Translation with notes of Euler's paper E796,

Recherches sur le problem de trois nombres carres tels que la somme de deux quelconques moins le troisieme fasse un nombre carre

"Research into the problem of three square numbers such that the sum of any two less the third one provides a square number."

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Translator's Introduction

The following translation is the happy collaboration of student and professor. Kristen McKeen, a senior undergraduate mathematics major at Rowan University made the original translation from French. Tom Osler, professor of mathematics for 45 years, worked with her in trying to comprehend Euler's ideas and writing the final version presented here.

We tried to imagine how Euler would express himself if he was fluent in modern English and modern mathematical jargon. Euler often wrote in very long sentences, and we sometimes replaced these by several shorter ones. We did not convert Euler's mathematical notion into modern form, since in all cases the older notation is easy to understand. When we found typographical errors, we did not change them but made note of the error in parenthesis. Other errors are probably ours.

Modern readers will find that Euler's work reads more like a diary than a finished mathematical paper. He sometimes describes paths of thought that he ultimately abandons. At times he appears to explore an idea that he leaves incomplete. A few notes follow the paper in which we make comments on Euler's work, explain related computer experiments, and complete a few elementary mathematical steps that he omits.

1. Let x, y, z be the roots of three squares and require that the following three equations be satisfied

$$\begin{aligned} yy + zz - xx &= pp, \\ xx + zz - yy &= qq, \\ xx + yy - zz &= rr. \end{aligned} \tag{1}$$

If we add these equations two at a time, the following equations are produced:

$$\begin{aligned} pp + qq &= 2zz, \\ pp + rr &= 2yy, \\ qq + rr &= 2xx. \end{aligned}$$

Thus while solving our first problem, we also solve a second problem: *Find three square numbers such that the half sum of any two of them is also a square*. This follows from

$$\frac{pp+qq}{2} = zz, \frac{pp+rr}{2} = yy \text{ and } \frac{qq+rr}{2} = xx.$$

2. Moreover, it is obvious, having found the three numbers x, y, z, all their multiples will also solve the same problem, namely: nx, ny, nz. Because the problem as stated does not have a unique solution, in the following, we will search for three numbers, having no common divisor. First we notice that all three of the unknown numbers cannot be even. We will now show that these three numbers must all be odd. All even squares are of the form $4aa$, and all odd squares are of the form $4(aa+a)+1$. Therefore if we assume that two of the squares are even, that's to say

$xx = 4aa$, $y = 4bb$ (*should be* $yy = 4bb$) and the third odd: $zz = 4(cc+c)+1$,

the expression $xx + yy - zz$ becomes $4(aa+bb-cc-c)-1$, which will never be a square. Next assume that only two are odd and the third is even, as

$xx = 4(aa+a)+1$, $y = 4(bb+b)+1$, (*should be* $yy = 4(bb+b)+1$), $z = 4cc$ (*should be* $zz = 4cc$).

The expression $xx + yy - zz$ now becomes $4(aa+a+bb+b-cc)+2$, which will not be a square either. Finally assume all three squares are odd, for example

$$xx = 4(aa + a) + 1,\ \ yy = 4(bb + b) + 1,\ \ z = 4(cc + c) + 1,\ (should\ be\ zz = 4(cc + c) + 1).$$

The expression $xx + yy - zz$ becomes $4(aa + a + bb + b - cc - c) + 1$, which may represent a square.

3. In this section we show, that if one wants to determine four square numbers such that the sum of three less the fourth one equals a perfect square, the search will be useless, because no solution is possible. To prove this, we will have to cover all the cases involving evens and odds. Indeed,

1) if three squares are even, namely:

$$xx = 4aa,\ \ yy = 4bb,\ \ zz = 4cc \text{ and } vv = 4(dd + d) + 1,$$

we will have for $xx + yy + zz - vv$ the value $4(aa + bb + cc - dd - d) - 1$, which will never be a square.

2) If there are two evens and two odds,

$$xx = 4aa,\ \ yy = 4bb,\ \ zz = 4(cc + c) + 1,\ \ vv = 4(dd + d) + 1,$$

we have for $xx - yy + zz + vv$ the expression $4(aa - bb + cc + c + dd + d) + 2$, which will never be a square.

3) Next consider the case of only one even square, namely

$$xx = 4aa,\ \ yy = 4(bb + b) + 1,\ \ zz = 4(cc + c) + 1,\ \ vv = 4(dd + d) + 1.$$

We have for $yy + zz + vv - xx$ the expression $4(bb + b + cc + c + dd + d - aa) + 3$, which again is never a perfect square.

4) Finally suppose the four numbers are odd,

$$xx = 4(aa + a) + 1,\ \ yy = 4(bb + b) + 1,\ \ zz = 4(cc + c) + 1,\ \ vv = 4(dd + d) + 1.$$

We have for $xx + yy + zz - vv$ the value $4(aa + a + bb + b + cc + c - dd - d) + 2$, which is not a square.

4. After these considerations, let us examine a method to arrive at a solution of the proposed problem. For this purpose first note that the first two equations may be expressed as the general equation

$$zz \pm (yy - xx) = \text{an ordinary square.}$$

Now $AA + BB \pm 2AB$ is always a perfect square; then, comparing this formula with the preceding one, we have $zz = AA + BB$, $yy - xx = 2AB$, and to make $AA + BB$ a perfect square it is not necessary to suppose $A = aa - bb$, $B = 2ab$, and we will have $zz = (aa + bb)^2$; therefore $z = aa + bb$. After these assumptions, the value of $yy - xx$ will be $4ab(aa - bb)$. But $yy - xx$ is the product of $y + x$ by $y - x$, and $4ab(aa - bb)$ the product of $2ab$ by $(2aa - 2bb)$, and to satisfy the equation $yy - xx = 4ab(aa - bb)$ take $y + x = 2ab$, $y - x = 2aa - 2bb$. Therefore

