STUDIES
ON
THE ROTATIONAL MOVEMENT
OF CELESTIAL BODIES

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I.

If celestial bodies were perfectly spherical, or their moments of inertia compared to their principal axes were equivalent, then some rotational movement they had once received would always be conserved without changing either their speed or rotational axis, which would always stay directed toward the same points in the sky. In addition, the attractive forces exerted on some body from other celestial bodies would not at all disrupt its rotational movement, since the resultant average force would pass through the inertial center of the body, as I demonstrated in a preceding Memoir. But if a celestial body is not spherical, or its moments of inertia compared to its three principal axes are not equal, and it began to turn around a different axis from its principal axes, then still there would not be applied forces, its rotational movement would be disturbed, and the rotational axis would change direction, as I demonstrated in another Memoir that preceded this one.

2. From this, it follows that if the rotational movement of a celestial body is not uniform, or the rotational axis is not always found to be directed toward the same points in the sky, then this body certainly does not have this property, that its moments of inertia compared to its principal axes are equivalent, but there will be an inequality between its principal moments of inertia. Therefore, since Earth's axis is not always directed toward the same points in the sky, although the diurnal movement seems uniform, we must conclude that Earth's moments of inertia are not equivalent. A similar inequality must take place in the Moon, since its rotational movement is not uniform, and a change in the position of its rotational axis has been observed.

3. On the matter of Earth's movement, it is necessary to observe that Earth's axis is different than its rotational axis. Since Earth's axis is in continuous movement due to its nutation and the precession of equinoxes, it never aligns with the rotational axis, which at each instant is absolutely stationary disregarding annual movement. To make this distinction clearer, let us consider a sphere around Earth's center on whose surface is Earth's pole *A*, which advances during a small time *dt* to *a*, creating the infinitely small angle $APa = d\omega$ around a fixed point *P*, but Earth nevertheless turns around the pole *A* by the small angle $ZAz = d\varphi$. With that established, there will be a point *O* on the arc *PA* that, by this dual movement, will stay at rest, because, by virtue of the pole, it describes the arc $O\omega = d\omega \sin PO$ and because of the diurnal movement, the arc $Oo = a\varphi \sin AO$. Let us therefore call these two arcs equivalent, and we will find $\tan AO = \dfrac{d\omega \sin AP}{d\varphi + d\omega \cos AP}$, where *O* will be the point on Earth which, for this instant, stays at rest.

Fig. 1

4. This therefore is not Earth's pole *A*, but another point *O* that is stationary during an instant, and hence the right line drawn to Earth's center from this point O will be the rotational axis and not Earth's axis, which passes through point *A*. It is certainly true that the difference, or the arc *AO*, is so small that it would not enter into any of our considerations, since the comparison of $d\omega$ to $d\varphi$ is $50''$ to $365\frac{1}{4} * 360°$, or $\frac{d\omega}{d\varphi} = \frac{1}{25920 * 365\frac{1}{4}}$, and since $\sin AP$ is about $\frac{2}{5}$, $\tan AO = \frac{1}{64800 * 365\frac{1}{4}}$, so that the interval *AO* is only $\frac{1}{115}$ of a second, or about a half of a sixtieth of a second. But, if the movement of the pole were more rapid compared to the diurnal movement, which could occur in other planets, then it would be necessary to carefully distinguish the planet's rotational axis. Because Earth's axis is a fixed line in Earth's body, but mobile with respect to the sky, the rotational axis is not a fixed line in Earth, but after time dt the rotational axis, currently the line derived from the center of Earth to the point *O*, will be the line derived to the point ω so that that the rotational axis continually changes both with regard to Earth and the sky.

5. There are then two manners of representing the diurnal movement of Earth. The first is the one used in Astronomy where a fixed line is conceived in Earth, called its axis, around which Earth turns while this line moves around the ecliptic poles, which are regarded as fixed points in the sky. The other manner is most appropriate for the Mechanical where the points in the sky are marked for each time while Earth turns around them; this manner is the only one of its kind and is perfectly determined by Earth's movement, yet the same movement could be represented by an infinity of different manners. Instead of the axis, some other line fixed anywhere in Earth could be considered and its movement assigned in the sky, then it would be necessary to define the movement by which Earth would turn around this line. But it is necessary to admit that the manner actually used is the simplest of its kind, which most reasonably represents to us the movement of Earth: it seems clearer than the other based on the rotational axis, although I was obliged to follow this one in the present studies.

