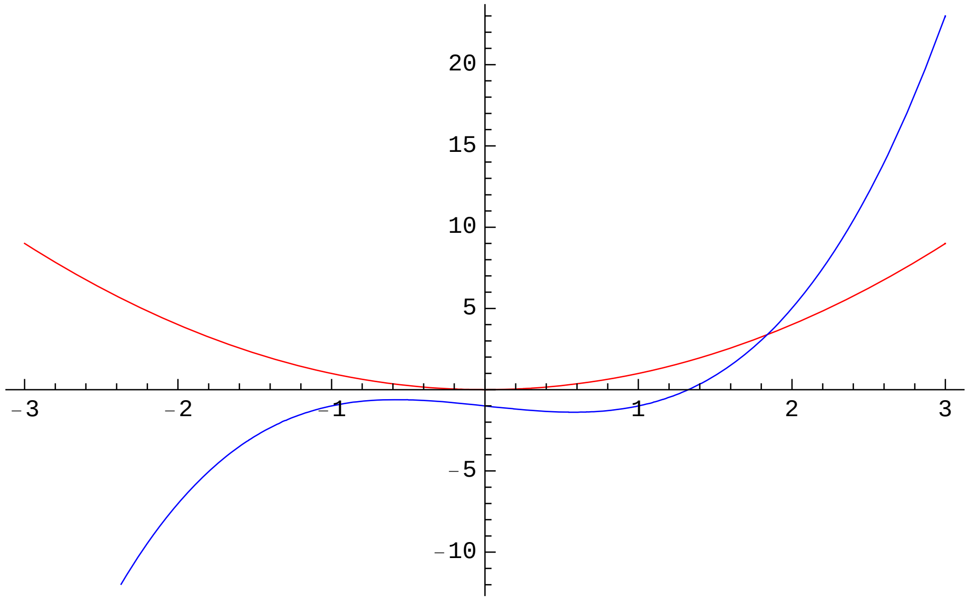
Out[29]=



PlotStyle can also be used when plotting graphs of two or more functions. Here are two examples.

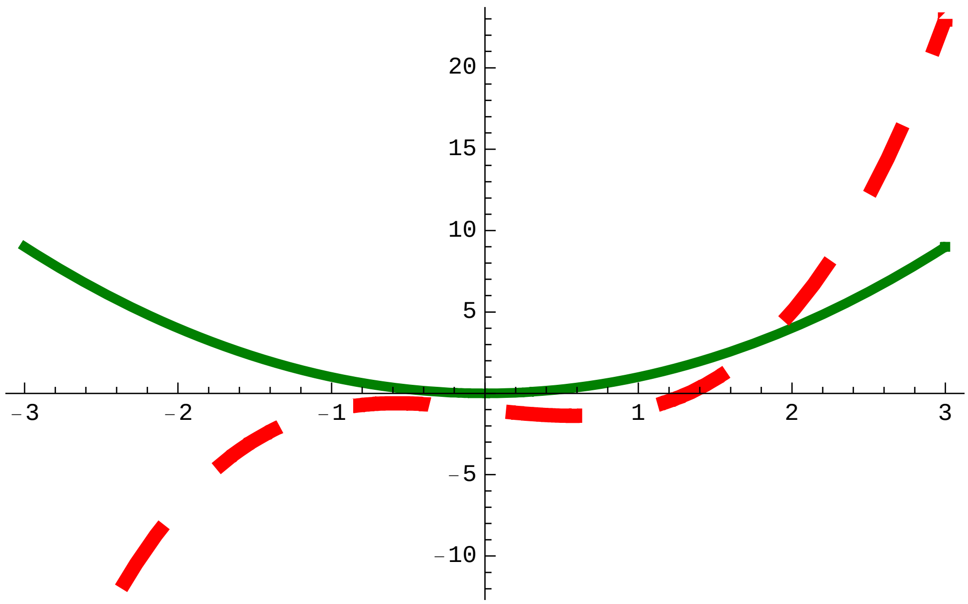
In[30]:=
```
Plot[{x^2, x^3 - x - 1}, {x, -3, 3},
 PlotStyle → {RGBColor[0.996109`, 0, 0], RGBColor[0, 0, 0.996109`]}]
```

Out[30]=



In[31]:=
```
Plot[{x^2, x^3 - x - 1}, {x, -3, 3}, PlotStyle →
  {{RGBColor[0, 0.500008`, 0], Thickness[0.01`]}, {RGBColor[0.996109`, 0, 0],
    Thickness[0.015`], Dashing[{0.08`, 0.08`, 0.08`}]}}]
```

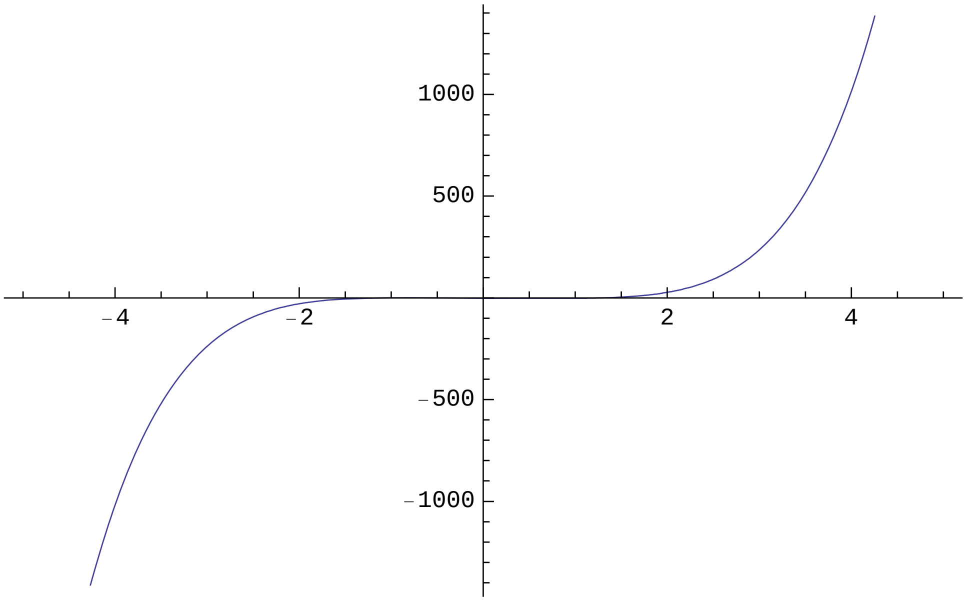
Out[31]=



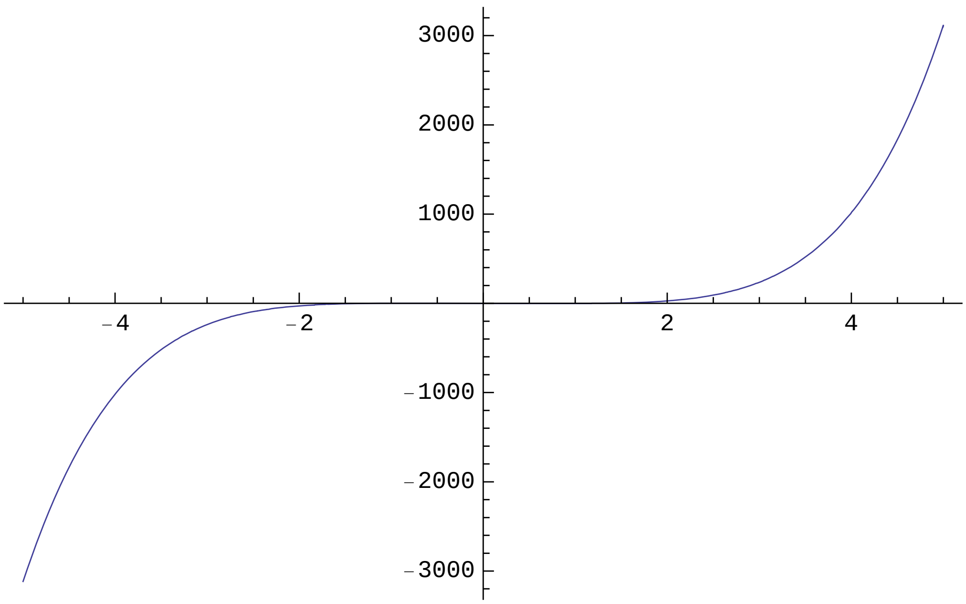
2.2.2 PlotRange

We have already used the **PlotRange** option in Section 2.1(see Example 3). This option specifies what points should be used to plot the graph. As observed in Example 3 of Section 2.1, some points of a graph may not be plotted unless we specify the y-range of the plot. The option **PlotRange** -> **All** includes all points correspond - ing to the specified values of x. Here is an example.

In[32]:=
```
Plot[x^5 - 2 x - 1, {x, -5, 5}]
```

Out[32]=


In[33]:=
```
Plot[x^5 - 2 x - 1, {x, -5, 5}, PlotRange → All]
```

Out[33]=


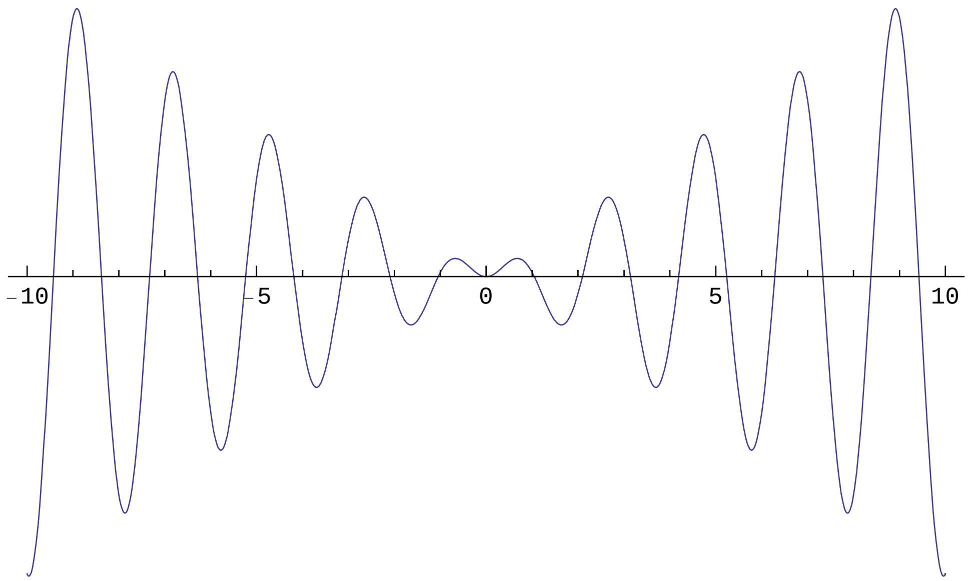
2.2.3 Axes

There are several options regarding axes. We consider four of them.

1. Axes: **Axes->True** draws both axes, **Axes->False** draws no axes, **Axes ->{True,False}** draws the x-axis only. An example of the last case is given below.

In[34]:=
```
Plot[x Sin[3 x], {x, -10, 10}, Axes → {True, False}]
```

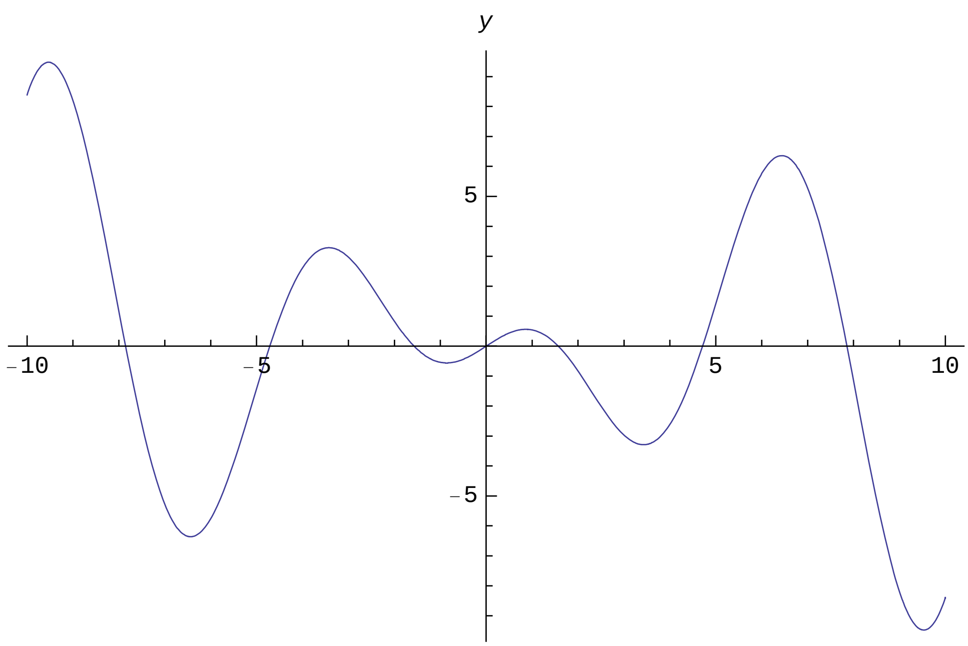
Out[34]=



2. **AxesLabel:** **AxesLabel->None** (which is the default option) does not specify the labels of the axes. **AxesLabel->y** will label the y-axis only. **AxesLabel->{ "x", "y" }** labels both axes. Examples of both cases are given below.