$$x = bb + ab - aa \text{ and } y = aa + ab - bb.$$

Thus, if we take for the values of x, y and z the expressions

$$bb + ab - aa, \ \ aa + ab - bb \text{ and } aa + bb,$$

the first two equations will be satisfied. We must now determine if these values will also satisfy the third equation. By substituting these values of x, y, z, we get

$$xx + yy - zz = a^4 + b^4 - 4aabb = rr.$$

5. We must now find numbers a and b such that the formula $a^4 + b^4 - 4aabb$ becomes a perfect square. It is easy to see that this condition is satisfied, if we take $a = 2b$. To find another solution of the equation $a^4 + b^4 - 4aabb = rr$, we set $a = b(z + 2)$, and we get $a^4 + b^4 - 4aabb = b^4(z^4 + 8z^3 + 20z^2 + 16z + 1)$. Let us suppose that the square root of this expression is $b^2(zz + 8z + 1)$. By comparing the square of

$$b^2(zz+8z+1) \text{ with } b^4(z^4+8z^3+20z^2+16z+1),$$

we find that

$$8z^3+46zz=0\text{; which has the solution } z=-\frac{23}{4}.$$

Consequently $z+2=-\frac{15}{4}$ and $a=-\frac{15b}{4}$. Now, since the values of a and b may be either positive or negative, we take $a=15,\ b=4,$ and we get $x=149,\ y=269,\ z=241,$ which appear to be the smallest desired numbers. From these we find $p=329,\ q=89,\ r=191.$

6. As this solution is obtained from the equation $yy-xx=4ab(aa-bb),$ by factoring the right side using $2ab$ and $2(aa-bb),$ it follow**s** that we can in general express the values of y and x of this manner: $y+x=\frac{2m}{n}ab$ and $y-x=\frac{2n}{m}(aa-bb)$. But, after some very tiresome calculations, we do not reach very useful solutions. The most simple assumption is $y+x=2a(a+b),\ \ y-x=2b(a-b)$; from which we find

$$yy+xx=2(a^4+2a^3b+2aabb-2ab^3+b^4).$$

From this, for the value of $rr=yy+xx-zz,$ we find $a^4+4a^3b+2aabb-4ab^3+b^4$ which is the complete square of $aa+2ab-bb.$ Therefore the values of a and b are entirely arbitrary. But when considering the values of x, y and z, which are $aa+bb,$ $aa+2ab-bb$ and $aa+bb,$ we see that x and z are equal, and for this reason this solution is not acceptable.

7. We can still use other methods for the solution of the problem. But all of the methods have the great defect of only giving solutions of limited usefulness, and that after very long and very difficult calculations. For this reason we explain here four absolutely remarkable methods, which, without a lot of difficulty, supply infinitely many

general formulas for expressing the three numbers x, y, and z, which, in turn, will give infinitely many solutions. However, the formulas do not contain all possible solutions.

Easy methods for finding more general solutions.

First method.

8. If we suppose $s = xx + yy + zz$, our equations (1) will become

$$s - 2xx = pp, \text{ or } s = pp + 2xx,$$

$$s - 2yy = qq, \text{ or } s = qq + 2yy,$$

$$s - 2zz = rr, \text{ or } s = rr + 2zz,$$

from which we can see that *s* has to be, in three different ways, the sum of a square and a double square.

9. Let us now carefully consider the numbers which can be expressed in the form $aa + 2bb$. First we inquire if, when such a number is prime, *a* and *b* might not be unique. If two solutions exist, then we have $s = aa + 2bb$ and also $s = cc + 2dd$. It follows then that $aa - cc = 2dd - 2bb$ and consequently $\frac{a+c}{d+b} = \frac{2(d-b)}{a-c}$. Now since these two fractions are equal, let us suppose that after having been reduced to smaller terms they are $\frac{m}{n}$. From that we have $\frac{a+c}{d+b} = \frac{m}{n}$, or $a + c = mf$, $d + b = nf$. Similarly $\frac{2(d-b)}{a-c} = \frac{m}{n}$ and $d - b = mg$, $a - c = 2ng$, and consequently $2a = mf + 2ng$, $2b = nf - mg$. But since $4s = 4aa + 8bb$, by substituting, in place of $2a$ and $2b$, their values, we get $4s = ff(mm + 2nn) + 2gg(mm + nn)$, or $4s = (ff + 2gg)(mm + 2nn)$, which is impossible since *s* is a prime number.

10. It follows then that if *s* is not known to be a prime number, and it is demonstrated that it is a number of the form $aa + 2bb$ then it may only be divisible by

numbers of the same form, with a and b relatively prime**.** So s is the product of two or more prime numbers of the same form $aa+2bb$. But it is easy to observe, that two prime factors are not sufficient for producing a triple resolution; therefore s must have at least three prime factors of the form $aa+2bb$.

11. Let us observe here that all odd numbers of the form $aa+2bb$ are always of the form $8n+1$ or $8n+3$, and that when the number is even and of the form $aa+2bb$, it is the double of one or the other of the two formulas. The form $aa+2bb$ brings us back to the first case, when a is odd and secondly, when a is even. Thus, all other odd numbers of the form $8n+5$ or $8n+7$ are entirely excluded from the number of divisors of the form $aa+2bb$. Therefore all numbers which are divisible by these: 5, 7, 13, 15, 21, 23, 29, 31, 37, 39, 45, 47, 53, 55, etc. may not be combined in the form $aa+2bb$, where we suppose a and b are relatively prime.