6. Before examining to what degree the rotational movement of a celestial body is disturbed by the forces applied on it by the other celestial bodies, it will be good to explain what their rotational movement must be if they had not been subjected to such forces. I therefore present the case where all of a celestial body's moments of inertia are equivalent, since then not only the rotational movement would be uniform and the rotational axis stationary, but the applied forces would never be disrupted. I view the celestial body, for which we are determining the rotational movement, as having its moments of inertia compared to its three principal axes to be inequivalent. Firstly, I remark that if this body had once received a rotational movement around one of its principal axes, then this axis would stay constantly directed toward the same points in the sky, and the angular speed would always stay the same. It is apparent that if Earth were not subjected to solar and lunar forces, then its rotational axis would stay at rest, from which it must conclude that the right line we call its axis is one of its three principal axes.

7. This observation leads me to a thought that seems important. Since the inertial center of Earth is situated on its axis, it is still undecided if it is found at the middle of the axis, if it is close to one or the other pole, or rather if it falls in the equatorial plane or some other parallel circle. It is easily understood that common phenomena would not provide the knowledge to decide the above, but maybe some effects of the action of the

Moon will be able to give us some clarity. Mr. Meyer, the talented Astronomer of Göttingen to whom Astronomy is indebted for many important discoveries, believes to have strong supporting evidence that the inertial center of Earth is not found at the middle of the axis or in the equatorial plane, but in a certain parallel circle whose determination merits, without doubt, all the possible cares of Astronomers. It is in this parallel circle that the other two principal axes of Earth must exist.

8. But, if Earth had not received a rotational movement around one of its three principal axes, the phenomenon of its diurnal movement would no longer be so simple and would demand addressing to justly represent it; even so, there would not be forces disturbing it. Although this case likely has not taken place in Earth, it could very well exist in some other planet, therefore meriting a more careful development; maybe this is the cause of the irregularities noted in the rotational movement of Venus; and hence it would be good to treat this more particularly, all as if it had taken place in Earth. In this case, it would not be a question of Earth's axis since some stationary points, around which the sky would seem to turn, would be seen in the sky for some time, but these points would continually change place, and the rotational movement would not even be uniform. These irregularities would, without doubt, hinder most Astronomers.

9. Let us return to the beginning, or to a fixed epoch, where the three principal Fig. 2 axes of Earth had been directed toward the points in the sky $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$. Let us also suppose then that Earth had had a rotational movement around the point $\mathfrak{D}$ in the direction $\mathfrak{ABC}$ with an angular speed $\mathfrak{c}$. Arcs of great circles are derived from the point $\mathfrak{D}$ to the points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and let us call $\mathfrak{DA} = \mathfrak{a}$, $\mathfrak{DB} = \mathfrak{b}$, and $\mathfrak{DC} = \mathfrak{c}$. For the constitution of Earth, its moment of inertia compared to the axis $\mathfrak{A} = Maa$, compared to the axis $\mathfrak{B} = Mbb$, and to the axis $\mathfrak{C} = Mcc$, which I assume are known. Now, to most simply represent that the rotational movement, by which Earth will subsequently be carried, would be feasible, it must always be compared to a sure point fixed in the sky from which are derived the arcs of great circles to points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$:

$$\cos \mathfrak{A}P = \frac{\mathfrak{c}\, aa \cos \mathfrak{a}}{\sqrt{G}}; \qquad \cos \mathfrak{B}P = \frac{\mathfrak{c}\, bb \cos \mathfrak{b}}{\sqrt{G}}; \qquad \cos \mathfrak{C}P = \frac{\mathfrak{c}\, cc \cos \mathfrak{c}}{\sqrt{G}},$$

where $\sqrt{G} = \mathfrak{c}\sqrt{(a^4 \cos \mathfrak{a}^2 + b^4 \cos \mathfrak{b}^2 + c^4 \cos \mathfrak{c}^2)}$.