In[35]:=
```
Plot[x Cos[x], {x, -10, 10}, AxesLabel → y]
```

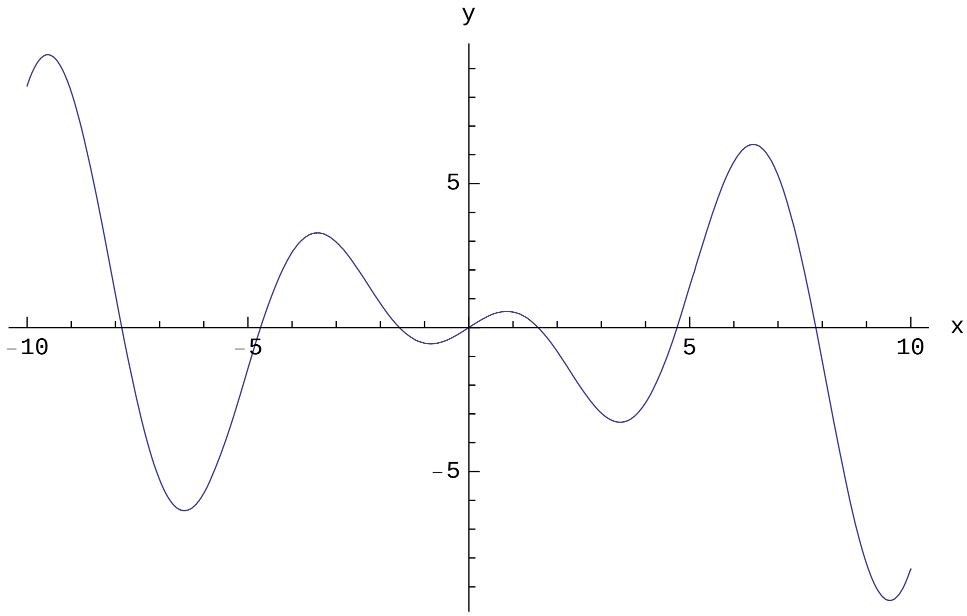
Out[35]=



In[36]:=

```
Plot[x Cos[x], {x, -10, 10}, AxesLabel → {"x", "y"}]
```

Out[36]=

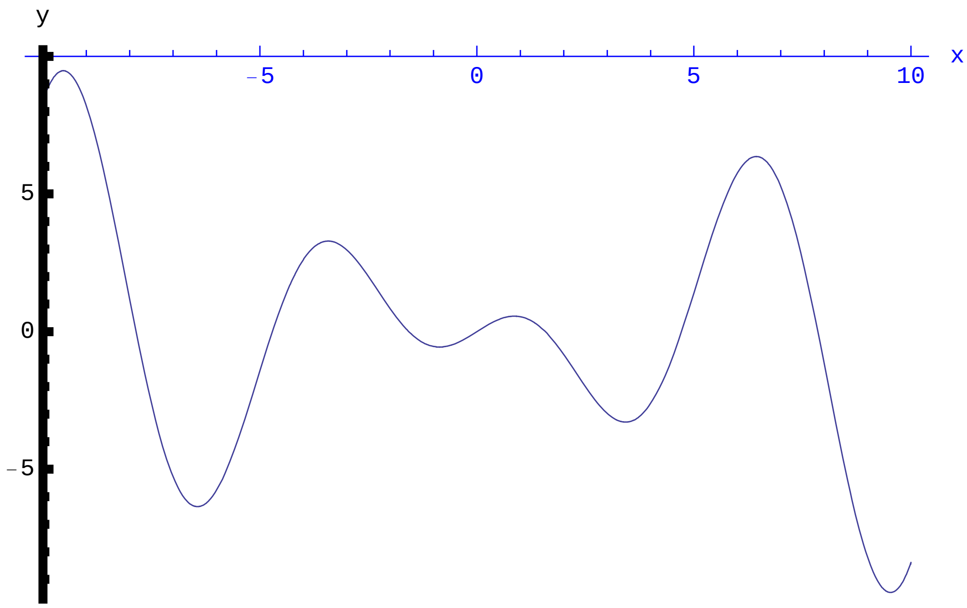


3. Origin: **AxesOrigin** specifies the location where the two axes should intersect. The default value given by **AxesOrigin->** **Automatic** chooses the intersection point of the axes based on an internal (*Mathematica)* algorithm. It usually chooses (0,0). The option **AxesOrigin->{a,b}** chooses the point **(a,b)** as the intersection point.

4. AxesStyle: This option specifies the style of the axes. Here is an example where we specify the thickness and the color of the axes. We also use the **AxesOrigin** option.

In[37]:=
```
Plot[x Cos[x], {x, -10, 10}, AxesOrigin → {-10, 10},
 AxesStyle → {RGBColor[0, 0, 0.996109`], Thickness[0.01`]},
 AxesLabel → {"x", "y"}]
```

Out[37]=

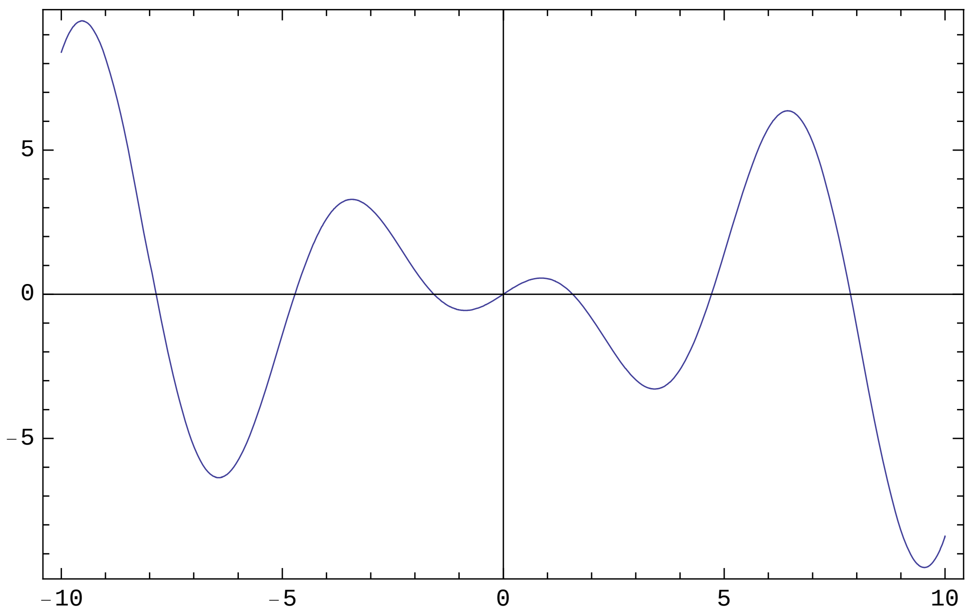


2.2.4 Frame

There are several options regarding the frame of a plot. We will show some of these by using the following examples.

In[38]:=
```
Plot[x Cos[x], {x, -10, 10}, Frame → True]
```

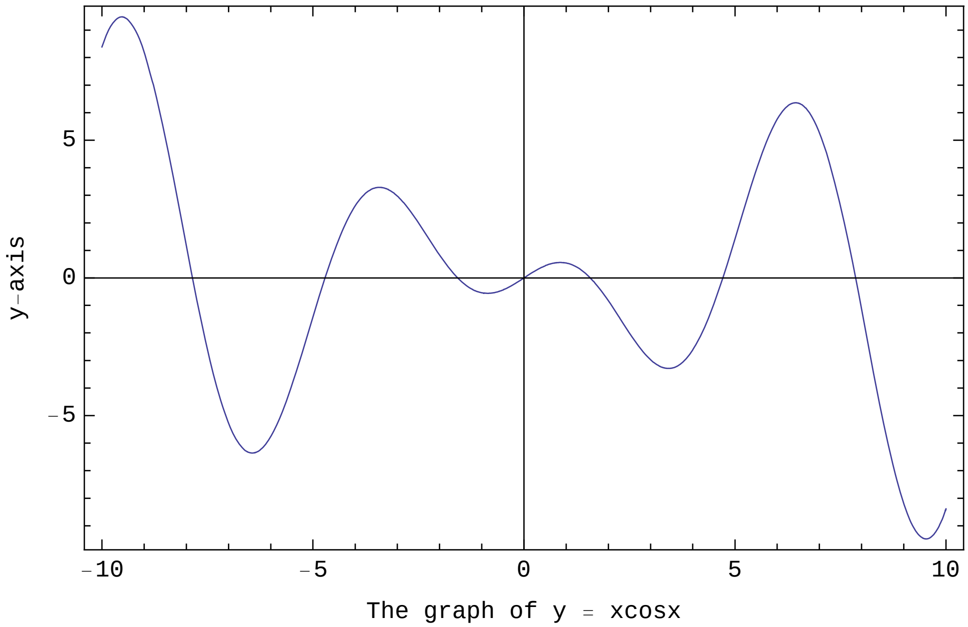
Out[38]=



In[39]:=

```
Plot[x Cos[x], {x, -10, 10}, Frame → True,
 FrameLabel → {"The graph of y = xcosx", "y-axis", None, None}]
```

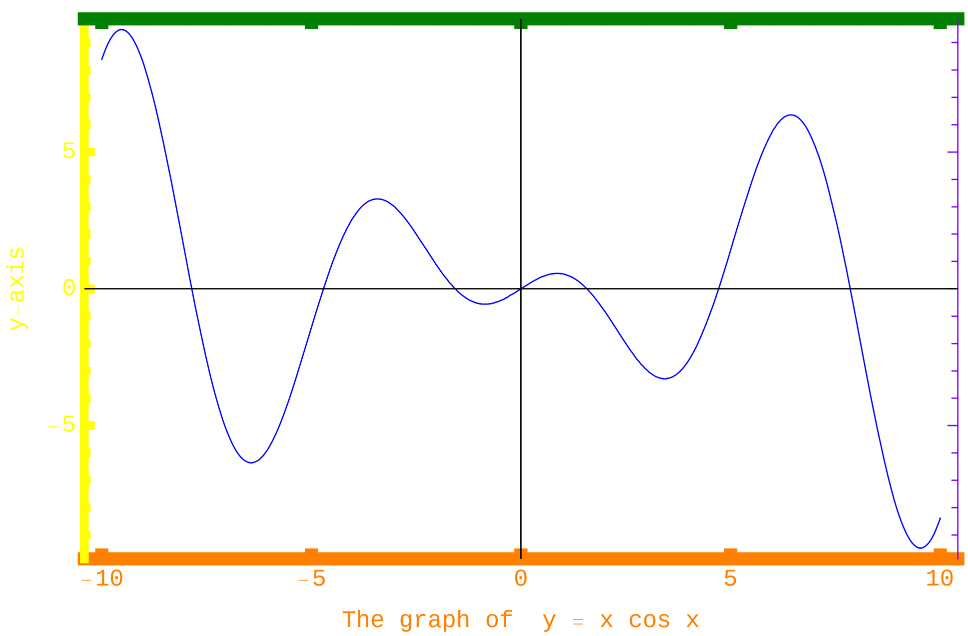
Out[39]=



In[40]:=

```
Plot[x Cos[x], {x, -10, 10},
 PlotStyle → {RGBColor[0, 0, 0.996109`]}, Frame → True,
 FrameLabel → {"The graph of  y = x cos x", "y-axis", None, None},
 FrameStyle → {{RGBColor[0.996109`, 0.5`, 0], Thickness[0.015`]},
   {RGBColor[0.996109`, 0.996109`, 0], Thickness[0.01`]},
   {RGBColor[0, 0.500008`, 0], Thickness[0.015`]},
   {RGBColor[0.500008`, 0, 0.996109`]}}]
```

Out[40]=



We recommend that the reader experiment with this example by changing the color specifications to see which option controls which color.

2.2.5 Show

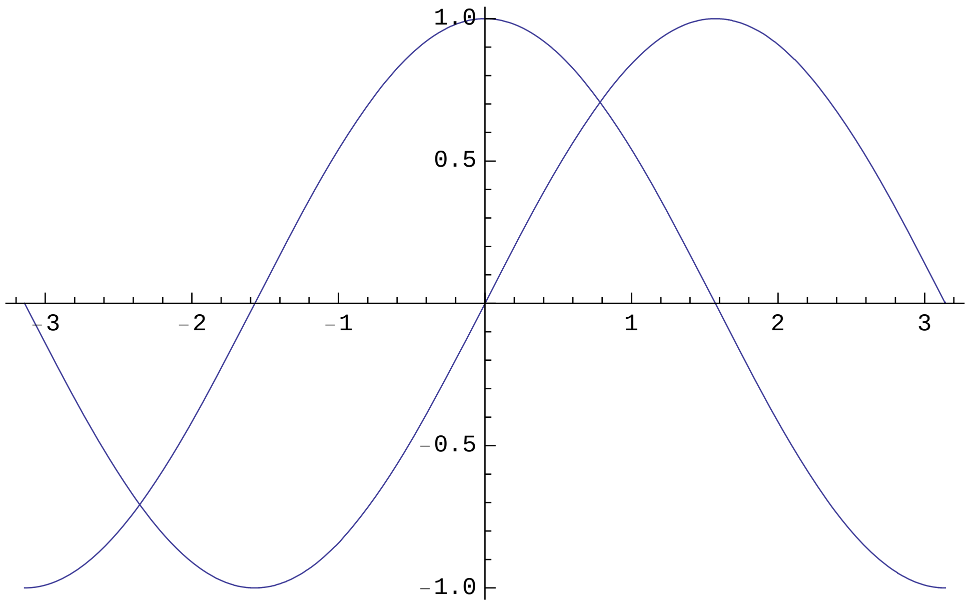
The other important option of **Plot** is **Show**. The command **Show[plot, options]** displays the graphics **plot** using the options specified by **options**. Also **Show[plot1,plot2, ...]** displays the graphics **plot1, plot2, ...** on one coordinate system. Before we give an example, we need to discuss the plot option **DisplayFunction.**
DisplayFunction -> Identity is an option that causes *Mathematica* to compute the graphics but not dispaly the graph. **DisplayFunction -> $DisplayFunction** displays the graph. Here is how it works.

```
In[41]:= plot1 = Plot[Sin[x], {x, -Pi, Pi}, DisplayFunction -> Identity];

In[42]:= plot2 = Plot[Cos[x], {x, -Pi, Pi}, DisplayFunction -> Identity];

In[43]:= Show[{plot1, plot2}, DisplayFunction → $DisplayFunction]
```

Out[43]=



■ Exercises 2.2

1. Plot the graphs of the following functions using at least one plot option discussed in this section.

Note: **ln** x *is one of the built-in Mathematica functions and is entered as* **Log[x]**. *The logarithmic function* $\log_a x$ *is entered as* **Log[a,x]**. *For the natural base* **e** *you either type* **E** *or you can obtain* *e* *from* **File-Palette-BasicInput**, *where you can also find* ∞ *among others.*

(a) $f(x) = x^4 + 2x^3 + 1$ for $-3 \le x \le 3$ (b) $f(x) = x \ln x$ for $0 \le x \le 4$

(c) $f(x) = 1 - \frac{1}{x^3} + \frac{1}{x}$ for $-20 \le x \le 20$

2. Plot the graphs of the following pairs of functions on the same axes. Use a plot style to identify the graphs.

(a) $f(x) = e^x$ and $g(x) = \ln x$

(b) $f(x) = \frac{2x}{x-5}$ and $g(x) = \frac{x-5}{2x}$

(c) $f(x) = x^2 - \sin x$ and $g(x) = \sqrt{x^4+1} - \sqrt{x^2+1}$

Chapter 3. Limits

Limit[f, x –> a] finds the limiting value of f as x approaches a. This command can also be found from the menu File/Palette/BasicCalculations/Calculus/Common Operation (notice the other operations that can be found in this way).

Limit[f, x –> a, Direction –> 1] computes the limit as x as approaches a from the left (i.e. x increases to a).

Limit[f, x –> a, Direction –> –1] computes the limit as x approaches a from the right (i.e. x decreases to a).

If the limit does not exist, then *Mathematica* will attempt to explain why or else return the limit expression unevaluated if it has insufficient information about the function.

Example 3.1. Evaluate $\lim_{x\to 1}\left(\frac{x^2+x+2}{x+1}\right)$.

In[44]:=
```
Limit[(x^2 + x + 2) / (x + 1), x -> 1]
```
Out[44]= 2

Compare the answer with the graph of this function obtained in Example 2.1.