12. It is very remarkable that all the following prime numbers, of the form $8n+1$ or $8n+3$, are always reducible to a square plus the double of a square, but only in one way, and here are some examples

$8n+1$	$8n+3$
$17=3^2+2\cdot 2^2$	$3=1^2+2\cdot 1^2$
$41=3^2+2\cdot 4^2$	$11=3^2+2\cdot 1^2$
$73=1^2+2\cdot 6^2$	$19=1^2+2\cdot 3^2$
$89=9^2+2\cdot 2^2$	$43=5^2+2\cdot 3^2$
$97=5^2+2\cdot 6^2$	$59=3^2+2\cdot 5^2$
$113=9^2+2\cdot 4^2$	$67=7^2+2\cdot 3^2$
$137=3^2+2\cdot 8^2$	$83=9^2+2\cdot 1^2$

$$107 = 3^2 + 2 \cdot 7^2$$

$$131 = 9^2 + 2 \cdot 5^2$$

$$139 = 11^2 + 2 \cdot 3^2$$

13. With all divisions we will not know how to find the smallest number to divide, and yet there is not a doubt that the division does occur for all numbers of the form $8n+1$ or $8n+3$, and we can demonstrate this rigorously. For this purpose, it's only a matter of proving that given a prime, of the form $8n+1$ or $8n+3$, one may always find a product of the form $aa+2bb$ which allows one or the other for a factor. This proof follows from a very beautiful theorem by Fermat, namely: that the form $c^{2m}-1$ is always divisible by the number $2m+1$, provided it is prime and is not a divisor of c. Consequently, if the number $8n+1$ is prime, it will always be a factor of the formula $c^{8n}-1$, provided c is not be a multiple of $8n+1$. But the quantity $c^{8n}-1$ has two factors which are $(c^{4n}+1)$, $(c^{4n}-1)$, it's therefore necessary that one or the other is divisible by $8n+1$. Consequently, if we take for c a number for which $c^{4n}-1$ is not a multiple of $8n+1$, the number $c^{4n}+1$ will necessarily be divisible by $8n+1$. But the formula $c^{4n}+1$ may be written then $(c^{2n}-1)^2+2c^{2n}$; therefore the number $8n+1$ is a divisor of a number of the form $aa+2bb$.

14. As for the formula $8n+3$, each prime number of the form $8n+3$ is a divisor of $c^{8n+2}-1$ and consequently of $c^{4n+1}+1$, or of $c^{4n+1}-1$. Let $c=2$, the formula $c^{4n+1}-1$ becomes $2\cdot 2^{4n}-1$, which will never by divisible by $8n+3$, because all the divisors of

the form $2ff-1$ are $8n+1$ or $8n-1$ and never $8n+3$. Therefore $2^{4n+1}+1$ or $2\cdot 2^{4n}+1$ which is of the form $aa+2bb$, will necessarily be divisible by $8n+3$.

After this useful digression, we return to our problem. We have seen that the sum *s* must have at least three factors, thus let us set it equal to

$$(aa+2bb)(cc+2dd)(ff+2gg)$$

and to shorten the calculations, let $(aa+2bb)(cc+2dd)=mm+2nn$, then we will have

$$m=ac\pm 2bd,\quad n=bc\mp ad.$$

Now our sum *s* can be expressed as $s=(mm+2nn)(ff+2gg)$, which we set equal to $zz+2vv$, and get $z=mf\pm 2ng$ and $v=nf\mp mg$.

15. Let us substitute now, in place of *m* and *n*, the values found, and we will have four different values for *z* and *v*, namely for *z*:

$$1)\quad f(ac+2bd)+2g(bc-ad),$$

$$2)\quad f(ac+2bd)-2g(bc-ad),$$

$$3)\quad f(ac-2bd)+2g(bc+ad),$$

$$4)\quad f(ac-2bd)-2g(bc+ad),$$

and for *v*:

$$1)\quad f(bc-ad)-g(ac+2bd),$$

$$2)\quad f(bc-ad)+g(ac+2bd),$$

$$3)\quad f(bc+ad)-g(ac-2bd),$$

$$4)\quad f(bc+ad)+g(ac-2bd).$$

16. There are therefore four different values of both z and v. But we only require three, because of the three conditions $s = pp + 2xx$, $s = qq + 2yy$ and $s = rr + 2zz$, that we have to fulfill. We only use the first three values of z and v; and get

$$\begin{aligned}
&f(ac+2bd)+2g(bc-ad)=p,\\
&f(ac+2bd)-2g(bc-ad)=q,\\
&f(ac-2bd)+2g(bc+ad)=r,\\
&f(bc-ad)-g(ac+2bd)=x,\\
&f(bc-ad)+g(ac+2bd)=y,\\
&f(bc+ad)-g(ac-2bd)=z.
\end{aligned}$$

17. We search now, using the values x, y, z, for the sum of their squares which will have the form $Aff + Bgg + 2Cfg$, where

$$\begin{aligned}
&A = 3bbcc - 2abcd + 3aadd,\\
&B = 3aacc + 4abcd + 12bbdd,\\
&C = -(bc+ad)(ac-2bd).
\end{aligned}$$

Now consider the difference between this expression for the sum and the earlier expression

$$s = (aa+2bb)(cc+2dd)(ff+2gg)$$

$$= ff(aacc+2bbcc+2aadd+4bbdd)+2gg(aacc+2bbcc+2aadd+4bbdd)$$

(To make these equal) we are lead to write

$$Fff + Ggg + 2Cfg = 0, \text{ where}$$

$$\begin{aligned}
&F = bbcc - 2abcd + aadd - aacc - 4bbdd,\\
&G = aacc + 4abcd + 4bbdd - 4bbcc - 4aadd,\\
&C = -(bc+ad)(ac-2bd).
\end{aligned}$$

We now have reached a solution of our problem. We must find values of the six letters a, b, c, d, f, g, which make $Fff + Ggg + 2Cfg = 0$, and from these we find x, y, z and also p, q, r.

