Euler, Goldbach and Exact Values of Trigonometric Functions

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I. Introduction

In a portion of a letter sent to Christian Goldbach on December 9, 1741, Leonard Euler writes (see [8]):

> *I have lately also found a remarkable paradox. Namely that the value of the expression* $\frac{2^{+\sqrt{-1}}+2^{-\sqrt{-1}}}{2}$ *is approximately equal to 10/13 and that this fraction differs only in parts per million from the truth. The true value of this expression however is the cosine of the arc .6931471805599 or the arc of 39 degrees 42 min. 51 sec. 52 tenths of sec. and 9 hundredths of sec. in a circle of radius one.*

This paradox can be seen more clearly if we compare numerically the two quantities mentioned by Euler (in modern notation):

$$\frac{2^i+2^{-i}}{2}=\cos(\ln 2)=0.7692389...$$

$$\frac{10}{13}=0.7692307...$$

Of course, Euler's paradox can be resolved by considering the continued fraction expansion of $\cos(\ln 2)=[0;1,3,2,1,726,1,...]$ or in fraction form:

$$\cos(\ln 2)=\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{726+\cfrac{1}{1+...}}}}}} \tag{1.1}$$

It is well known that truncations of a continued fraction α, called convergents (or partial quotients), provide rational approximations of α, and if a convergent precedes a large quotient in the continued fraction, then it gives a close approximation (see Kruschev [5] for an introduction to continued fractions based on Euler's point of view). This explains Euler's paradox since 10/13 is the fifth convergent preceding the quotient 1/726 in the continued fraction (1.1)[1]:

$$\frac{10}{13}=\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1}}}}$$

On February 13, 1742, Goldbach replies to Euler and poses the following problem in relation to the paradox raised in Euler's letter:

> *With the observation as it was communicated to me that* $\frac{2^{+\sqrt{-1}}+2^{-\sqrt{-1}}}{2}$ *is approximately equal to 10/13 I have noticed that if you wanted to make it so that* $2^{p\sqrt{-1}}+2^{-p\sqrt{-1}}=0$ *then p would have to be smaller than 3 and larger than 2. I confess that these limits are large but I do not have the curiosity to determine them any closer.*

Euler's replies back to Goldbach on March 6, 1742, providing him with the exact solution for p:

> *Now that I have the curiosity to investigate when* $2^{p\sqrt{-1}}+2^{-p\sqrt{-1}}=0$ *it has given me the opportunity to remark that such an infinite mode could happen. First observed that p is between 2 and 3, namely 2.26618021. The true value is* $p=\frac{\pi}{2l2}$ *where* $\pi=3.14159265$ *and* $l2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+etc.$*=.6931471803. All following values are derived out of this in that you multiply these with 3,5,7,9 etc.*

Euler and Goldbach would exchange four more letters on this topic with portions in each letter discussing their own approaches and generalizations to the above problem, although they never resolve Euler's paradox. What makes these letters interesting, besides the mathematics contained in them, is that they seem to indicate Euler's earliest application of his famous formula

$$e^{ix}=\cos x+i\sin x, \tag{1.2}$$

which was first published by Euler in 1748 in his pre-calculus textbook, *Introductio in analysin infinitorum* [3]. Although (1.2) never appears explicitly in Euler's letters, it is not farfetched to conclude from his use of the formula $(a^{ix}+a^{-ix})/2=\cos(x\ln a)$ that Euler certainly knew of (1.2) as early as 1741 (if not earlier).[2] Moreover, in reading these letters one recognizes many of Euler's trademark techniques for exploring and generalizing mathematical problems. He is clearly considered the master in comparison to Goldbach, answering all of Goldbach's questions at depth and even pointing out some of Goldbach's mathematical mistakes.

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[1] Euler most likely derived 10/13 in this manner since he had previously developed the modern theory of continued fractions in his work *De Fractionlous Continious* published in 1737.