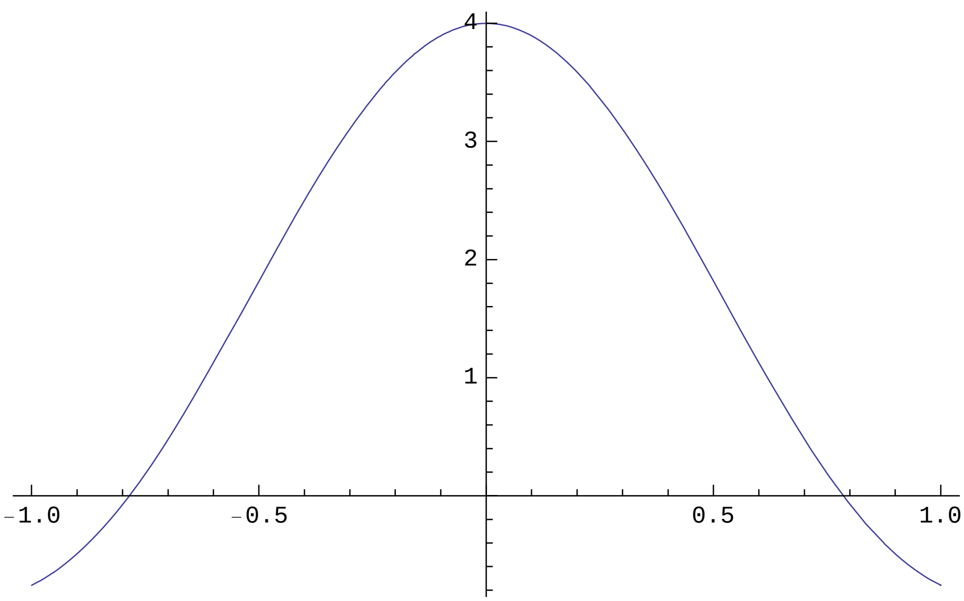
Example 3.2. Evaluate $\lim_{x\to 0}\frac{\sin(4x)}{x}$.

In[45]:=
```
Limit[Sin[4 x] / x, x -> 0]
```
Out[45]= 4

Let us check the answer by graphing the function up close at $x = 0$:

In[46]:=
```
Plot[Sin[4 x] / x, {x, -1, 1}]
```

Out[46]=



Example 3.3. Evaluate $\lim_{x\to\infty} \frac{2^x}{x}$

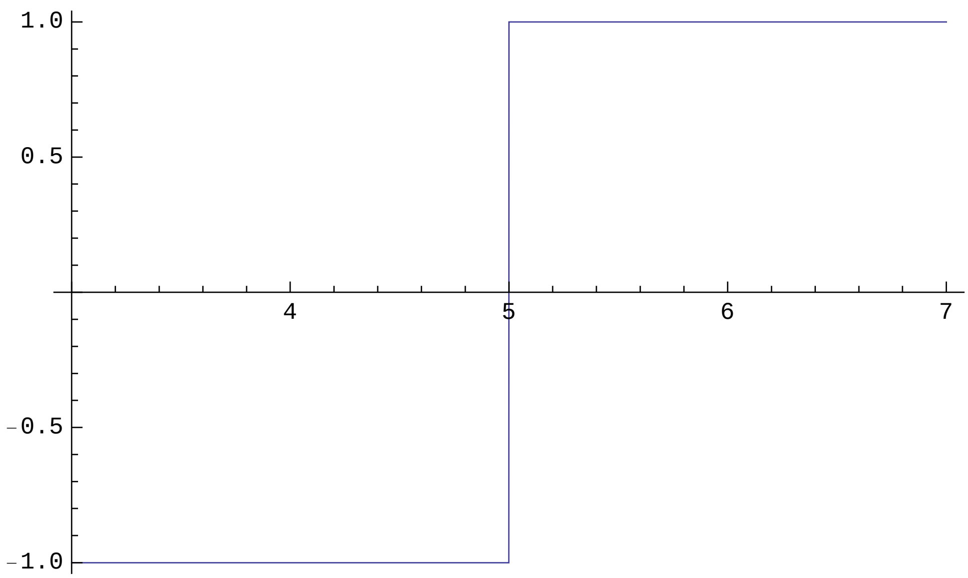
In[47]:=
```
Limit[2^x / x, x -> Infinity]
```
Out[47]= ∞

Example 3.4. Evaluate $\lim_{x\to 5^+} \frac{|x-5|}{x-5}$

In[48]:=
```
Limit[Abs[x - 5] / (x - 5), x -> 5, Direction -> -1]
```
Out[48]= 1

Again, we can check the answer by plotting the graph of the function:

In[49]:=
```
Plot[Abs[x - 5] / (x - 5), {x, 3, 7}]
```
Out[49]=



Example 3.5. Evaluate $\lim_{x\to\infty} (3x-2)\Big/\sqrt{2x^2+1}$.

In[50]:=
```
Limit[(3 x - 2) / Sqrt[2 x^2 + 1], x -> Infinity]
```
Out[50]= $\frac{3}{\sqrt{2}}$

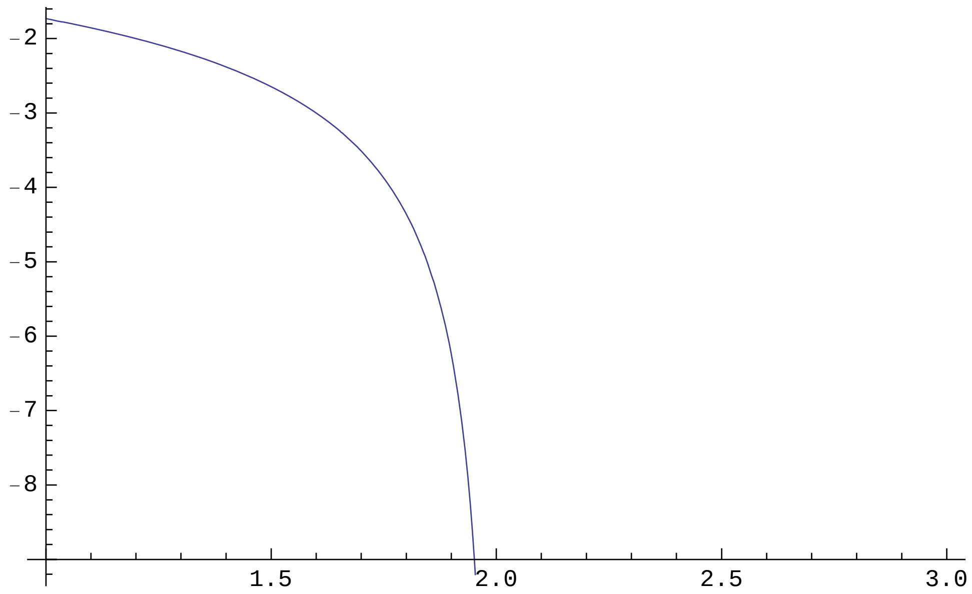
In[51]:=
```
N[%]
```
Out[51]= 2.12132

Example 3.6. Evaluate $\lim_{x\to 2^-} \frac{\sqrt{4-x^2}}{x-2}$.

In[52]:=
```
Limit[Sqrt[4 - x^2] / (x - 2), x -> 2, Direction -> 1]
```
Out[52]= $-\infty$

Again, let us plot the function to visually understand why the answer is $-\infty$.

In[53]:=
```
Plot[Sqrt[4 - x^2] / (x - 2), {x, 1, 3}]
```

Out[53]=


The messages prior to the plot informs us that *Mathematica* was unable to evaluate the function for values of x greater than 2 because this results in taking the square root of a negative number.

Example 3.7. Evaluate $\lim_{x\to\infty} \sin x$ (this is an example where the limit does not exist).

In[54]:=
```
Limit[Sin[x], x -> Infinity]
```

Out[54]= Interval[{-1, 1}]

Here, *Mathematica* is telling us that the limit does not exist by returning the range of values for $\sin x$ as x approaches infinity.

Example 3.8. Evaluate $\lim_{x \to \infty} \left(\frac{e^x}{x}\right)$

Note: For the natural base e you either type **E** or you can obtain *e* from **File-Palette-BasicInput**, where you can also find ∞ among others. **ln** x is one of the built-in *Mathematica* functions and is entered as **Log[x]**. The logarithmic function $\log_a x$ is entered as **Log[a,x].**

In[55]:=
```
Limit[ⅇ^(x)/x, x -> ∞]
```

Out[55]= ∞

Example 3.9. Evaluate $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

In[56]:=
```
Limit[(1/Log[x]) - (1/(x-1)), x->1, Direction->-1]
```

Out[56]= $\frac{1}{2}$

Again, we can graph the function near $x = 1$ to visually understand why the answer is $\frac{1}{2}$. We leave this to the student.

Let us end with an example where the **Limit** command is used to evaluate the derivative of a function (in anticipation of commands introduced in the next section for computing derivatives).

Example 3.11. Find the derivative of $f(x) = \frac{1}{x}$ according to the limit definition.

By definition, the derivative of a function $f(x)$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

In[57]:=
```
Limit[(1 / (x + Delta) - 1 / x) / Delta, Delta -> 0]
```

Out[57]=
$$-\frac{1}{x^2}$$

■ Exercise 3.1

1. Compute the following limits:

a) $\lim_{x \to 1} x^2 - \frac{1}{x-1}$ b) $\lim_{x \to -5} \frac{100}{x+5}$ c) $\lim_{x \to \infty} \frac{1+x+x^2}{\sqrt[3]{x^{10}-x}}$ d) $\lim_{x \to 0} \frac{\sin x}{x}$

2. Find the derivative of $f(x) = \cos x$ according to the limit definition and evaluate the answer at $x = \pi$.

3. Evaluate each of the following limits:

(a) $\lim_{x \to 2} \left(\frac{2x-1}{4-3x}\right)$ (b) $\lim_{x \to 0^+} \left(\frac{1-\ln x}{e^{\frac{1}{x}}}\right)$

(c) $\lim_{x \to 0^+} \left(\frac{1}{x} - \ln x\right)$ (d) $\lim_{x \to \left(\frac{\pi}{2}\right)^-} (\sec 3x \cos 5x)$

(e) $\lim_{x \to 0^+} (\sin x)^{1/x}$

Chapter 4. Differentiation

■ 4.1 The Derivative

All the rules of differentiation are programmed into *Mathematica*. This makes it easy to differentiate complicated expressions. The command **D[*f*, *x*]** calculates the derivative of the function f with respect to the variable x. You can obtain higher order derivatives of order n by using the command **D[*f*, {*x*, *n*}]**. Again, all these commands are available from the palette menu File/Palette-BasicCalculations/Common Operations. The symbol for the derivative of a function of one variable from the palette is $\partial_{\square}\,\square$.

Example 4.1. Find the derivative of $x \sin x$ and evaluate it at $x = \frac{\pi}{4}$.

There are a number of ways to compute the derivative. We demonstrate five methods, including how to evaluate expressions by using the substitution command **/. *x* -> *a***.

Method 1:

```
In[58]:= D[x * Sin[x], x]
         D[x * Sin[x], x] /. x -> Pi / 4
Out[58]= x Cos[x] + Sin[x]
```

$$\text{Out[59]=}\quad \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$$

```
In[60]:= N[%]
Out[60]= 1.26247
```

Method 2: Click on the palette button $\partial_{\square}\,\square$ (the symbol ∂ traditionally refers to partial differentiation, but just think of it as the symbol $\frac{d}{dx}$ for now), then enter x in the lower box for the variable of differentiation, press the Tab key to move between boxes and enter the function into the upper box.

```
In[61]:= ∂x (x * Sin[x]) /. x -> π / 4
```

$$\text{Out[61]=}\quad \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$$

Method 3: If the function $x \sin x$ and its derivative need to be referred to often, then it is more convenient to label them, say by f and $D f$, when performing the computation:

In[62]:=
```
f = x * Sin[x] (* This defines an expression in Mathematica *)
Df = D[f, x]
Df /. x -> Pi / 4
```

Out[62]= $x \operatorname{Sin}[x]$

Out[63]= $x \operatorname{Cos}[x] + \operatorname{Sin}[x]$

Out[64]= $$\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$$

Method 4: If the function $x \sin x$ will be evaluated for different values of x, then labeling it as $f[x]$ by using the argument x is more suitable. In this case, the derivative can easily be obtained by evaluating $f'[x]$ (*Mathematica* is programmed to understand prime notation for derivatives):

In[65]:=
```
Clear[f]
f[x_] := x * Sin[x]
f'[x]
f'[Pi / 4]
```

Out[67]= $x \operatorname{Cos}[x] + \operatorname{Sin}[x]$

Out[68]= $$\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$$

Example 4.2. Differentiate $\frac{x}{(x^2+4)^{1/3}}$.

In[69]:= $$\partial_x \frac{x}{(x^2+4)^{\frac{1}{3}}}$$

Out[69]= $$-\frac{2x^2}{3(4+x^2)^{4/3}} + \frac{1}{(4+x^2)^{1/3}}$$

In[70]:=
```
Together[%]
```

Out[70]= $$\frac{12+x^2}{3(4+x^2)^{4/3}}$$

Example 4.3. Differentiate $\arcsin x + x\sqrt{1-x^2}$.

In[71]:=
```
D[ArcSin[x] + x Sqrt[1 - x^2], x]
```

Out[71]= $$\frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$

Example 4.4. Find the second derivative of $x\sqrt{x^2-1}$.

In[72]:=
```
D[x * Sqrt[x^2 - 1], {x, 2}]
```

Out[72]= $$\frac{2x}{\sqrt{-1+x^2}} + x\left(-\frac{x^2}{(-1+x^2)^{3/2}} + \frac{1}{\sqrt{-1+x^2}}\right)$$

Recall that we can also use prime notation to obtain the second derivative as follows:

In[73]:=
```
g[x_] := x Sqrt[x^2 - 1]
g''[x]
```

Out[74]= $-\frac{x^3}{\left(-1+x^2\right)^{3/2}}+\frac{3\,x}{\sqrt{-1+x^2}}$

Example 4.5. (Implicit Differentiation) Find an equation of the line tangent to the graph of $x^2\left(x^2+y^2\right)=y^2$ at the point $P\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

The solution here boils down to finding the slope of the tangent line at P, which can be found by evaluating the derivative $\frac{dy}{dx}$ at $(x, y)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

In[75]:=
```
Clear[eq, deq, x, y] (* This clears needed arguments *)
eq = (x^2 (x^2 + y[x]^2) == y[x]^2)
```

Out[76]= $x^2\left(x^2+y[x]^2\right)==y[x]^2$

In[77]:=
```
deq = D[eq, x]
```

Out[77]= $2\,x\left(x^2+y[x]^2\right)+x^2\left(2\,x+2\,y[x]\,y'[x]\right)==2\,y[x]\,y'[x]$

In[78]:=
```
soln = Solve[deq, y'[x]]
```

Out[78]= $\left\{\left\{y'[x]\to\frac{-2\,x^3-x\,y[x]^2}{\left(-1+x^2\right)y[x]}\right\}\right\}$

Lastly, the slope can be found by evaluating the output at the coordinates of P.