18. Therefore we study $Fff + Ggg + 2Cfg = 0$, which implies

$$\frac{f}{g} = \frac{-C \pm \sqrt{(CC - FG)}}{F}.$$

It is necessary to search for numbers *a*, *b*, *c*, *d*, *f*, *g*, such that $CC - FG$ becomes a square. But this leads us towards three great difficulties, which we want to avoid. Happily we come across a case, where the equation $Fff + Ggg + 2Cfg = 0$ reduces easily to the first degree, namely when *F* is equal to 0. In this case we have $Ggg + 2Cfg = 0$, or $\frac{f}{g} = -\frac{G}{2C}$.

Thus, after reducing $-\frac{G}{2C}$ to smaller terms; if we take the numerator for *f* and the denominator for *g*, all the formulas above will be expressed in rational numbers. This is the merit of this method.

19. Let us now remark that the value $bbcc - 2abcd + aadd - aacc - 4bbdd$, found for *F*, may be expressed as the product of two factors in the following manner:

$$F = \{(b+a)c + (a+2b)d\}\{(b-a)c + (a-2b)d\}.$$

Thus if one or the other of these two factors is made zero, then *F* becomes zero. First we will take $\frac{c}{d} = \frac{-a-2b}{b+a}$, and second, we will use $\frac{c}{d} = \frac{2b-a}{b-a}$. There will therefore be two values for the letters *c* and *d*, and consequently two solutions of the problem.

20. In the same manner we want to make the value of *G* vanish. Since it is equal to $aacc + 4abcd + 4bbdd - 4bbcc - 4aadd$, which is the product of the two factors

$$(a+2b)c - (2b+2a)d,\ \ (a-2b)c - (2b-2a)d,$$

we will have, for the determination of *c* and *d*, the equation $\frac{c}{d} = \frac{2b+2a}{a+2b}$, or $\frac{c}{d} = \frac{2b-2a}{a-2b}$. But these values do not result in new solutions; thus it's sufficient of us to use values obtained from $F = 0$.

21. There is therefore a solution simple enough for the proposed problem, and which provides at the same time infinitely many numerical values. For achieving this, we give the following rules:

1) After having taken at will the two numbers *a* and *b*, we search for the values of *c* and *d* by one or the other of the two formulas $\frac{c}{d} = \frac{-a-2b}{b+a}$, or $\frac{c}{d} = \frac{2b-a}{b-a}$, since each will lead to a solution.

2) We then find the values of *C* and *G* from the formulas

$$C = -(bc+ad)(ac-2bd),$$
$$G = (aa-4bb)cc + (4bb-4aa)dd + 4abcd,$$

and we will have

$$\frac{f}{g} = \frac{(aa-4bb)cc + 4(bb-aa)dd + 4abcd}{2(bc+ad)(ac-2bd)}.$$

After reducing this fraction to lowest terms, we take *f* equal to the numerator, and *g* to the denominator.

3) Having thus found the values of *f* and *g*, we immediately have those of *x*, *y*, *z* by the formulas

$$x = f(bc-ad) - g(ac+2bd),$$
$$y = f(bc-ad) + g(ac+2bd),$$
$$z = f(bc+ad) - g(ac-2bd),$$

which are the three desired numbers.

4) Finally the letters p, q, r are also found by following these formulas

$$p = f(ac+2bd)+2g(bc-ad),$$
$$q = f(ac+2bd)-2g(bc-ad),$$
$$r = f(ac-2bd)+2g(bc+ad).$$

Let us clarify these rules by some examples.

Example 1. Let $a=1$ and $b=1$, thus $\frac{c}{d}$ will be equal, in the first case to $-\frac{3}{2}$, and in the second to $\frac{1}{0}$, which does not lead to a solution. Thus suppose $c=3$ and $d=-2$; $\frac{f}{g}$ will be $=-\frac{51}{14}$; let $f=51$ and $g=-14$. Thus

$x = 51(3+2)+14(3-4) = 241,$ $\quad\quad$ $p = -51-28\cdot 5 = -191,$

$y = 51(3+2)-14(3-4) = 269,$ $\quad\quad$ $q = -51+28\cdot 5 = 89,$

$z = 51(3-2)+14(3+4) = 149,$ $\quad\quad$ $r = 51\cdot 7-28 = 329,$

finally $s = 3\cdot 17\cdot 2993$ or $3\cdot 17\cdot 41\cdot 73$.

Example 2. Let $a=1$ and $b=2$, thus $\frac{c}{d} = -\frac{5}{3}$ or $=\frac{3}{1}$, and thus we have two cases.

Case 1. Let $c=3$ and $d=1$, and get $\frac{f}{g} = \frac{99}{14}$ and, consequently, $f=99$ and $g=14$. From these values we obtain

$x = 99\cdot 5-14\cdot 7 = 397,$ $\quad\quad$ $p = 99\cdot 7+28\cdot 5 = 833,$

$y = 99\cdot 5+14\cdot 7 = 593,$ $\quad\quad$ $q = 99\cdot 7-28\cdot 5 = 553,$

$z = 99\cdot 7+14 = 707,$ $\quad\quad$ $r = -99+28\cdot 7 = 97,$

$s = 9\cdot 11\cdot 10193.$

Case 2. Let $c=5$ and $d=-3,$ and calculate $\frac{f}{g}=-\frac{387}{238}$. Thus we have