10. To better understand this important point in the sky P, knowing the position of the principal axes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ at this instant with respect to axis $\mathfrak{A}$, we will have

$$\cos \mathfrak{BA}P = \frac{bb \cos \mathfrak{b}}{\sqrt{(b^4 \cos \mathfrak{b}^2 + c^4 \cos \mathfrak{c}^2)}}, \text{ and}$$

$$\sin \mathfrak{BA}P = \frac{-\,cc \cos \mathfrak{c}}{\sqrt{(b^4 \cos \mathfrak{b}^2 + c^4 \cos \mathfrak{c}^2)}}, \text{ so that}$$

$$\tan \mathfrak{BA}P = \frac{-\,cc \cos \mathfrak{c}}{bb \cos \mathfrak{b}} \qquad \text{having}$$

$$\cos \mathfrak{A}P = \frac{aa \cos \mathfrak{a}}{\sqrt{(a^4 \cos \mathfrak{a}^2 + b^4 \cos \mathfrak{b}^2 + c^4 \cos \mathfrak{c}^2)}},$$

from where the position of point P is most conveniently determined. Here it is necessary to note that if the principal moments of Earth were equivalent, or $aa = bb = cc$, because of $\sqrt{G} = \mathfrak{v}\, aa$, since $\cos\mathfrak{a}^2 + \cos\mathfrak{b}^2 + \cos\mathfrak{c}^2 = 1$, then we would have $\cos\mathfrak{A}P = \cos\mathfrak{a}$, $\cos\mathfrak{B}P = \cos\mathfrak{b}$, and $\cos\mathfrak{C}P = \cos\mathfrak{c}$; and hence the point P would coincide with point $\mathfrak{D}$. Therefore, if Earth's moments of inertia are approximately equal, then we would know that the point P will not be very far from point $\mathfrak{D}$; this is why it is necessary to conceive the point P placed inside of the triangle $\mathfrak{ABC}$ containing point $\mathfrak{D}$, because since the principal axes pass through two opposed points of the sphere, we can always form a triangle $\mathfrak{ABC}$ containing the point $\mathfrak{D}$.

11. Let us introduce this point P in the calculation, and let us establish for the beginning, or our epoch, the arcs

$$P\mathfrak{A} = \mathfrak{l},\ P\mathfrak{B} = \mathfrak{m},\ \text{and}\ B\mathfrak{C} = \mathfrak{n},$$

and let $\mathfrak{G} = a^4\cos\mathfrak{a}^2 + b^4\cos\mathfrak{b}^2 + c^4\cos\mathfrak{c}^2$, so that $\sqrt{G} = \mathfrak{v}\sqrt{\mathfrak{G}}$, and

$$\cos\mathfrak{l} = \frac{aa\cos\mathfrak{a}}{\sqrt{\mathfrak{G}}};\qquad \cos\mathfrak{m} = \frac{bb\cos\mathfrak{b}}{\sqrt{\mathfrak{G}}};\qquad \cos\mathfrak{n} = \frac{cc\cos\mathfrak{c}}{\sqrt{\mathfrak{G}}},$$

and $\mathfrak{G}\left(\frac{\cos\mathfrak{l}^2}{a^4} + \frac{\cos\mathfrak{l}^2}{a^4} + \frac{\cos\mathfrak{l}^2}{a^4}\right) = 1$; logically, for the rotational pole $\mathfrak{D}$ at the same time:

$$\cos\mathfrak{a} = \frac{\sqrt{\mathfrak{G}}}{aa}\cos\mathfrak{l};\qquad \cos\mathfrak{b} = \frac{\sqrt{\mathfrak{G}}}{Bb}\cos\mathfrak{m};\qquad \cos\mathfrak{c} = \frac{\sqrt{\mathfrak{G}}}{cc}\cos\mathfrak{n},$$

and for the angle $P\mathfrak{AB}$, if we wanted to use it:

$$\cos P\mathfrak{AB} = \frac{\cos\mathfrak{m}}{\sin\mathfrak{l}},\quad \text{and}\quad \sin P\mathfrak{AB} = -\frac{\cos\mathfrak{n}}{\sin\mathfrak{l}}$$

so that in calling this angle $P\mathfrak{AB} = \mathfrak{r}$, we have $\tan\mathfrak{r} = -\frac{\cos\mathfrak{n}}{\cos\mathfrak{m}}$. These quantities are for the initial state, or the fixed epoch, and depend on the position of the rotational axis $\mathfrak{D}$ compared to the body's principal axes. Next, I assume that the body had turned then in the direction $\mathfrak{ABC}$ with angular speed $\mathfrak{v}$, where $\mathfrak{v}$ marks the described angle in one second.

12. Having established these elements, the state and movement of the body after some time has passed, which I call t seconds, will be investigated. The principal axes will then have reached some A, B, C, and the body will presently turn around the rotational axis O in the direction ABC with angular speed $\mathfrak{v}$. To this effect, let us, for brevity, call $\frac{bb-cc}{aa} = A$; $\frac{cc-aa}{bb} = B$; $\frac{aa-bb}{cc} = C$, as in the preceding Memoir, where I had given the solution to this problem, but instead of the letter u, I use Gv here. Firstly, it is necessary to construct this differential equation:

$$\mathfrak{r}\, dt \sqrt{\mathfrak{G}} = \frac{aabbcc\, dv}{\sqrt{((\cos \mathfrak{l}^2 + 2Aa^4 v)(\cos \mathfrak{m}^2 + 2Bb^4 v)(\cos \mathfrak{n}^2 + 2Cc^4 v))}},$$

so that for each proposed time, the quantity v, which disappears at time $t = 0$, can be assigned. It is noticed that for the letters A, B, C, at least one must be a negative value, and hence this construction can be derived from the movement of a pendulum that is driven in a circle. At the least, it will not be difficult for each case to fill the tables for the values of v at each time.

13. Then, calling the angle $\mathfrak{A}PA = \lambda$, which marks how much the principal axis A has advanced since the beginning in the opposite direction of the rotational movement, we have

$$d\lambda = \mathfrak{r}\, dt \sqrt{\mathfrak{G}}\ \frac{bb \cos \mathfrak{n}^2 + cc \cos \mathfrak{m}^2 - 2Aaabbccv}{bbcc\, (\sin \mathfrak{l}^2 - 2Aa^4 v)},$$

so that the angular speed by which point A presently advances around the fixed point P is:

$$\mathfrak{r} \sqrt{\mathfrak{G}}\, \frac{bb \cos \mathfrak{n}^2 + cc \cos \mathfrak{m}^2 - 2Aaabbccv}{bbcc\, (\sin \mathfrak{l}^2 - 2Aa^4 v)},$$

which had been at the beginning:

$$\mathfrak{r} \sqrt{\mathfrak{G}}\, \frac{bb \cos \mathfrak{n}^2 + cc \cos \mathfrak{m}^2}{bbcc \sin \mathfrak{l}^2}.$$

In the same manner, we could assign how much the other two principal axes B and C will have advanced around the fixed point P since their initial positions $\mathfrak{B}$ and $\mathfrak{C}$. Now then, we will have for the arcs PA, PB, PC

$\cos PA = \sqrt{(\cos \mathfrak{l}^2 + 2Aa^4 v)}$; $\cos PB = \sqrt{(\cos \mathfrak{m}^2 + 2Bb^4 v)}$; $\cos PC = \sqrt{(\cos \mathfrak{n}^2 + 2Cc^4 v)}$,
$\sin PA = \sqrt{(\sin \mathfrak{l}^2 - 2Aa^4 v)}$; $\sin PB = \sqrt{(\sin \mathfrak{m}^2 - 2Bb^4 v)}$; $\sin PC = \sqrt{(\sin \mathfrak{n}^2 - 2Cc^4 v)}$,
from which the true position of the three principal axes A, B, C will be known.