In[79]:=
```
m = soln /. {x -> Sqrt[2] / 2, y[x] -> Sqrt[2] / 2}
```

Out[79]= $\left\{\left\{y'\left[\frac{1}{\sqrt{2}}\right]\to 3\right\}\right\}$

The slope of 3 here can now be substituted into the slope-intercept equation of a line:

In[80]:=
```
Clear[x, y]
y = 3 (x - Sqrt[2] / 2) + Sqrt[2] / 2
```

Out[81]= $\frac{1}{\sqrt{2}}+3\left(-\frac{1}{\sqrt{2}}+x\right)$

In[82]:=
```
Expand[%]
```

Out[82]= $-\sqrt{2}+3\,x$

The equation of the tangent line at P is therefore $y=3\,x-\sqrt{2}$.

■ Exercises 4.1

1. Compute the derivatives of the following functions:

a) $f(x) = 3x^2 + 1$ b) $g(x) = \frac{1}{x^3}$ c) $h(x) = \frac{\sin x}{\cos x}$

2. Evaluate the derivatives of the following functions at the specified values of x:

a) $f(x) = (x-1)(x+1)$ at $x = 1$ b) $g(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$ at $x = 9$

3. Compute the second derivatives of the functions given in Exercise 2.

4. Find an equation of the line tangent to the graph of $x - y^2 = 0$ at the point $P(9, -3)$.

5. Find all relative extrema of the function $f(x) = x^4 - 7x^2 + 10$.

6. Differentiate the following

(a) $y = \sqrt[4]{3x-5}$ (b) $y = \frac{1}{\sqrt{\cos(x^2)}}$

(c) $y = e^{x \ln x}$ (d) $y = \ln(\cos(e^x))$

(e) $y = e^x \operatorname{arcsec} x$ (f) $y = \sec^{-1} x + \csc^{-1} x$

(g) $y = \sinh(1 - x^2)$ (h) $y = x \tanh^{-1} x + \ln\sqrt{1 - x^2}$

7. Find $\frac{dy}{dx}$ if $x^2 y + 3xy^2 - x = 3$. Also find the equation of the tangent line to the graph of the equation at the point $(1, 1)$.

8. Let $f(x) = e^{\sin x}$ for $-2\pi \le x \le 2\pi$

(a) Define f in *Mathematica*

(b) Find the solutions of $f'(x) = 0$. (Hint: Use **FindRoot**)

(c) Identify the relative extrema of f.

(d) Find the inflection points of f.

(e) Verify your answers in (b), (c), and (d) by drawing the graph of f.

4.2 Applications of Derivative

Example 4.6. Use the second derivative test to find the relative extrema of $f(x) = -3x^5 + 5x^3$.

Recall that the second derivative test involves evaluating the second derivative $f''(x)$ at the critical points of $f(x)$, i.e. points where the derivative is zero. Therefore, we begin by solving the equation $f'(x) = 0$ for x.

In[83]:=
```
Clear[f]
f[x_] := -3 x^5 + 5 x^3
Solve[f'[x] == 0, x]
```
Out[85]=
```
{{x → -1}, {x → 0}, {x → 0}, {x → 1}}
```

Let us evaluate $f''(x)$ at these critical values: $x = -1,\ 0,\ 1$.

In[86]:=
```
f''[{-1, 0, 1}]
```
Out[86]=
```
{30, 0, -30}
```

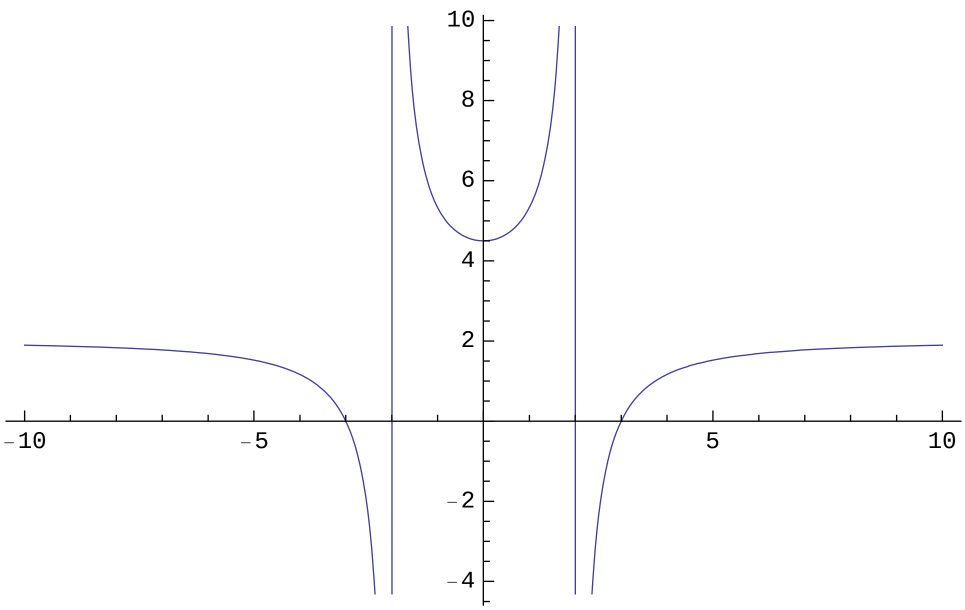
Hence, there is a relative minimum at $x = -1$ since $f''(-1) > 0$ and a relative maximum at $x = 1$ since $f''(1) < 0$. What happens at $x = 0$? Try to use the first derivative test or plot the graph of $f(x)$.

Example 4.7. Determine all horizontal or vertical asymptotes of $f(x) = 2\left(\frac{x^2-9}{x^2-4}\right)$.

We begin our analysis by plotting the graph of $f(x)$:

In[87]:=
```
f[x_] := 2 (x^2 - 9) / (x^2 - 4)
Plot[f[x], {x, -10, 10}]
```

Out[88]=



To get a precise hold of where the horizontal asymptotes are, we need to compute $\lim_{x \to \pm\infty} f(x)$ (recall that a function $f(x)$ has a horizontal asymptote at $y = c$ if either $\lim_{x \to +\infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$) :

In[89]:=
```
Limit[f[x], x -> Infinity]
Limit[f[x], x -> -Infinity]
```
Out[89]= 2

Out[90]= 2

Therefore, there is a horizontal asymptote at $y = 2$. Next, the vertical asymptotes can be found at points that are infinite discontinuities of $f(x)$ (recall that a function $f(x)$ has a vertical asymptote at $x = d$ if either $\lim_{x \to d^+} f(x) = \pm\infty$ or $\lim_{x \to d^-} f(x) = \pm\infty$. As $f(x)$ is a rational function, this reduces the work to determining the zeros of its denominator, $x^2 - 4$:

In[91]:=
```
Solve[x^2 - 4 == 0, x]
```
Out[91]= {{x → -2}, {x → 2}}

Indeed, the graph of $f(x)$ reveals that there should be vertical asymptotes at $x = 2$ and $x = -2$. We can verify this mathematically by computing

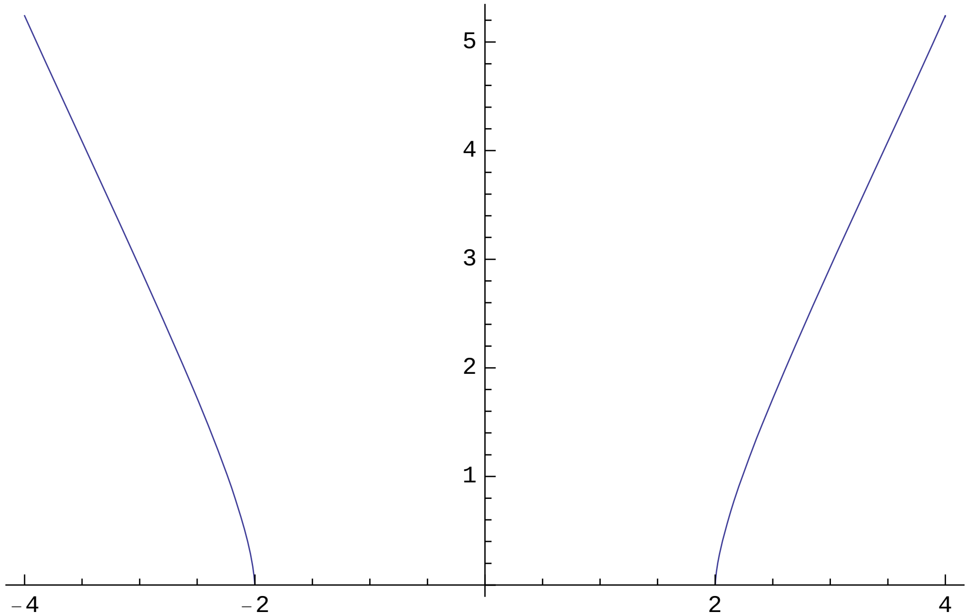
In[92]:=
```
Limit[f[x], x -> -2, Direction -> 1]
Limit[f[x], x -> 2, Direction -> 1]
```
Out[92]= $-\infty$

Out[93]= ∞

Example 4.8. Sketch the graph of $f(x) = (x^2 - 4)^{2/3}$ and determine its relative extrema (if any exists at all).

Again, we begin our analysis by plotting the graph of $f(x)$:

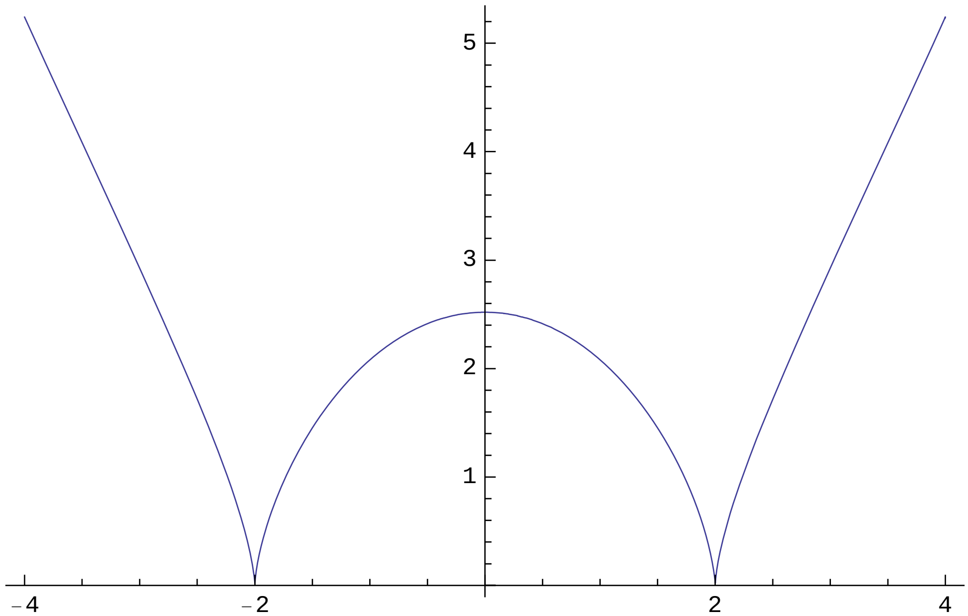
In[94]:=
```
Clear[f]
f[x_] := (x^2 - 4)^(2/3)
Plot[f[x], {x, -4, 4}]
```
Out[96]=



The output messages here tell us that *Mathematica* cannot evaluate $f(x)$ on the open interval $(-2, 2)$ because the base $x^2 - 4 < 0$ there; this makes $f(x)$ ill-defined as an *exponential complex* function. Therefore, we should explain to *Mathematica* that $f(x)$ should be treated as a *real radical* function $\sqrt[3]{(x^2-4)^2}$ and ask it to plot $f(x)$ again with the function rewritten in radical form:

In[97]:=
```
f[x_] := ∛((x² - 4)²)
Plot[f[x], {x, -4, 4}]
```

Out[98]=



The relative extrema of $f(x)$ can now be found by using the first derivative test. If $x = c$ is a critical point of $f(x)$, then the test works as follows:

(1) If $f'(x)$ changes from negative to positive at $x = c$, then $f(x)$ has a relative minimum at $x = c$, and (2) If $f'(x)$ changes from positive to negative at $x = c$, then $f(x)$ has a relative maximum at $x = c$.

From the graph, it seems that the critical points where $f'(x)$ is undefined are at $x = -2, 2.$ We can verify this by evaluating $f'(x)$ at these values:

In[99]:=
```
f'[-2]
f'[2]
```

```
Power::infy : Infinite expression 1/0^(2/3) encountered. >>

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. >>
```

Out[99]= Indeterminate

```
Power::infy : Infinite expression 1/0^(2/3) encountered. >>

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. >>
```

Out[100]= Indeterminate

Now, the critical points where $f'(x)$ vanishes can be found with the Solve command:

In[101]:=
```
Solve[f'[x] == 0, x]
```
Out[101]=
```
{{x → 0}}
```

Based on the graph, we can conclude that $f(x)$ has relative minima at $x = -2, 2$ since the derivative changes from negative to positive there and a relative maximum at $x = 0$ by a similar argument.

■ Exercises 4.2

1 . Determine all horizontal and vertical asymptotes of $y = 5\left(\frac{1}{x-4} - \frac{1}{x+2}\right)$.

2 . Determine all relative extrema of $f(x) = \frac{2x^2-3x+5}{x^2+1}$.

Chapter 5. Integration

■ 5.1 The Integral

Integrate[*f*, *x*] gives the indefinite integral of f with respect to x. On the other hand,

Integrate[*f*, {*x*, *a*, *b*}] gives the definite integral of f on the interval $[a, b]$.

NIntegrate[*f*, {*x*, *a*, *b*}] gives a numerical approximation of the definite integral of f on $[a, b]$.

The command Integrate can evaluate all rational functions and a host of transcendental functions, including exponential, logarithmic, trigonometric and inverse trigonometric functions. It can also be obtained from the palette menu File/Palette/BasicCalculations/Calculus/Common Operations.

Example 5.1. Evaluate $\int x(x^2+1)^2 \, dx$.

Method 1: (Palette buttons)

In[102]:=
$$\int x\,(x^2+1)^2\,dx$$

Out[102]=
$$\frac{x^2}{2}+\frac{x^4}{2}+\frac{x^6}{6}$$

Note: *Mathematica* does not explicitly include the constant of integration C in its answers for indefinite integrals; the user should always include the constant of integration.

Method 2:

In[103]:=
```
Integrate[x (x^2 + 1)^2, x]
```

Out[103]=
$$\frac{x^2}{2}+\frac{x^4}{2}+\frac{x^6}{6}$$

Question: if the substitution method with $u = x^2 + 1$ is used to solve this integral, then the answer becomes $\frac{1}{6}\left(1+x^2\right)^3$. How does one reconcile this answer with the one obtained in the output above?