$f=387,\ \ g=-238;$ from which we get

$$x=387\cdot 13-238\cdot 7=3365, \qquad p=387\cdot -7-2\cdot 238\cdot 13=-8897,$$

$$y=387\cdot 13+238\cdot 7=6697, \qquad q=387\cdot -7+2\cdot 238\cdot 13=3479,$$

$$z=387\cdot 7+238\cdot 17=6755, \qquad r=387\cdot 17-2\cdot 238\cdot 7=3247,$$

$$s=9\cdot 43\cdot 263057.$$

Example 3. Let $a=3$ and $b=1,$ and get $\frac{c}{d}=-\frac{5}{4}$ or $=\frac{1}{2}.$ It must be remarked here that the last case is already handled in the preceding example, since *a*, *b*, *c*, *d* are changeable. This is why we only consider the first case, where $c=5$ and $d=-4$. We get $\frac{f}{g}=\frac{627}{322},$ and so $f=627,\ \ g=322$. Thus we have

$$x=627\cdot 17-322\cdot 7=8405, \qquad p=627\cdot 7+644\cdot 17=15337,$$

$$y=627\cdot 17+322\cdot 7=12913, \qquad q=627\cdot 7-644\cdot 17=-6559,$$

$$z=627\cdot -7-322\cdot 23=-11795, \qquad r=627\cdot 23-644\cdot 7=9913,$$

$$s=11\cdot 57\cdot 600497.$$

22. These examples are sufficient for showing how, by the rules, we may easily find as many solutions as desired. We are content here to show the smaller results for which the numbers *x*, *y*, *z* do not surpass one thousand.

I	II	III	IV	V
$x=241$	397	425	595	493
$y=269$	593	373	769	797
$z=149$	707	205	965	937

$p=191$	833	23	1081	1127
$q=89$	553	289	833	697
$r=329$	97	527	119	289

Second Method.

23. The solution of our problem has been reduced to this quadratic equation

$$Fff+Ggg+2Cfg=0, \text{ where}$$

$$C=-(bc+ad)(ac-2bd),$$
$$F=(bb-aa)cc+(aa-4bb)dd-2abcd,$$
$$G=(aa-4bb)cc+(4bb-4aa)dd+4abcd,$$

and finally to the formula $\frac{f}{g}=\frac{-C\pm\sqrt{CC-FG}}{F}$, in which $CC-FG$ must be a square.

Let us therefore write $CC-FG=VV$, so that $\frac{f}{g}=\frac{-C\pm V}{F}$. By substituting in the expression $CC-FG$ the values of C, F, and G, we get the expression

$$VV=(aa-2bb)^2c^4+8(aa-2bb)abc^3d-4(aa-2bb)^2ccdd-16(aa-2bb)abcd^3+4(aa-2bb)^2d^4,$$

which is divisible by $(aa-2bb)^2$. By the substitution of m in place of $\frac{ab}{aa-2bb}$, this becomes simply

$$\frac{VV}{(aa-2bb)^2}=c^4+8mc^3d-4ccdd-16mcd^3+4d^4.$$

24. Since this formula must be a square, let us assume its square root is equal to

$$\frac{V}{aa-2bb}=cc-4mcd+2dd\,.$$

By comparing , we find that: $2mc-d-2mmd=0$ and, consequently, $\frac{c}{d}=\frac{2mm+1}{2m}.$

Thus, we let $c=2mm+1$ and $d=2m,$ and our formula becomes

$$\frac{V}{aa-2bb}=(2mm+1)^2-8mm(2mm+1)+8mm=4mm+1-12m^4.$$

25. Now we need only start with arbitrary numbers for a and b, compute $m=\frac{ab}{aa-2bb}$, C, F, V, and, then get $\frac{f}{g}=\frac{-C\pm V}{F}$. Thus the numbers f and g may be determined in two ways in each case. However, having found these numbers, one will be able to determine easily the values of x, y, z, that those of p, q, r. The simplest cases were examined in our first method of solution. Here are other examples:

Example 1. Let $a=2$ and $b=1,$ then $m=1,$ $c=3,$ $d=2,$ $f=28,$ $g=51,$ and get

$$x=482, \qquad p=382,$$

$$y=-538, \qquad q=178,$$

$$z=298, \qquad r=-658.$$

Example 2. Let $a=3$ and $b=2,$ then $m=6,$ $c=73,$ $d=12,$ $f=-7,$ $g=17,$ and obtain

$$x=5309, \qquad p=1871,$$

$$y=3769, \qquad q=5609,$$

$$z=4181, \qquad r=4991.$$

Third method

26. As in our first method, we let the sum of three unknown squares $s=(aa+2bb)(cc+2dd)(ff+2gg)$. However we suppose that the first factor $aa+2bb$ can be expressed in two ways as a square plus twice a square, namely $\alpha\alpha+2\beta\beta=aa+2bb.$ We use the expression $aa+2bb$ to determine the numbers $x, y,$ $p, q,$ as we have done (16), and the last expression $\alpha\alpha+2\beta\beta$ to determine z and $r,$ so that

$$x=f(bc-ad)-g(ac+2bd), \qquad p=f(ac+2bd)+2g(bc-ad),$$

$$y=f(bc-ad)+g(ac+2bd), \qquad q=f(ac+2bd)-2g(bc-ad),$$

$$z=f(\beta c+\alpha d)-g(\alpha c+2\beta d), \qquad r=f(\alpha c-2\beta d)+2g(\beta c+\alpha d).$$

We now calculate the sum of the three squares $xx+yy+zz=s,$ and we get this formula

$$s=Aff+Bgg-2Cfg, \text{ where}$$

$$A=2bbcc-4abcd+2aadd+\beta\beta cc+2\alpha\beta cd+\alpha\alpha dd,$$

$$B=2aacc+8abcd+8bbdd+\alpha\alpha cc+4\alpha\beta cd+4\beta\beta dd,$$

$$C=(\alpha c-2\beta d)(\beta c+\alpha d).$$

Let $\quad D=(aa+2bb)(cc+2dd)=aacc+2bbcc+2aadd+4bbdd;$

And observe that $\quad s=(aa+2bb)(cc+2dd)(ff+2gg)=Dff+2Dgg.$

Subtracting this value of s from the expression $Aff+Bgg-2Cfg,$ we obtain the equation

$$Fff+Ggg-2Cfg=0, \text{ where } F=A-D,\ G=B-2D,$$

and consequently $\quad F=(\beta\beta-aa)cc+(\alpha\alpha-4bb)dd-4abcd+2\alpha\beta cd,$

$$G=(\alpha\alpha-4bb)cc+8abcd-4\alpha\beta cd+4(\beta\beta-aa)dd\ .$$

These may be written as follows:

$$F=\big((\beta+a)c+(\alpha+2b)d\big)\big((\beta-a)c+(\alpha-2b)d\big),$$

$$G=\big((\alpha+2b)c-2(\beta+a)d\big)\big((\alpha-2b)c-2(\beta-a)d\big).$$

27. Using these equations, as before we can make $F=0$, by setting

$$\frac{c}{d}=\frac{-\alpha-2b}{\beta+a} \quad \text{or} \quad =\frac{-\alpha+2b}{\beta-a}.$$