[2] Roger Cotes had discovered the inverse formula $\log(\cos x+i\sin x)=ix$ in 1714 (see http://www-history.mcs.st-and.ac.uk/Biographies/Cotes.html)

In this paper we explain the mathematics stemming from Euler and Goldbach's consideration of the equation $2^i + 2^{-i} = 0$ and show how it leads to connections with certain exact values of trigonometric functions in terms of Fibonacci-Lucas sequences. In particular, given $a^{pi} + a^{-pi} = b$, Euler claims that for any real value *r*,

$$a^{rpi} + a^{-rpi} = \left(\frac{b+\sqrt{b^2-4}}{2}\right)^r + \left(\frac{b-\sqrt{b^2-4}}{2}\right)^r \tag{1.3}$$

Following up on Euler's result, if we now define

$$x_n = a^{npi} + a^{-npi} = \cos(np \log a) \tag{1.4}$$

for all positive integers *n*, then x_n satisfies the second-order linear recurrence

$$x_{n+2} = bx_{n+1} - x_n, \tag{1.5}$$

with $x_0 = 0$ and $x_1 = b$. Thus, (1.4) and (1.5) provide us with a recursive formula for calculating the values $\cos(np \log a)$. We mention that analogous formulas involving hyperbolic trigonometric functions were derived by T. Osler in [6] and [7].

II. Fibonacci Sequence and Binet's Formula

Recall that it is Goldbach who initiates the problem of finding solutions to $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$. Since $\cos(p) = (2^{pi} + 2^{-pi})/2$, it follows that $\cos(pi) = 0$, which forces

$$p = \frac{2n+1}{2}\pi,$$

where *n* is an integer. This is the solution given by Euler in his reply to Goldbach.

Goldbach next considers solutions to the equation

$$2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 3 \tag{1.6}$$

and without explicitly mentioning the solution for *p*, he states the formula below without proof:

$$2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}} = \frac{(1+\sqrt{5})^{2x+1} - (-1+\sqrt{5})^{2x+1}}{2^{2x+1}} - \frac{(1+\sqrt{5})^{2x-1} - (-1+\sqrt{5})^{2x-1}}{2^{2x-1}}. \tag{1.7}$$

This formula Goldbach considers remarkable and it unclear how he obtains it or why he considers it. One way to derive (1.7) is to solve (1.6) for $2^{p\sqrt{-1}}$, which equals $\frac{3\pm\sqrt{5}}{2}$, and then use algebra to manipulate the expression

$$\begin{aligned} 2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}} &= (\frac{3+\sqrt{5}}{2})^x + (\frac{3-\sqrt{5}}{2})^x \\ &= \frac{(1+\sqrt{5})^{2x+1} - (-1+\sqrt{5})^{2x+1}}{2^{2x+1}} - \frac{(1+\sqrt{5})^{2x-1} - (-1+\sqrt{5})^{2x-1}}{2^{2x-1}} \end{aligned}$$

Observe that the left hand side of (1.7) resembles Binet's formula for Fibonacci numbers F_n:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{1.8}$$

where F_n satisfies the recurrence $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$ and $F_1 = 1$. Indeed, the following identities hold for any integer *n*:

$$\begin{aligned} &\frac{(1+\sqrt{5})^{2n+1} - (-1+\sqrt{5})^{2n+1}}{2^{2n+1}} - 2F_{2n} = F_{2n+1}, \\ &2F_{2n} - \frac{(1+\sqrt{5})^{2n-1} - (-1+\sqrt{5})^{2n-1}}{2^{2n-1}} = F_{2n-1}. \end{aligned} \tag{1.9}$$

It follows from adding the two equations in (1.9) and equating with (1.7) that

$$x_n = 2^{np\sqrt{-1}} + 2^{-np\sqrt{-1}} = F_{2n+1} + F_{2n-1},$$

i.e., each x_n is a sum of Fibonacci numbers (or a bisection of Lucas numbers)[3]. Here are the first few values of x_n:

$$x_0 = F_1 + F_{-1} = 1 + 1 = 2, \quad x_1 = F_3 + F_1 = 2 + 1 = 3$$
$$x_2 = F_5 + F_3 = 5 + 2 = 7, \quad x_3 = F_7 + F_5 = 13 + 5 = 18$$

We note that Euler never mentions to Goldbach the connection between (1.7) and the Fibonacci numbers. This is surprising given the recursive nature of the Fibonacci numbers and the circumstances suggesting that Euler knew of formula (1.8) before then. In particular, Daniel Bernoulli had published formula (1.8) in 1728 in [2] (Section 7) and Bernoulli is known to have had many correspondences with Euler during the period 1728-1742. Euler himself published a variation of (1.8), but much later in 1748, in [3] and well before Binet's independently discovery of it in 1843 in [1]. In fact, De Moivre seems to be the first person to have discovered this formula. In his 1722 paper [4], he implicitly derives the following well-known power series expansion for the reciprocal of $1 - x - x^2$ by partial fraction decomposition:

$$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n .$$

The bridge connecting the two sides of this equation is of course (1.8).