Example 5.2. Evaluate $\int \frac{x^2}{\left(x^2+1\right)^2}\, dx$.

In[104]:=
```
Integrate[x^2 / (x^2 + 1)^2, x]
```

Out[104]=
$$-\frac{x}{2\left(1+x^2\right)}+\frac{\text{ArcTan}[x]}{2}$$

Example 5.3. Evaluate $\int x^2 \sin x \, dx$.

In[105]:= $\int x^2 \,\text{Sin}[x]\, dx$

Out[105]= $-\left(-2+x^2\right)\text{Cos}[x]+2\,x\,\text{Sin}[x]$

Example 5.4. Evaluate $\int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$.

In[106]:= $\int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$

Out[106]= $\frac{16}{3}$

Example 5.5. Evaluate $\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}\, dx$.

In[107]:= `Integrate[Sqrt[x^2 - 3] / x, {x, Sqrt[3], 2}]`

Out[107]= $1-\frac{\pi}{2\sqrt{3}}$

In[108]:= `Simplify[%]`

Out[108]= $1-\frac{\pi}{2\sqrt{3}}$

In[109]:= `N[%]`

Out[109]= 0.0931003

Example 5.6. Approximate $\int_0^1 \tan x^2\, dx$.

Here is an example of an integral that *Mathematica* will not evaluate but return the integral unevaluated because the precise answer is not easily expressible in simple form. However, a numerical approximation is still possible through the command **N**.

In[110]:= `Integrate[Tan[x^2], {x, 0, 1}]`

Out[110]= $\int_0^1 \text{Tan}\left[x^2\right] dx$

In[111]:= `N[%]`

Out[111]= 0.398414

Or we can just use the command **NIntegrate** to perform both steps at once:

In[112]:= `NIntegrate[Tan[x^2], {x, 0, 1}]`

Out[112]= 0.398414

■ Exercises 5.1

1 . Evaluate the following integrals:

a) $\int\left(x^2+2\right)dx$ b) $\int\cos 3x\,dx$ c) $\int_0^1\sqrt{1-x^2}\,dx$ d) $\int_{-\pi}^{\pi}\sin^2 x\,dx$

2. Integrate each of the following. Simplify your answers.

(a) $\int\frac{x^5+3x^4-2x^2+1}{x^3}\,dx$ (b) $\int\frac{1}{1+\sin x}\,dx$

(c) $\int\sec^2 5x\tan^3 5x\;dx$ (d) $\int e^{3x}\cos 2x\;dx$

3. Evaluate the following integrals.

(a) $\int_0^3\left(x^3-4x^2+x\right)dx$ (b) $\int_1^4\left(\frac{1}{\sqrt{x}}+2\sqrt{x}\right)dx$

(c) $\int_0^{\frac{\pi}{4}}\sec x\;dx$ (d) $\int_0^{\frac{\sqrt{2}}{4}}\frac{2}{\sqrt{1-4x^2}}\,dx$

■ 5.2 Applications and Improper Integrals

Example 5.7. Compute the arclength of the graph of $y=\ln(\cos x)$ on the interval $\left[0,\frac{\pi}{4}\right]$.

Recall that the formula for the arclenth s of a function $f(x)$ on the interval $[a,b]$ is given by

$$s=\int_a^b\sqrt{1+[f'(x)]^2}\;dx.$$

We enter this formula into *Mathematica* to obtain

In[113]:=
```
Clear[f]
f[x_] := Log[Cos[x]]
s = Integrate[Sqrt[1 + f'[x]^2], {x, 0, Pi/4}]
```

Limit::ztest : Unable to decide whether numeric quantities

$$\left\{-\frac{1}{8\left(1+\frac{i(\ll1\gg)}{\text{Plus}[\ll2\gg]}\right)^2}-\frac{3}{4\left(e^{-\ll1\gg}+e^{\ll1\gg}\right)^4\left(1+\frac{i\ll1\gg}{\ll1\gg}\right)^2}+\frac{e^{-\frac{\ll1\gg}{4}}}{2\ll1\gg^4(\ll1\gg)^2}+\ll5\gg+\ll1\gg-\frac{3i\ll1\gg}{\ll1\gg}-\frac{i\,e^{-\frac{3\ll1\gg\ll1\gg\pi}{8}}}{4(\ll1\gg)^3\left(1+\frac{i(\ll1\gg+\ll1\gg)}{\text{Plus}[\ll2\gg]}\right)}+\ll3\gg\right\}$$

are equal to zero. Assuming they are. ≫

Out[115]= $2\,\text{ArcTanh}\left[\text{Tan}\left[\frac{\pi}{8}\right]\right]$

In[116]:= `N[%]`

Out[116]= 0.881374

The arclength is therefore $S = 0.881374$.

Example 5.8. Evaluate $\int \frac{x^5+x^2+x+2}{(x^2+1)^2}\,dx$

Solution:

In[117]:= $\int \frac{x^5+x^2+x+2}{(x^2+1)^2}\,dx$

Out[117]= $\frac{1}{2}\left(x^2+\frac{-2+x}{1+x^2}+3\,\text{ArcTan}[x]-2\,\text{Log}\left[1+x^2\right]\right)$

Note: All functions in an output appear in *Mathematica*'s notation. To convert a function in an output to a more familiar form the command is **TraditionalForm**. For the above example the "traditional" form is obtained as follows. (Note that $\log x$ is the same as $\ln x$.)

In[118]:= $\int \frac{x^5+x^2+x+2}{(x^2+1)^2}\,dx$ `// TraditionalForm`

Out[118]//TraditionalForm=

$$\frac{1}{2}\left(x^2+3\tan^{-1}(x)-2\log\left(x^2+1\right)+\frac{x-2}{x^2+1}\right)$$

Example 5.9. Evaluate the improper integral: $\int_1^{\infty}(1-x)\,e^{-x}\,dx$

In[119]:= $\int_1^{\infty}(1-x)\,e^{-x}\,dx$

Out[119]= $-\frac{1}{e}$

Example 5.10. Determine the convergence of the improper integral: $\int_0^1 \frac{1}{x^3}\,dx$

In[120]:=
$$\int_0^1 \frac{1}{x^3}\,dx$$

Integrate::idiv : Integral of $\frac{1}{x^3}$ does not converge on {0, 1}. >>

Out[120]=
$$\int_0^1 \frac{1}{x^3}\,dx$$

The above output shows that the integral is divergent.

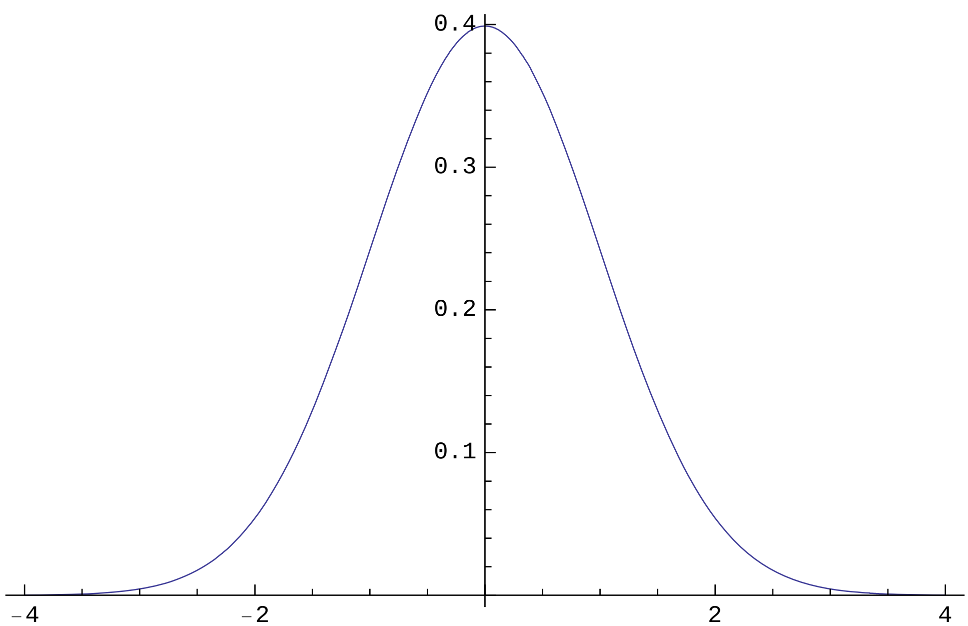
Example 5.11. Show that the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ has points of inflection at $x = -1$ and $x = 1$. Also find the total area of the region under the curve and above the x-axis. (This function is very useful in probablity and statistics. The graph is called the normal (or bell) curve.

Solution: We will first define a function that represents the given function using *Mathematica* as follows.

In[121]:=
```
Clear[f]
```
$$f[x\_] := \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

Here is the graph of f.

In[123]:=
```
Plot[f[x], {x, -4, 4}]
```

Out[123]=


To find the inflection points we enter:

In[124]:=
```
Solve[f''[x] == 0, x]
```

Out[124]=
```
{{x → -1}, {x → 1}}
```

We leave it to the student to verify that $x = -1$ and $x = 1$ are indeed inflection points. To find the area of the region in question we enter:

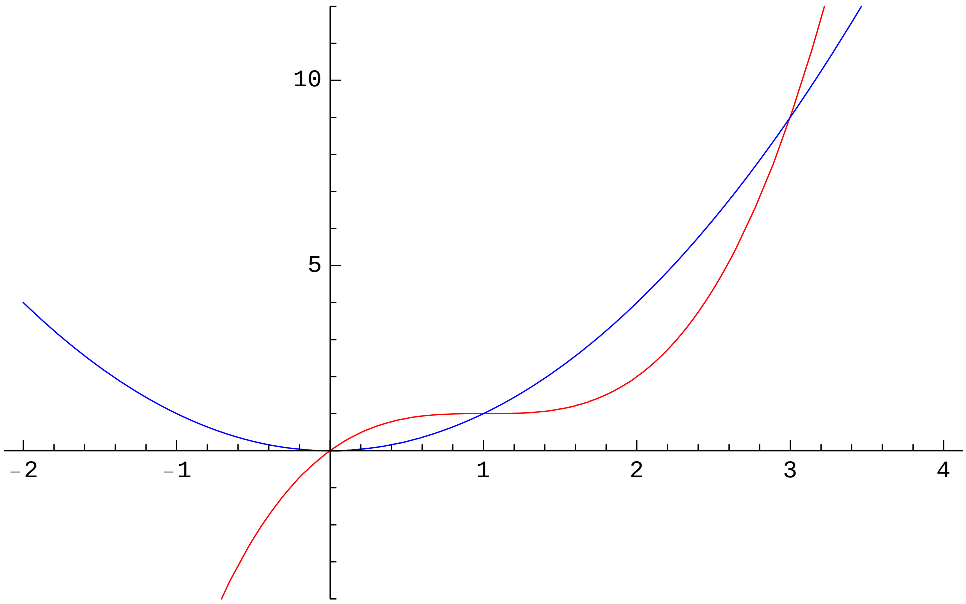
In[125]:=
```
Integrate[f[x],{x,-Infinity,Infinity}]
```
Out[125]= 1

Example 5.12: Sketch the region enclosed by the graphs of $f(x) = x(x^2 - 3x + 3)$ and $g(x) = x^2$ and find the area of the region.

Solution: We define these functions in *Mathematica* and draw their graphs on the same coordinate system. Then we will find the intersection points of the graphs and identify the region. Finally, we evaluate the appropriate integrals.

In[126]:=
```
Clear[f, g]
f[x_] := x (x^2 - 3 x + 3)
    g[x_] := x^2
```

In[129]:=
```
Plot[{f[x], g[x]}, {x, -2, 4},
 PlotStyle → {{RGBColor[0.996109`, 0, 0]}, {RGBColor[0, 0, 0.996109`]}},
 PlotRange → {-4, 12}]
```

Out[129]=


In[130]:=
```
Solve[f[x] == g[x], x]
```
Out[130]= {{x → 0}, {x → 1}, {x → 3}}

In[131]:=
```
Integrate[f[x] - g[x], {x, 0, 1}] + Integrate[g[x] - f[x], {x, 1, 3}]
```
Out[131]= $\frac{37}{12}$

■ Exercises 5.2

1. Approximate the area under the curve $y = \sqrt{1 - x^2}$ and above the interval $[0, 1]$.

2. Compute the arclength of the curve in Exercise 1 along the interval $[0, 1]$.

3. Compute the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

4. Find the area of the region enclosed by the curves $y = x^3 - 2x^2,\ y = 2x^2 - 3x,\ x = 0,\ x = 3.$ Also skecth

the region.

In[132]:=

5. Evaluate the following improper integrals :

(a) $$\int_1^{\infty} \frac{\ln x}{x^3}\, dx$$ (b) $$\int_0^1 \frac{2}{x\sqrt{1+4x^2}}\, dx$$

Chapter 6. Sequences and Series

■ 6.1 Sequences

We define a sequence a_n exactly the way we define a function. Thus, instead of a_n we use a(n). The limit of a sequence may be obtained by using the command **Limit** . When **Limit[a[n], n->∞]** is used, *Mathematica* automatically assumes that **n** is a real number(instead of a positive integer). However, for an arbitrary sequence, **a[n]** may not be well-defined if **n** is a real number other than a positive integer. In such cases *Mathematica* returns the unevaluated limit. The command **NLimit[a[n], n->∞]** gives a numerical approximation of the limit. Before we use **NLimit** we need to download the package **NumericalMath`NLimit.** Here are some examples:

Example 6.1. Find the limit of the following sequences.

(a) $a_n = \frac{4n+1}{3n-100}$ (b) $b_n = \frac{(-1)^n}{n}$

(c) $c_n = \cos\left(\frac{n\pi}{2}\right)$ (d) $d_n = \left(1 + \frac{a}{n}\right)^n$

Solution: We will use the letter corresponding to the sequence to define a sequence in *Mathematica*. Thus, a_n will be defined as **a[n]**.

In[133]:=

```
Clear[a]
a[n_] := (4 n + 1)/(3 n - 100)
```

In[135]:=

```
Limit[a[n], n -> Infinity]
```

Out[135]= $\frac{4}{3}$

In[136]:=
```
Clear[b]
b[n_] := (-1)^n / n
```

In[138]:=
```
Limit[b[n], n -> Infinity]
```

Out[138]= 0

In[139]:=
```
<< "NumericalCalculus`"
```

In[140]:=
```
NLimit[b[n], n -> Infinity]
```

Out[140]= $-3.76335\times10^{-6} + 3.67562\times10^{-16}\, i$

The above output suggests that the limit is zero. To show that this is true, we use the fact that if $f(x)$ is a function such that $a_n = f(n)$ for all positive integers n, then $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{n\to\infty} a_n = L$. We also recall that $\lim_{x\to\infty} f(x) = 0$ if and only if $\lim_{x\to\infty} |f(x)| = 0$

In[141]:=
```
Limit[Abs[b[n]], n -> Infinity]
```

Out[141]= 0

Thus, $\lim_{n\to\infty} b_n = 0$

In[142]:=
```
Clear[c]
c[n_] := Cos[n π / 2]
```

In[144]:=
```
Limit[c[n], n -> Infinity]
```

Out[144]= Interval[{-1, 1}]

Thus, $\lim_{n\to\infty} c_n$ does not exist.