Then our equation becomes $Ggg-2Cfg=0$, from which we get $\frac{f}{g}=\frac{G}{2C}$. This formula is complicated because of the value of *G*, but we can simplify it. Notice that since *F* is equal to zero, the quantity *G* may be replaced by 2*F*+*G,* and this quantity, according to the preceding equations, is equal to

$$\big(2(\beta\beta-aa)+\alpha\alpha-4bb\big)(cc+2dd)=-(aa+2bb)(cc+2dd);$$

consequently
$$\frac{f}{g}=-\frac{(aa+2bb)(cc+2dd)}{2(\alpha c-2\beta d)(\beta c+\alpha d)};$$

from which it follows that $f=(aa+2bb)(cc+2dd)$; $g=-2(\alpha c-2\beta d)(\beta c+\alpha d)$.

28. If one wants to substitute the values of *c*, *d*, *f*, *g* in the final formulas of *x*, *y*, *z* and *p*, *q*, *r*, they would become complicated. But we can give a very simple rule for finding the numbers *x*, *y*, *z* and *p*, *q*, *r*.

Rule

for finding as many solutions as one would want of our problem.

29. Having selected at will two numbers *m* and *n*, in which *m* must be odd, we obtain from these three quantities $s=mm+2nn$, $t=mm-2nn$ and $u=2mn$. Now the values of the six variables *x*, *y*, *z*, *p*, *q* and *r* become

$$x = s(s+u)(3s+4u) - 2tt(s+2u), \qquad p = st(3s+4u) + 4t(s+u)(s+2u),$$

$$y = s(s+u)(3s+4u) + 2tt(s+2u), \qquad q = st(3s+4u) - 4t(s+u)(s+2u),$$

$$z = st(3s+4u) + 2t(s+2u)^2, \qquad r = s(s+2u)(3s+4u) - 4tt(s+2u).$$

30. Considering these six formulas, notice that changing the sign of t only changes the signs of z, p and q and does not give us new solutions to our problem. But if we take u negative, these formulas will undergo a significant change. It follows that each pair of numbers m and n gives, two different solutions, depending on whether we take m and n positive or negative. Here are some examples.

Example 1. Let $m = 1$ and $n = \pm 1$; then $s = 3$, $t = 1$, $u = \pm 2$. First let $u = -2$, and get $s + u = 1$, $s + 2u = -1$, $3s + 4u = 1$ and, consequently,

$$x = 3 \cdot 1 \cdot 1 + 2 = 5, \qquad p = 3 - 4 = -1,$$

$$y = 3 - 2 = 1, \qquad q = 3 + 4 = 7,$$

$$z = 3 + 2 = 5, \qquad r = -3 + 4 = 1.$$

But here two of the desired numbers are equal, that's why this solution will not be admitted.

If we select $u = 2$, then $s + u = 5$, $s + 2u = 7$, $3s + 4u = 17$ and, consequently,

$$x = 3 \cdot 5 \cdot 17 - 2 \cdot 7 = 241, \qquad p = 3 \cdot 17 + 4 \cdot 5 \cdot 7 = 191,$$

$$y = 3 \cdot 5 \cdot 17 + 2 \cdot 7 = 269, \qquad q = 3 \cdot 17 - 4 \cdot 5 \cdot 7 = -89,$$

$$z = 3 \cdot 17 + 2 \cdot 49 = 149, \qquad r = 3 \cdot 7 \cdot 17 - 4 \cdot 7 = 329.$$

Example 2. Let, in this example, $m = 1$ and $n = 2$; then $s = 9$, $t = -7$, $u = \pm 4$. Let us take first $u = -4$; and get $s + u = 5$, $s + 2u = 1$, $3s + 4u = 11$ and finally obtain

$x = 9 \cdot 5 \cdot 11 - 98 = 397,$ $\qquad$ $p = -9 \cdot 7 \cdot 11 - 4 \cdot 7 \cdot 5 \cdot 1 = -833,$

$y = 9 \cdot 5 \cdot 11 + 98 = 593,$ $\qquad$ $q = -9 \cdot 7 \cdot 11 + 4 \cdot 7 \cdot 5 \cdot 1 = -553,$

$z = -7 \cdot 9 \cdot 11 - 2 \cdot 7 = -707,$ $\qquad$ $r = 9 \cdot 1 \cdot 11 - 4 \cdot 7 \cdot 7 \cdot 1 = -97.$

For the second case, let $u = 4;$ then $s + u = 13,$ $s + 2u = 17,$ $3s + 4u = 43$ and, consequently,

$x = 9 \cdot 13 \cdot 43 - 98 \cdot 17 = 3365,$ $\qquad$ $p = -7 \cdot 9 \cdot 43 - 4 \cdot 7 \cdot 13 \cdot 17 = -8897,$

$y = 9 \cdot 13 \cdot 43 + 98 \cdot 17 = 6697,$ $\qquad$ $q = -7 \cdot 9 \cdot 43 + 4 \cdot 7 \cdot 13 \cdot 17 = 3479,$

$z = -7 \cdot 9 \cdot 43 - 2 \cdot 7 \cdot 17^2 = -6755,$ $\qquad$ $r = 9 \cdot 17 \cdot 43 - 4 \cdot 7^2 \cdot 17 = 3247.$

Demonstration

of the preceding rule.