II. Linear Recurrences and Special Values of Trigonometric Functions

In response to Goldbach's solution of the equation $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 3$, Euler considers the more general situation: If $a^{pi} + a^{-pi} = b$, then

$$a^{rpi} + a^{-rpi} = \left(\frac{b + \sqrt{b^2 - 4}}{2}\right)^r + \left(\frac{b - \sqrt{b^2 - 4}}{2}\right)^r$$

This follows from solving the quadratic equation $a^{pi} + a^{-pi} = b$ for a^{pi}:

$$a^{pi} = \frac{b \pm \sqrt{b^2 - 4}}{2}$$

If we define $x_n = a^{npi} + a^{-npi}$, where *n* is a non-negative integer, then x_n satisfies the linear recurrence

[3] Sequence A005248 in The Online Encyclopedia of Integer Sequences: http://oeis.org

$$x_{n+2} = bx_{n+1} - x_n,$$
$$x_0 = 2, x_1 = b \tag{1.10}$$

To prove this, we first show that the elementary solution $y_n = a^{npi}$ satisfies the same recurrence. This follows from multiplying the identity $a^{pi} + a^{-pi} = b$ by $a^{(n+1)pi}$ to obtain

$$a^{(n+2)pi} = ba^{(n+1)pi} - a^{npi},$$

or equivalently,

$$y_{n+2} = by_{n+1} - y_n. \tag{1.11}$$

It is easy to check that the other elementary solution a^{-npi} also satisfies the same recurrence. By linearity, the sequence $x_n = a^{npi} + a^{-npi}$ satisfies the recurrence as well. Thus,

$$x_n = \left(\frac{b+\sqrt{b^2-4}}{2}\right)^n + \left(\frac{b-\sqrt{b^2-4}}{2}\right)^n$$

Since x_n can be written in the trigonometric form

$$x_n = a^{npi} + a^{-npi} = 2\cos[np\ln a],$$

we obtain as a result the following formula for special values of cosine:

$$\cos[np\ln a] = \frac{1}{2}\left[\left(\frac{b+\sqrt{b^2-4}}{2}\right)^n + \left(\frac{b-\sqrt{b^2-4}}{2}\right)^n\right] \tag{1.12}$$

It is not difficult to show that a corresponding formula holds for the sine function as well. In particular, if the equation $a^{pi} - a^{-pi} = b$ holds, then

$$a^{rpi} + (-1)^r a^{-rpi} = \left(\frac{b+\sqrt{b^2+4}}{2}\right)^r + (-1)^r\left(\frac{-b+\sqrt{b^2+4}}{2}\right)^r.$$

We claim that $x_n = a^{npi} + (-1)^n a^{-npi}$ satisfies the following recurrence for integer values of n:

$$x_{n+2} = bx_{n+1} + x_n,$$
$$x_0 = 0, x_1 = b \tag{1.13}$$

which is analogous to (1.10). This follows from the fact that $y_n = a^{npi}$ and $z_n = (-1)^n a^{-npi}$ are both elementary solutions of the same recurrence. For y_n, this can be proven using the same argument as that used to establish (1.11). For z_n, we multiply $a^{pi} - a^{-pi} = b$ by $(-1)^{n+1} a^{-(n+1)pi}$ to obtain

$$(-1)^{n+1} a^{-npi} - (-1)^{n+1} a^{-(n+2)pi} = b(-1)^{n+1} a^{-(n+1)pi},$$

or equivalently,

$$z_{n+2} = bz_{n+1} + z_n.$$

Thus, by linearity the general solution $x_n = y_n + z_n$ satisfies the same recurrence as well, which we write in the trigonometric form

$$x_n = a^{npi} + (-1)^n a^{-npi} = \begin{cases} 2\cos[np\ln a] & \text{if } n \text{ even} \\ 2\sin[np\ln a] & \text{if } n \text{ odd} \end{cases}$$

This results in the following formula for special values of sine for n an odd integer:

$$\sin[np\ln a]=\frac{1}{2}\left[\left(\frac{b+\sqrt{b^2+4}}{2}\right)^n+\left(\frac{b-\sqrt{b^2+4}}{2}\right)^n\right]. \qquad (1.14)$$

We leave it for the reader to show that for n an even integer,

$$\sin[np\ln a]=\frac{1}{2}\left[\left(\frac{b+\sqrt{b^2+4}}{2}\right)^n-\left(\frac{b-\sqrt{b^2+4}}{2}\right)^n\right]. \qquad (1.15)$$

We conclude by observing that Osler's hyperbolic version of formulas (1.14) and (1.15) derived in [6] can be obtained by replacing the quantities a^{pi} and b by a and M, respectively.

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