In[145]:=
```
Clear[d]
d[n_] := (1 + a/n)^n
```

In[147]:=
```
Limit[d[n], n -> Infinity]
```

Out[147]= e^a

■ Exercises 6.1

1. Find the limit of the following sequences.

(a) $a_n = \frac{\sqrt{n}}{\ln n}$ (b) $b_n = \frac{1+(-1)^n}{n}$

(c) $c_n = \sin\left(\frac{n\pi}{6}\right)$ (d) $d_n = n\,e^{-n^2}$

■ 6.2 Series

Sum[a_n, **{n,** n_1, n_2**}]** is the sum of a_n as **n** goes from n_1 to n_2. Using the **BasicInput Palette**, we can also enter this as $\sum_{n=n_1}^{n_2} a_n$. Similarily we enter the infinite series $\sum_{n=n_1}^{\infty} a_n$.

Example 6.2. Find the following finite sums

(a) $\sum_{n=1}^{10} \frac{(-1)^n}{n}$ (b) $\sum_{k=0}^{5} (k+1)(k-1)$

(c) $\sum_{k=0}^{20} \binom{20}{k} k^2$ (d) $\sum_{j=1}^{n} j$

Solution: We will use the letters corresponding to the question for the sum. The binomial coefficient $\binom{n}{m}$ is given by **Binomial[n, m].**

In[148]:= $a = \sum_{n=1}^{10} \frac{(-1)^n}{n}$

Out[148]= $-\frac{1627}{2520}$

In[149]:= $b = \sum_{k=0}^{5} (k+1)(k-1)$

Out[149]= 49

In[150]:= $c = \sum_{k=0}^{20} \text{Binomial}[20, k]\, k^2$

Out[150]= 110 100 480

In[151]:= $d = \sum_{j=1}^{n} j$

Out[151]= $\frac{1}{2} n (1 + n)$

Example 6.3. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Let S_n be its nth partial sum.

(a) Find S_{100}. (b) Compute every 10th partial sum up to n=100.

Solution: First we define S_n. For (b) the command we need is **Table[s[n] , {n, 1, J, k }],** which gives the list of values **s[1], s[1+k], s[1+2k], s[1+3k], ... , s[1+m*k]** , where **1+m*k** is the largest integer less than or equal to **J** . **N[Table[s[n] , {n, 1, J, K }]]** gives the list in decimals and **TableForm[N[Table[s[n] , {n, 1, J, K }]]]** lists the values in a column. We will use all the three commands.

In[152]:= **Clear[s]**

$s[n\_] := \sum_{j=1}^{n} \frac{1}{j}$

In[154]:= **a = s[100]**

Out[154]= $\frac{14466636279520351160221518043104131447711}{2788815009188499086581352357412492142272}$

In[155]:= **N[%]**

Out[155]= 5.18738

In[156]:= **b = Table[s[n], {n, 10, 100, 10}]**

Out[156]= $\left\{\frac{7381}{2520}, \frac{55835135}{15519504}, \frac{9304682830147}{2329089562800}, \frac{2078178381193813}{485721041551200}, \frac{13943237577224054960759}{3099044504245996706400}, \frac{15117092380124150817026911}{3230237388259077233637600}, \frac{42535343474848157886823113473}{8801320137209899102584580800}, \frac{4880292608058024066886120358155997}{982844219842241906412811281988800}, \frac{3653182778990767589396015372875328285861}{718766754945489455304472257065075294400}, \frac{14466636279520351160221518043104131447711}{2788815009188499086581352357412492142272}\right\}$

In[157]:= **b = N[Table[s[n], {n, 10, 100, 10}]]**

Out[157]= {2.92897, 3.59774, 3.99499, 4.27854, 4.49921, 4.67987, 4.83284, 4.96548, 5.08257, 5.18738}

In[158]:=
```
b = TableForm[N[Table[s[n], {n, 10, 100, 10}]]]
```
Out[158]//TableForm=
```
2.92897
3.59774
3.99499
4.27854
4.49921
4.67987
4.83284
4.96548
5.08257
5.18738
```

Example 6.4. Determine whether the series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ c) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

d) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(\text{Log}[n+1])^3}$ e) $\sum_{n=0}^{\infty} r^n$

Solution: We will use the letter of the question for the solution. Thus, in an input $e = \sum_{n=0}^{\infty} r^n$ is the solution of e).

In[159]:= $a = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Out[159]= $-\text{Log}[2]$

In[160]:= $b = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

Out[160]= $\frac{1}{e}$

In[161]:= $c = \sum_{n=1}^{\infty} \frac{n^n}{n!}$

Sum::div : Sum does not converge. ≫

Out[161]= $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Note that *Mathematica* returned the last series unevaluated and without comment. This means that *Mathematica* could not decide whether the series is divergent or not. We apply the Ratio Test as follows.

In[162]:=
```
Clear[a]
```
$a[n\_] := \frac{n^n}{n!}$

In[164]:=
```
r = NLimit[a[n + 1] / a[n], n -> ∞ ]
```
Out[164]= 2.71828

Thus, the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by Ratio Test.

In[165]:= $d = \sum_{n=1}^{\infty} \frac{1}{(n+1)\,(\text{Log}[n+1])^3}$

Out[165]= $\sum_{n=1}^{\infty} \frac{1}{(1+n)\,\text{Log}[1+n]^3}$

The above series is convergent. But its closed form is unknown. In such cases, *Mathematica* can be used to approximate the nth partial sum of the series.

In[166]:=
```
Clear[a, s]
```
$a[n\_] := \frac{1}{(n+1)\,(\text{Log}[n+1])^3}$

$s[n\_] := \sum_{j=1}^{n} a[j]$

In[169]:= $r = \text{NLimit}\left[\frac{a[n+1]}{a[n]}, n \to \infty\right]$

Out[169]= 1.

Thus, the Ratio Test is inconclusive. Let us try the Integral Test. First observe that $a(x)$ is a positive valued function. To verify that it is decreasing, we evaluate

In[170]:=
```
a'[x]
```
Out[170]= $-\frac{3}{(1+x)^2\,\text{Log}[1+x]^4} - \frac{1}{(1+x)^2\,\text{Log}[1+x]^3}$

In[171]:=
```
Together[%]
```
Out[171]= $\frac{-3 - \text{Log}[1+x]}{(1+x)^2\,\text{Log}[1+x]^4}$

Clearly $a'(x) < 0$ for all $x > 1$. Thus we can apply the Integral Test. We will use the command **NIntegrate[a[x], { x ,1, ∞ }]** .

In[172]:=
```
NIntegrate[a[x], {x, 1, ∞}]
```
Out[172]= 1.04068

Thus, by the Integral Test the series is convergent.

In[173]:=
```
Clear[r]
```
$e = \sum_{n=0}^{\infty} r^n$

Out[174]= $\frac{1}{1-r}$

The above example shows that we must be careful when using *Mathematica* to determine the convergence of a series. However, as the following two examples show, when we specify the common ratio of the geometric series, we get the expected result.

In[175]:= $\sum_{n=0}^{\infty} 2^n$

Sum::div : Sum does not converge. >>

Out[175]= $\sum_{n=0}^{\infty} 2^n$

This shows that that the series $\sum_{n=0}^{\infty} 2^n$ is divergent.

In[176]:= $\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$

Out[176]= $\frac{5}{4}$

■ Exercises 6.2

1 . Find the following finite sums

(a) $\sum_{n=1}^{100} \frac{n}{n^2+1}$ (b) $\sum_{k=0}^{15} \frac{k^3}{n^3}$

(c) $\sum_{k=0}^{100} \binom{100}{k} (-1)^k$ (d) $\sum_{j=1}^{n} (2j-1)$

2 . Consider the geometric series $\sum_{n=1}^{\infty} (0.3)^{n-1}$. Let S_n be its nth partial sum.

(a) Find S_{100}. (b) Compute every 10th partial sum up to n=100.

3 . Determine whether the series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

c) $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$ d) $\sum_{n=1}^{\infty} \frac{n!}{n\, 3^n}$

6.3 Power Series

The Taylor polynomial of a given function f at a point x_0 is given by the command **Series[f , { x, x_0, n}]**, which gives the nth Taylor polynomial $P_n(x)$ plus a term of the form $O[x]^{n+1}$. To get the exact Taylor polynomial the command is **Normal[Series[f , { x, x_0, n}]].**

The command to obtain the Taylor series of a function f at a point x_0 is the same as that of its Taylor polynomial. Thus, *Mathematica* can only give the first "few" terms of the Taylor series of a function.

SeriesCoefficient[s, n] gives the coefficient of the **nth** power of the series **s**.

Example 6.5. Let f(x) $(= e)^x$. Find its 5th Taylor polynomial at x = 0

In[177]:=
```
Series[E^x, {x, 0, 5}]
```

Out[177]=
$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O[x]^6$$

In[178]:=
```
Normal[Series[E^x, {x, 0, 5}]]
```

Out[178]=
$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}$$

Example 6.6. Find the nth coefficient of the Taylor series of $f(x)$ at $x = c$

(a) $f(x) = \sqrt{x}, \quad c = 1, \quad n = 6$

(b) $f(x) = \sin x, \quad c = \frac{\pi}{2}, \quad n = 10$

Solution: Note that we must first find the Taylor series of these functions at the given points and that we must evaluate the series to the order greater than or equal to the given value of n. However, unless we want to, we do not have to see the actual power series of the function. For the first function we will give both the series and the desired coefficient and for the second we calculate the coefficient only.

In[179]:=
```
a = Series[√x , {x, 1, 8}]
```

Out[179]=
$$1+\frac{x-1}{2}-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3-\frac{5}{128}(x-1)^4+\frac{7}{256}(x-1)^5-\frac{21(x-1)^6}{1024}+\frac{33(x-1)^7}{2048}-\frac{429(x-1)^8}{32768}+O[x-1]^9$$

In[180]:=
```
a6 = SeriesCoefficient[Series[√x , {x, 1, 8}], 6]
```
Out[180]= $-\frac{21}{1024}$

In[181]:=
```
b = SeriesCoefficient[Series[Sin[x], {x, π/2, 12}], 10]
```
Out[181]= $-\frac{1}{3628800}$

Example 6.7. Let $f(x) = \frac{3 + \sin x}{2 + x^2}$.

a) Find the Taylor polynomials $P_n(x)$ of f at $x = 0$ for $n = 1, 2, \ldots 6.$

b) Draw the graphs of the function f and its Taylor Polynomials found in (a)

c) Use the graph in (b) to estimate the radius of convergence of the Taylor series of f at $x = 0$

Solution: Recall that **Normal[Series[f[x], {x,0, n}]]** is the command to obtain the **n** th Taylor polynomial of f at 0. We will use the command **Table[Normal[Series[f[x], {x,0, n}]] ,{n, 0, 6}]** to generate the first 6 Taylor polynomials of f. For a better visualization of these polynomials we will use the command **TableForm.** The simplest way to draw the graphs of the Taylor polynomials is by using the **Evaluate** command. To use this command we let **a=Table[Normal[Series[f[x], {x,0, n}]] ,{n, 0, 6}]** Then **Plot[Evaluate[a], {x, -2, 2}]** plots the graphs of the polynomials. To make the comparison between the graph of f and its Taylor polynomials we will use the **PlotStyle** option for the graph of f. Finally to estimate the radius of convergence of the Taylor series of f, we use **Show** to draw the two graphs on the same axis.

In[182]:=
```
Clear[f]
f[x_] := (3 + Sin[x])/(2 + x^2)
```

In[184]:=
```
a = Table[Normal[Series[f[x], {x, 0, n}]], {n, 0, 6}];
```

In[185]:=
```
TableForm[%]
```

Out[185]//TableForm=

$\frac{3}{2}$

$\frac{3}{2} + \frac{x}{2}$

$\frac{3}{2} + \frac{x}{2} - \frac{3x^2}{4}$

$\frac{3}{2} + \frac{x}{2} - \frac{3x^2}{4} - \frac{x^3}{3}$

$\frac{3}{2} + \frac{x}{2} - \frac{3x^2}{4} - \frac{x^3}{3} + \frac{3x^4}{8}$

$\frac{3}{2} + \frac{x}{2} - \frac{3x^2}{4} - \frac{x^3}{3} + \frac{3x^4}{8} + \frac{41x^5}{240}$

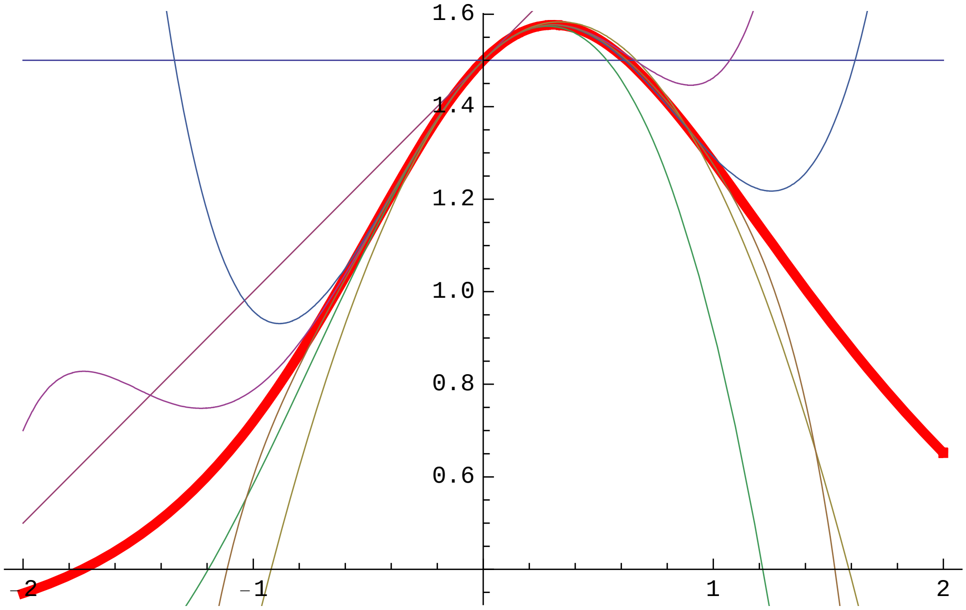
$\frac{3}{2} + \frac{x}{2} - \frac{3x^2}{4} - \frac{x^3}{3} + \frac{3x^4}{8} + \frac{41x^5}{240} - \frac{3x^6}{16}$

In[186]:=
```
b1 = Plot[f[x], {x, -2, 2},
    PlotStyle -> {RGBColor[0.996109, 0, 0], Thickness[0.01]},
    DisplayFunction -> Identity];
```

In[187]:=
```
b2 = Plot[Evaluate[a], {x, -2, 2},
        DisplayFunction -> Identity];
```

In[188]:=
```
Show[b1, b2, DisplayFunction → $DisplayFunction]
```

Out[188]=



This clearly suggests that the radius of convergence is 1. We leave it to the student to redo the above example with larger values of n to confirm this and find the interval of convergence..