31. Let us put $aa + 2bb = \alpha\alpha + 2\beta\beta = s,$ $a\alpha - 2b\beta = t$ and $a\beta + b\alpha = u;$ we then have $ss = tt + 2uu.$ Let us take the values found above (27) of c and d, namely $c = -\alpha - 2b,$ $d = \beta + a.$ Concerning the other two values (*of c and d*), they are derived by taking a and b negative. We get $cc + 2dd = 3s + 4u,$ $ac + 2bd = -t,$ $\alpha c - 2\beta d = -(s + 2u),$ $bc - ad = -(s + u),$ and $\beta c + \alpha d = t,$ and finally, according to (27), $f = s(3s + 4u)$ and $g = 2t(s + 2u).$

32. Let us substitute now the values in the retrieved formulas above (26); we will find the following expressions:

$$x = s(s + u)(3s + 4u) - 2tt(s + 2u),$$

$$y = s(s + u)(3s + 4u) + 2tt(s + 2u),$$

$$z = st(3s + 4u) + 2t(s + 2u)^2,$$

$$p = st(3s+4u) + 4t(s+u)(s+2u),$$

$$q = st(3s+4u) - 4t(s+u)(s+2u),$$

$$r = s(3s+4u)(s+2u) - 4tt(s+2u),$$

which were obtained in the rule.

33. Finally, since the three numbers *s*, *t*, *u* are subject only to the condition $ss = tt + 2uu$, we must find the numbers *s*, *t*, *u* which fulfill this condition; then the preceding formulas will immediately give the values of the desired numbers. As for those of *s*, *t*, *u* which fill the condition $ss = tt + 2uu$, here are the simplest:

s	3	9	17	19	27	33	33	41	43
t	1	7	1	17	23	17	31	23	7
u	2	4	12	6	10	20	8	24	30.

Fourth method.

34. We have seen, at the beginning of this paper, that the equations

$$yy + zz - xx = pp, \qquad zz + xx - yy = qq$$

will be satisfied, if we take

$$z = aa + bb, \quad yy - xx = 4ab(aa - bb), \quad p = aa + 2ab - bb, \quad q = aa - 2ab - bb\,.$$

It is easy to see that these equation will be satisfied, if we take

$$z = mn(aa+bb), \qquad yy - xx = 4mmnn(aa-bb)$$

and $\qquad p = mn(aa + 2ab - bb), \qquad q = mn(aa - 2ab - bb).$

Therefore it remains to fulfill the third condition of our problem, namely:

$xx + yy - zz = rr.$

35. Now, so that the three numbers x, y, z do not have a common factor, we take $y+x=2mma(a+b)$ and $y-x=2nnb(a-b)$. To simplify the expressions let $aa+ab=A$ and $ab-bb=B,$ so that $y+x=2mmA$ and $y-x=2nnB;$ and since $A-B=aa+bb$ we find $z=mn(A-B).$ The sum of the squares of $y+x$ and $y-x$ is

$$2yy+2xx=4m^4AA+4n^4BB; \text{ therefore } yy+xx=2m^4AA+2n^4BB\,.$$

Subtracting from this the value of zz, we get this expression for rr

$$rr=2m^4AA+2n^4BB-mmnn(A-B)^2.$$

36. To make this formula more manageable, let us suppose $m=f+g,$ $n=f-g;$ from which we get $rr=\alpha f^4+\beta f^3g+\gamma ffgg+\beta fg^3+\alpha g^4,$ where

$$\alpha=2AA+2BB-(A-B)^2=(A+B)^2,$$

$$\beta=8AA-8BB,$$

$$\gamma=12(AA+BB)+2(A-B)^2.$$

By substituting these values of $\alpha,$ $\beta,$ γ in the preceding equation, we will have

$$rr=(A+B)^2f^4+8(AA-BB)f^3g+\left[12(AA+BB)+2(A-B)^2\right]ffgg+8(AA-BB)fg^3+(A+B)^2g^4.$$

37. To make this expression a square, let us suppose that its root is

$$r=(A+B)ff+4(A-B)fg-(A+B)gg,$$

Thus it follows that

$$rr = (A+B)^2 f^4 + 8(AA-BB) f^3 g - 2(A+B)^2 ffgg + 16(A-B)^2 ffgg - 8(AA-BB) fg^3 + (A+B)^2 g^4.$$

Subtracting from this expression the preceding one we will obtain

$$0 = 32ABffg + 16(AA-BB) fg^3,$$

from which we get $\qquad \dfrac{f}{g} = \dfrac{AA-BB}{-2AB}$

and, consequently, $\qquad f = AA - BB,$

$$g = -2AB.$$

Thus we find the numbers *f* and *g* according to the values of *A* and *B* which are determined by the equation $A = aa + ab$, $B = ab - bb$. Then we take $m = f + g$, $n = f - g$, and obtain the values of *x*, *y*, *z*, *p*, *q*, which, according to the preceding equations, are

$$x = mmA - nnB, \qquad y = mmA + nnB, \qquad z = mn(A-B),$$

$$p = mn(aa + 2ab - bb), \qquad q = mn(aa - 2ab - bb).$$

As for *r*, we had

$$r = (A+B) ff + 4(A-B) fg - (A+B) gg,$$

and this equation, because $m = f + g$, $n = f - g$, becomes

$$r = mn(A+B) + (mm - nn)(A-B).$$

Therefore, it is easy to develop the values of *x*, *y*, *z*, and *p*, *q*, *r* for each value of the letters *a* and *b*.

38. We present a method for finding as many solutions as desired. After arbitrarily selecting *a* and *b*, we form $A = aa + ab$, $B = ab - bb$, then $f = AA - BB$ and

$g=-2AB$. From these we get $m=f+g$, $n=f-g$. Thus, having determined these values, the desired numbers are given by the following formulas:

$$x = mmA - nnB, \qquad p = mn(aa+2ab-bb),$$

$$y = mmA + nnB, \qquad q = mn(aa-2ab-bb),$$

$$z = mn(A-B), \qquad r = mn(A+B)+(mm-nn)(A-B).$$

We show a few examples.

Example 1. Let $a=1$, $b=2$; we have $A=3$, $B=-2$; From these we have $f=5$, $g=12$, $m=17$, $n=-7$, and finally the desired numbers are:

$$x = 17\cdot 17\cdot 3 + 7\cdot 7\cdot 2 = 965, \qquad p = -17\cdot 7\cdot 1 = -119,$$

$$y = 17\cdot 17\cdot 3 - 7\cdot 7\cdot 2 = 769, \qquad q = -17\cdot 7\cdot -7 = 833,$$

$$z = -17\cdot 7\cdot 5 = -595, \qquad r = -7\cdot 17\cdot 1 + 240\cdot 5 = 1081.$$

This solution has already been found reported previously (22).

Example 2. Let $a=2$, $b=1$; and get $A=6$, $B=1$, $f=35$, $g=-12$. Finally we have $m=23$, $n=47$, and consequently,

$$x = 23\cdot 23\cdot 6 - 47\cdot 47\cdot 1 = 965, \qquad p = 23\cdot 47\cdot 7 = 7567,$$

$$y = 23\cdot 23\cdot 6 + 47\cdot 47\cdot 1 = 5383, \qquad q = 23\cdot 47\cdot -1 = -1081,$$

$$z = 23\cdot 47\cdot 5 = 5405, \qquad r = 23\cdot 47\cdot 7 - 1680\cdot 5 = -833.$$

39. It should be observed that it would be superfluous to take both the numbers *a* and *b* odd, since then the numbers *A* and *B* will be even, and consequently reducible to the smaller numbers.

Example 3. Let $a=2$, $b=3$; we have $A=10$, $B=-3$, $f=91$, $g=60$, $m=151$, $n=31$; from which results

$x = 151 \cdot 151 \cdot 10 + 31 \cdot 31 \cdot 3 = 230893,$ $p = 151 \cdot 31 \cdot 7 = 32767,$

$y = 151 \cdot 151 \cdot 10 - 31 \cdot 31 \cdot 3 = 225127,$ $q = 151 \cdot 31 \cdot -17 = -79577,$

$z = 151 \cdot 31 \cdot 13 = 60853,$ $r = 151 \cdot 31 \cdot 7 + 21840 \cdot 13 = 316687.$

We observe here that all the solutions found with this method, are essentially different from all those calculated from the preceding methods.

Notes by section

Sections 1 to 7. In the first section Euler introduces the central problem to be examined. In sections 2 to 7 he describes explorations that he later abandoned. He presents four successful solutions beginning in section 8

Section 2. Euler assumes that the reader knows the following:

Lemma 1.: *Numbers of the form* $4n+2$ *and* $4n+3$ *cannot be squares*.

Proof: If $x^2 = 4n+2$, then x is even. Let $x = 2p$ and we have $4p^2 = 4n+2$. Thus $n = p^2 - 1/2$ which is impossible. This shows that $4n+2$ cannot be a square. If $x^2 = 4n+3$, then x is odd. Let $x = 2p+1$ and we have $4p^2 + 4p + 1 = 4n+3$. Thus $n = p^2 + p - 1/2$ which is impossible. This shows that $4n+3$ cannot be a square.

Section 9. The following is used without proof:

Lemma 2: The set T of numbers of the form $a^2 + 2b^2$ is closed under multiplication.

Proof By direct multiplication we see that

$$\left(a^2 + 2b^2\right)\left(c^2 + 2d^2\right) = (ac + 2bd)^2 + 2(ad - bc)^2.$$

Section 11. Euler states the following without proof:

Lemma 3: If $a^2 + 2b^2$ is odd, then $a^2 + 2b^2 \equiv 1$ *or* $3 \bmod 8$.

Proof: It is clear that *a* must be odd, and therefore $a^2 = (2m+1)^2 = 4m(m+1)+1$. Thus $a^2 \equiv 1 \bmod 8$. If *b* is even, then $2b^2 \equiv 0 \bmod 8$ and we have $a^2 + 2b^2 \equiv 1 \bmod 8$. If *b* is odd then $2b^2 = 2(2m+1)^2 = 8m(m+1)+2 \equiv 2 \bmod 8$ and $a^2 \equiv 1+2 \equiv 3 \bmod 8$. The lemma is proved.

The converse is not true. The number 35 is the smallest example, since $35 \equiv 3 \bmod 8$ but cannot be expressed in the form $a^2 + 2b^2$.

<u>Section 12.</u> We used a computer to extend Euler's list to all primes congruent to 1 or 3 mod 8 less than 800,000. In all cases these primes were of the form $a^2 + 2b^2$. Is Euler conjecturing that this is true for all such primes? (See (e) below.)

<u>Sections 9 to 14</u>. Let *T* denote the set of all numbers of the form $a^2 + 2b^2$ ..In these sections Euler studies the nature of numbers in *T*. We find that:

(a) The set *T* is closed under multiplication. (See Lemma 2 above.)

(b) If *p* is a prime in *T*, then $p = a^2 + 2b^2$, where *a* and *b* are unique.(See section 9.)

(c) If $s = x^2 + y^2 + z^2$, where the numbers *x, y, z* are the solution of the main problem of this paper, then *s* is in *T* and has at least 3 prime factors. (See section 10.)

(d) If *n* is odd and in *T*, then *n* is congruent to 1 or 3 mod 8. (See section 11.)

(e) All primes equal to 1 or 3 mod 8 divide some number in *T*. (See sections 13 and 14.)