■ Exerciese 6.3

1 . Let $f(x) = x^2 \cos x$. Find its 5th Taylor polynomial at $x = \pi$.

2 . Find the nth coefficient of the Taylor series of $f(x)$ at $x = c$

(a) $f(x) = \cos(\pi x^2), \quad c = 1, \quad n = 6$

(b) $f(x) = \dfrac{1}{\sqrt{1 + \sin x}}, \quad c = \frac{\pi}{2}, \quad n = 10$

3 . Let $f(x) = \dfrac{3}{\sqrt{4 + x^2}}$.

a) Find the Taylor polynomials $P_n(x)$ of f at $x = 0$ for $n = 1, 2, \ldots 6$.

b) Draw the graphs of the function f and its Taylor Polynomials found in (a)

c) Use the graph in (b) to estimate the radius of convergence of the Taylor series of f at $x = 0$

Chapter 7. Parametric Equations and Polar Graphs

■ 7.1 Parametric Equations

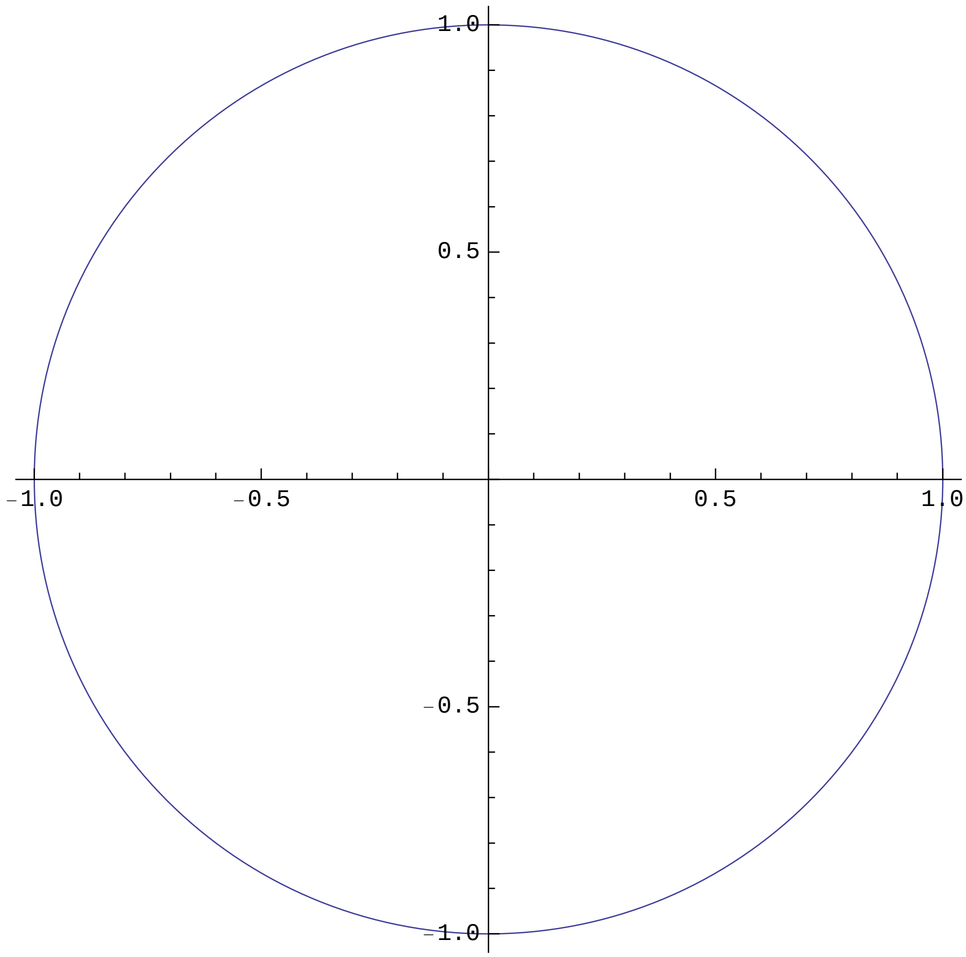
Example 7.1. Sketch the curve described by the parametric equation $x = \cos t$; and $y = \sin t$, $0 \leqslant t \leqslant 2\pi$.

Solution:

In[189]:=

```
ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2π}]
```

Out[189]=

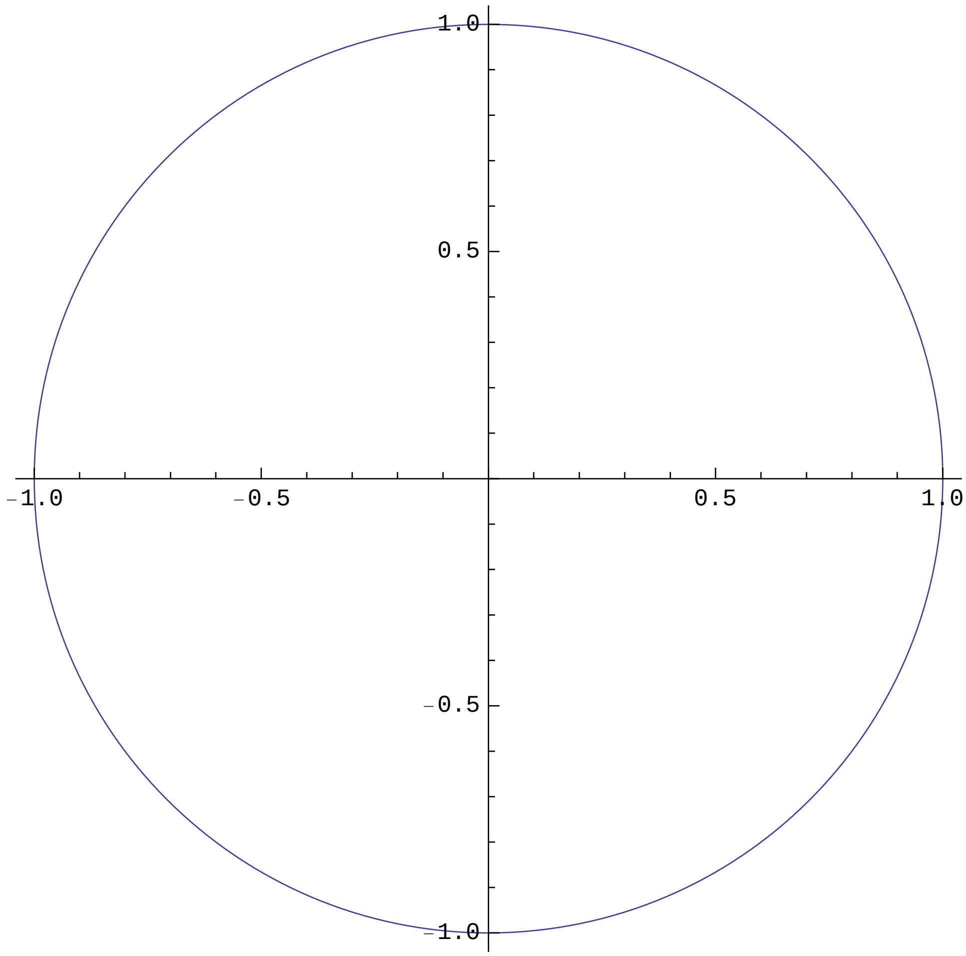


Recall that the above parametric equations represent the unit circle. However, *Mathematica* produced a graph that visually looks like an ellipse due to the different settings of the x and y-axes. An important option to fix this is to use the option **AspectRatio.** The default value of aspect ratio, which is the ratio of the height to the width of a plot, is **1/GoldenRatio** and its value is

In[190]:=
```
N[1 / GoldenRatio]
```
Out[190]=
```
0.618034
```

This value was chosen because it produces plots that are pleasing to the eye. However, to get a plot that accurately represents the graph of the given equations we need to set aspect ratio to automatic. We do this as follows.

In[191]:=
```
ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 π}, AspectRatio → Automatic]
```
Out[191]=



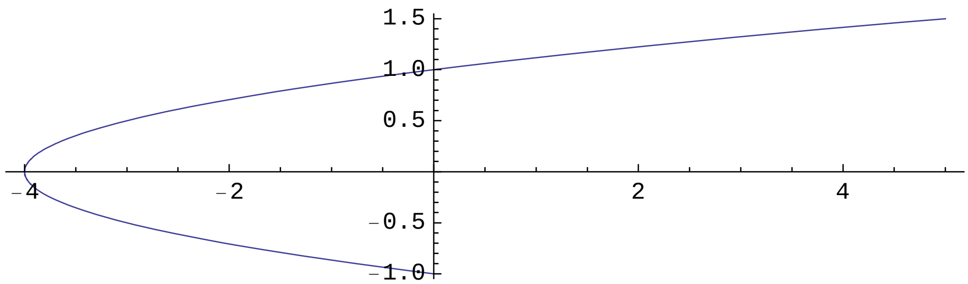
Example 7.2. Sketch the curve described by the parametric equation

$$x = t^2 - 4; \text{ and } y = \frac{t}{2}, \ -2 \leqslant t \leqslant 3.$$

Solution :

In[192]:=
```
ParametricPlot[{t^2 - 4, t/2}, {t, -2, 3}]
```

Out[192]=



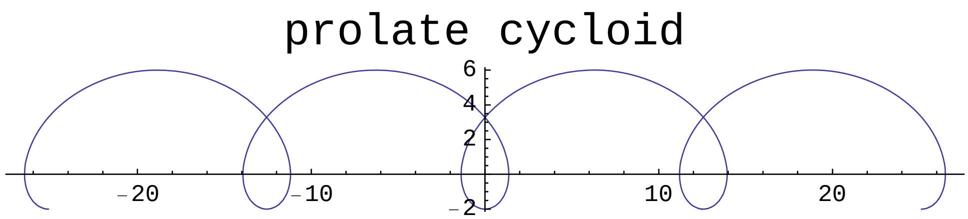
Example 7.3. Sketch the curve (*prolate cycloid*) described by the parametric equations

$x = 2\theta - 4\sin\theta$ and $y = 2 - 4\cos\theta$.

Solution:

In[193]:=
```
ParametricPlot[{2 θ - 4 Sin[θ], 2 - 4 Cos[θ]},
  {θ, -4 π, 4 π}, PlotLabel → "prolate cycloid"]
```

Out[193]=



Note that in the above input we have used the the command **PlotLabel**. In general, the command **PlotLabel ->"text"** prints the title **text** for the given plot.

Example 7.4. *Folium of Descartes* Given the parametric equations

$$x = \frac{4t}{1+t^3} \text{ and } y = \frac{4t^2}{1+t^3}, \text{ perform each of the following.}$$

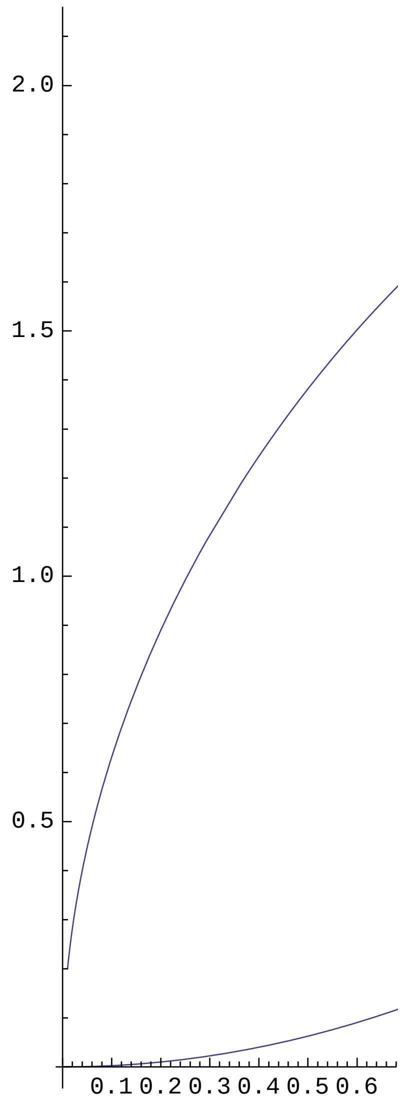
a. Sketch the curve described by the parametric equations.

b. Find the points of horizontal tangency to the curve.

Solution :

In[194]:=
```
ParametricPlot[{4t/(1+t^3), 4t^2/(1+t^3)}, {t, 0, 20},
 AspectRatio → Automatic, PlotLabel → "Folium of Descartes"]
```

Out[194]=



b) In order to find the points of horizontal tangency, we find the points where the slope of the tangent line is zero. Hence, we evaluate

In[195]:=
```
Solve[D[4t^2/(1+t^3), t] == 0, t]
```

Out[195]= $\{\{t \to 0\}, \{t \to -(-2)^{1/3}\}, \{t \to 2^{1/3}\}, \{t \to (-1)^{2/3}\, 2^{1/3}\}\}$

Thus, there are two points, corresponding to $t = 0$ and $t = 2^{\frac{1}{3}}$, at which the tangent lines are horizontal.

Example 7.5. Find the arc length of the curve $x = e^{-t}\cos t$, $y = e^{-t}\sin t$; $0 \leq t \leq \frac{\pi}{2}$.

Solution:

In[196]:=

$$\int_0^{\pi/2} \sqrt{((D[E^{\wedge}(-t) Cos[t], t])^{\wedge}2 + (D[E^{\wedge}(-t) Sin[t], t])^{\wedge}2)}\, dt$$

Out[196]=

$$\sqrt{2}\left(1 - e^{-\pi/2}\right)$$

In[197]:=

```
N[%]
```

Out[197]=

```
1.12023
```

■ Exercise 7. 1

1 . Sketch the curve represented by the parametric equations.

a) x= t^3, y = $t^2/2$ b) x=2(θ-sin θ), y= 1-cos θ c) x=3 $\cos^3\theta$, y=3 $\sin^3\theta$

2 . Find all points of horizontal and vertical tangency to the curve

x= cos θ + θ sin θ, y= sin θ - θ cos θ, 0 $\leq$ θ $\leq 2\pi$.

3 . Consider the parametric equations given by x= 3 cos (t/3) - cos t and y = 3 sin (t/3) - sin t.

a) Graph the curve represented by the parametric equations.

b) Find the slope of the tangent line at the point where t = π/4.

c) Find the length of the arc from t=0 to t = 3π/2.

■ 7.2 Polar Equations

In order to graph polar equations in *Mathematica*, we must load the package for polar plotting, We do this prior to plotting and by typing:

In[198]:=

```
Needs["BarCharts`"]; Needs["Histograms`"]; Needs["PieCharts`"]
```

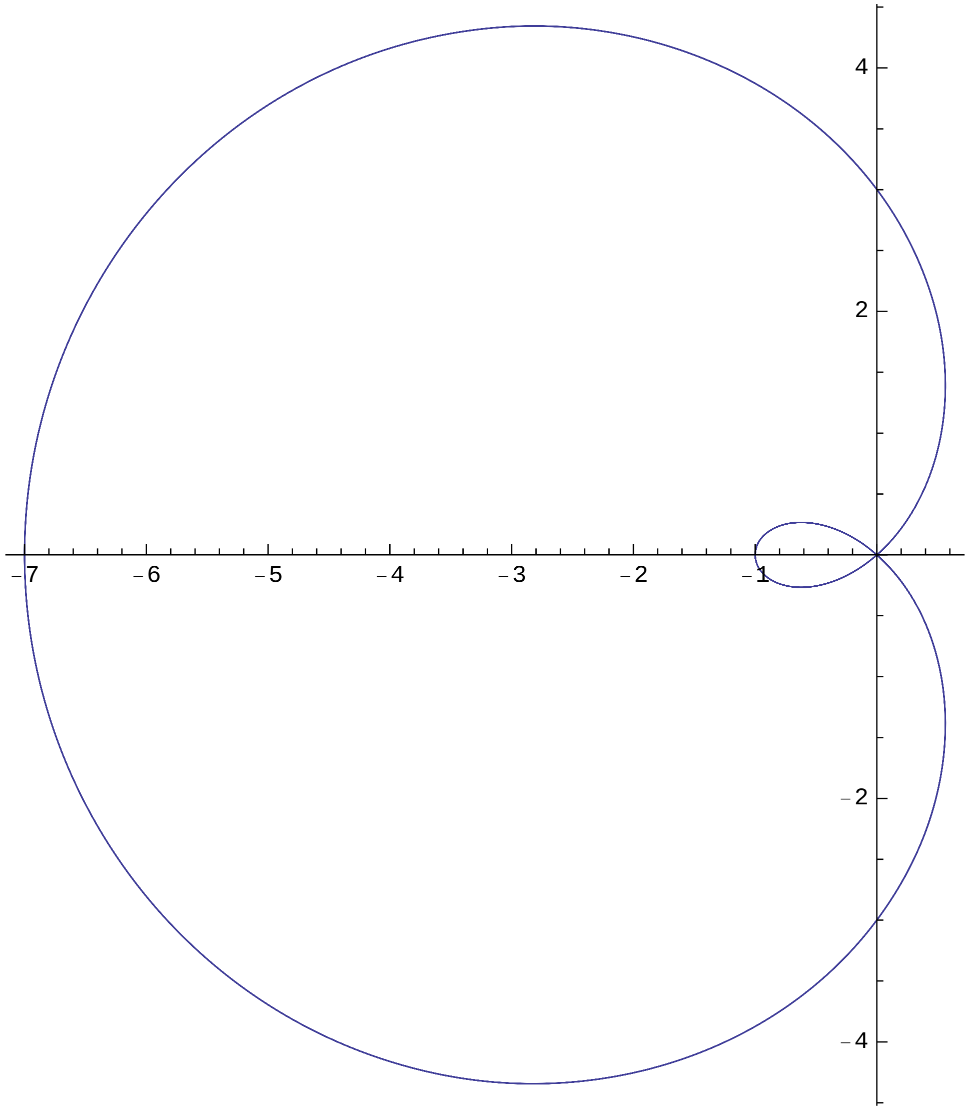
Example 7.6. Sketch the graph of r = 3 - 4 cosθ.

Solution:

In[199]:=

```
PolarPlot[3 - 4 Cos[θ], {θ, -4 π, 4 π}, AspectRatio → Automatic]
```

Out[199]=



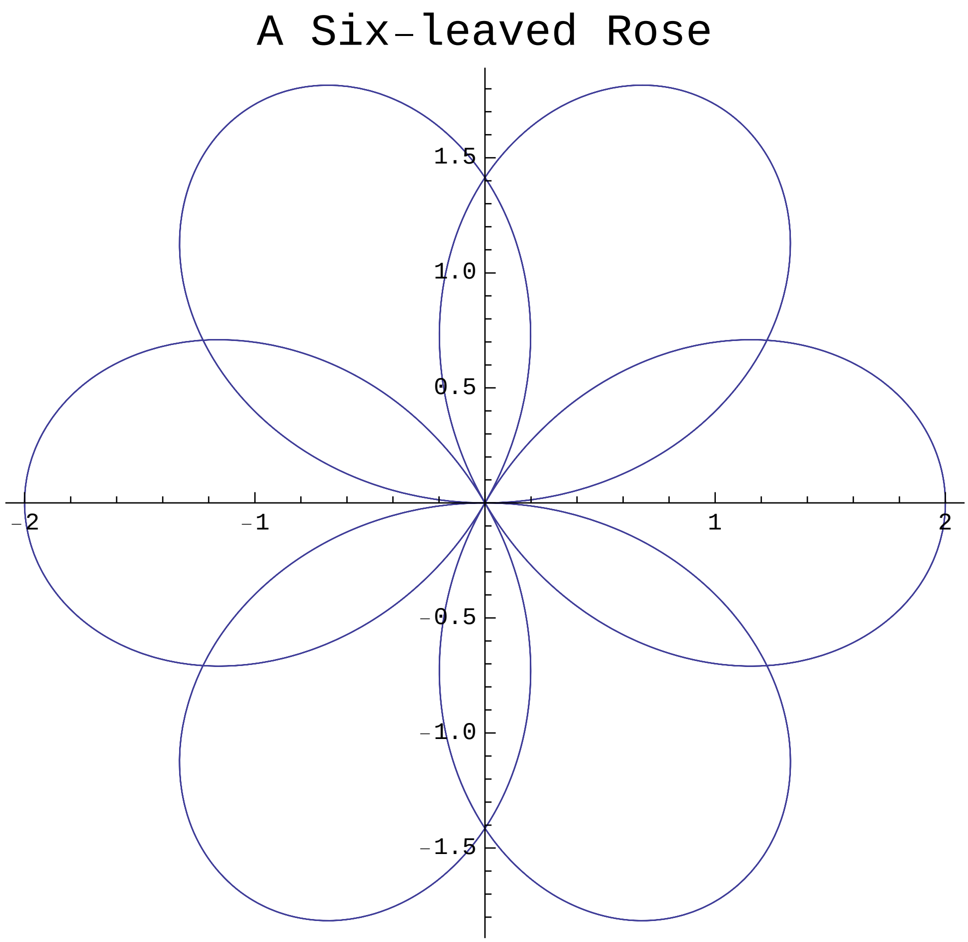
Example 7 : Sketch the graph of the six – leaf rose $r = 2\,\text{Cos}\left(\frac{3\,\theta}{2}\right)$.

Solution:

In[200]:=

```
PolarPlot[2 Cos[3 θ/2], {θ, -4 π, 4 π},
 PlotLabel → "A Six-leaved Rose", AspectRatio → Automatic]
```

Out[200]=

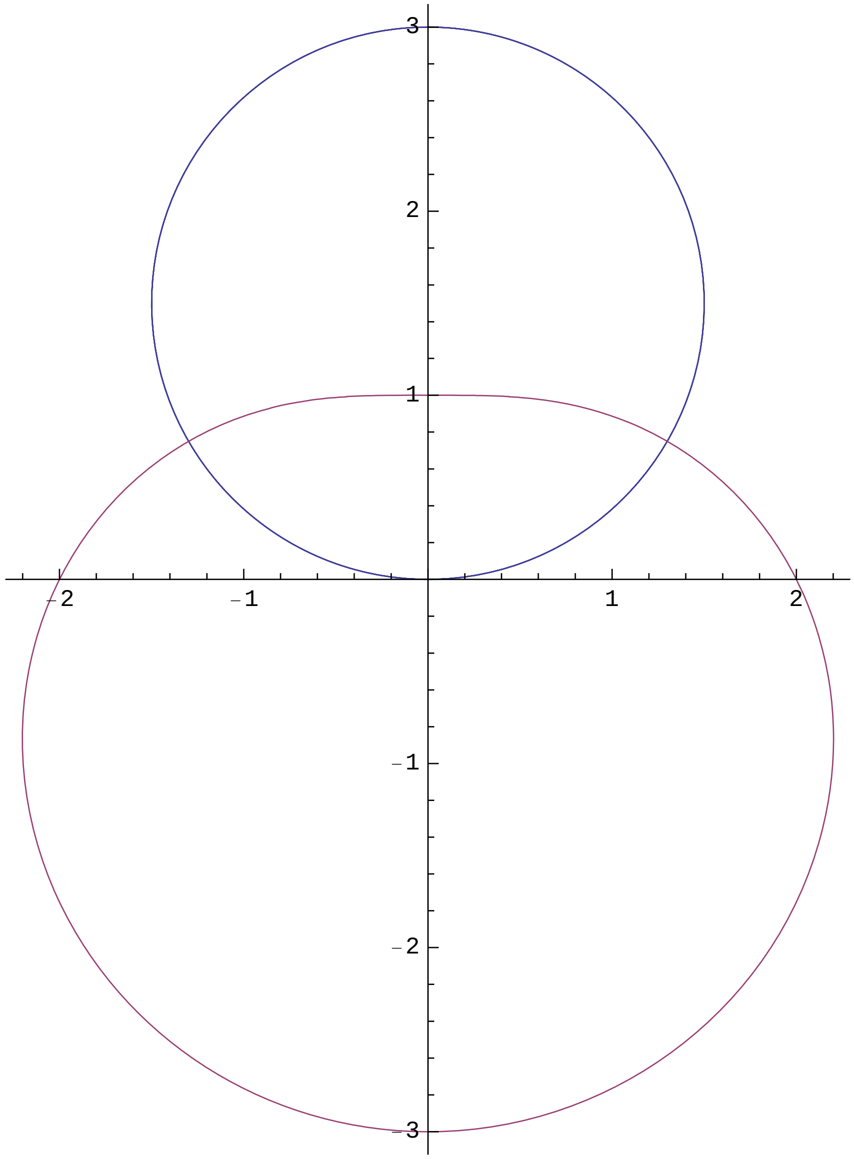


Example 7.8. Find the area of the region that is inside the circle r = 3 sinθ and outside the convex limacon r = 2- sinθ.

Solution: First, we plot the two polar curves on the same axes.

In[201]:=
```
PolarPlot[{3 Sin[θ], 2 - Sin[θ]}, {θ, 0, 2 π}, AspectRatio → Automatic]
```

Out[201]=



Next, we find the values of θ that correspond to the two points of intersection of the above curves.

In[202]:=
```
Solve[3 Sin[θ] == 2 - Sin[θ], θ]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[202]= $\left\{\left\{\theta \to \frac{\pi}{6}\right\}\right\}$

Note that *Mathematica* gives only the solution $\theta = \frac{\pi}{6}$ which lies in the first quadrant since inverse functions are being used. We can see from the above graph that the other point of intersection must be at $\theta = \frac{5\pi}{6}$. Thus, the area enclosed is given by (using symmetry)

In[203]:=
```
Integrate[(3 Sin[θ])^2, {θ, Pi/6, Pi/2}] -
Integrate[(2 - Sin[θ])^2, {θ, Pi/6, Pi/2}]
```

Out[203]= $-\frac{3}{8}\left(-5\sqrt{3}+4\pi\right)+\frac{3}{8}\left(3\sqrt{3}+4\pi\right)$

In[204]:=
```
N[%]
```

Out[204]= 5.19615

■ Exercise 7. 2

1. Graph the given polar equation and find an interval for θ over which the graph is traced only once.

a) $r = 3 - 4\cos\theta$ b) $r = 2 + \sin\theta$ c) $r = 3\cos(3\theta/2)$ d) $r = 5\sin 2\theta$

2. Consider the rose curve $r = \cos(2\theta)$ for $-2\pi \leqslant \theta \leq 2\pi$.

a) Graph the curve. b) Find the area of one petal of the curve.

3. Generate the butterfly curve $r = e^{\cos\theta} - 2\cos(4\theta) + \sin^5(\theta/12)$.

4. Graph and find the area of each of the following regions.

a) The common interior of $r=3-2\sin\theta$ and $r=-3+2\sin\theta$

b) Inside $r=2(1+\cos\theta)$ and outside $r=2\cos\theta$

c) Inner loop of $r=3+4\sin\theta$

5. Find the length of the given curve on the specified interval.

a) $r=1+\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$. b) $r=6(1+\cos\theta)$, $0 \leqslant \theta \leqslant 2\pi$.

6. Consider the polar equations $r = 4\sin\theta$ and $r = 2(2 - \sin^2\theta)$.

a) Graph the polar equations on the same axes.

b) Find the points of intersection of the curves.

c) Find the circumference of each curve.

Appendices

A. Traditional Notation versus *Mathematica* Notation

Traditional	Mathematica
$f(x) = x^2$	f = x^2 or f[x_] := x^2
$f(1)$	f /. x -> 1 or f[1]
$\sqrt{f(x)}$	Sqrt[f(x)]
$\lvert f(x) \rvert$	Abs[f(x)]
$\lim_{x \to a} f(x)$	Limit[f[x], x -> a]
$f'(x)$	f'[x] or D[f[x], x]
$\int f(x)\, dx$	Integrate[f[x], x]
$\int_a^b f(x)\, dx$	Integrate[f[x], {x, a, b}] or NIntegrate[f[x], {x, a, b}]
Plot $f(x)$ on $[a, b]$	Plot[f[x], {x, a, b}]
Solve $f(x) = g(x)$ for x	Solve[f(x) == g(x), x]
e (Euler number)	E
∞	Infinity
$\sin x$	Sin[x]
$\ln x$	Log[x]
$\log_a x$	Log[x, a]

■ B. Formatting Cells in a Notebook

Mathematica organizes a notebook in terms of data boxes called cells. The size of a cell is indicated by the corresponding size of the right bracket symbol attached to the right hand margin of each cell. A new cell can always be created by moving the cursor to any position between cells and begin typing. To edit a cell, just move the cursor to the desired position within that cell.

Each cell can be formatted to perform a specified function. By default, a new cell is always formatted as an input cell, which are used to evaluate *Mathematica* expressions. *Mathematica* outputs are contained within output cells, naturally. Other cell formats included title, section, subsection, text, formula, etc. The format of a cell is indicated by the left-most box on the toolbar. To change its format, first highlight the cell by clicking on the right bracket symbol attached to it. Then click on the indicator box and choose the desired format.

■ C. Saving and Printing a Notebook

Saving or printing a notebook can be accomplished by going to the File menu and highlighting the desired option. To print a selected portion of a notebook that has been highlighted, choose the Print Selection option instead.