14. But, having found this for a single axis A, the other two will be more easily determined by the angle PAB, which gives us:

$$\cos PAB = \sqrt{\frac{\cos \mathfrak{m}^2 + 2Bb^4 v}{\sin \mathfrak{l}^2 - 2Aa^4 v}}, \text{ and}$$

$$\sin PAB = -\sqrt{\frac{\cos \mathfrak{m}^2 + 2Cc^4 v}{\sin \mathfrak{l}^2 - 2Aa^4 v}}.$$

Therefore, this angle is variable; its increment for the element of time dt is found to be

$$\frac{dPAB}{dt} = \frac{-\mathfrak{r} \sqrt{\mathfrak{G}}\, (Cc^4 \cos \mathfrak{m}^2 - Bb^4 \cos \mathfrak{n}^2) \sqrt{(\cos \mathfrak{l}^2 + 2Aa^4 v)}}{aabbcc\, (\sin \mathfrak{l}^2 - 2Aa^4 v)},$$

which is the angular speed by which the angle PAB decreases, so that at the beginning, the angular speed by which the angle PAB diminished was

$$\frac{\mathfrak{r} \sqrt{\mathfrak{G}}\, (Cc^4 \cos \mathfrak{m}^2 - Bb^4 \cos \mathfrak{n}^2) \cos \mathfrak{l}}{aabbcc \sin \mathfrak{l}^2}.$$

It is necessary to note here that because of the assumed values of the letters A, B, C, both $Aa^2 + Bb^2 + Cc^2 = 0$ and $Aa^4 + Bb^4 + Cc^4 = 0$ are true.

15. This could suffice to know the movement, having determined the angle $\mathfrak{A}PA$, the arc PA, and the angle PAB, from where it would be known for each proposed time the position of the body with regard to the sky, and hence reciprocally, the apparent position of the sky. But this observation can be furthered. The body will then turn with angular speed $\mathfrak{V} = \mathfrak{r}\sqrt{(1 + 2(A + B + C)\mathfrak{G}v)}$ in the direction ABC around the rotational axis whose position is such that

$$\cos AO = \frac{\sqrt{\mathfrak{G}(\cos\mathfrak{l}^2 + 2Aa^4v)}}{aa\sqrt{(1 + 2(A + B + C)\mathfrak{G}v)}} = \frac{\sqrt{(\cos\mathfrak{a}^2 + 2A\,\mathfrak{G}v)}}{\sqrt{(1 - 2ABC\,\mathfrak{G}v)}}$$

$$\cos BO = \frac{\sqrt{\mathfrak{G}(\cos\mathfrak{m}^2 + 2Bb^4v)}}{bb\sqrt{(1 + 2(A + B + C)\mathfrak{G}v)}} = \frac{\sqrt{(\cos\mathfrak{b}^2 + 2B\,\mathfrak{G}v)}}{\sqrt{(1 - 2ABC\,\mathfrak{G}v)}}$$

$$\cos CO = \frac{\sqrt{\mathfrak{G}(\cos\mathfrak{n}^2 + 2Cc^4v)}}{cc\sqrt{(1 + 2(A + B + C)\mathfrak{G}v)}} = \frac{\sqrt{(\cos\mathfrak{c}^2 + 2C\,\mathfrak{G}v)}}{\sqrt{(1 - 2ABC\,\mathfrak{G}v)}}$$

where O will be the point in the sky that appears to stay at rest for this instant.

16. This representation of the movement becomes much simpler if the body's moments of inertia compared to the two axes B and C are equivalent, or $cc = bb$, since then $A = 0$, $B = -C = 1 - \frac{aa}{bb}$, and $\mathfrak{G} = a^4\cos\mathfrak{a}^2 + b^4(\cos\mathfrak{b}^2 + \cos\mathfrak{c}^2) = a^4\cos\mathfrak{a}^2 + b^4\sin\mathfrak{a}^2$

or $\mathfrak{G}\left(\frac{\cos\mathfrak{l}^2}{a^4} + \frac{\sin\mathfrak{l}^2}{b^4}\right) = 1$, the arc $P\mathfrak{A} = \mathfrak{C}$ always remains the same quantity,

or $PA = P\mathfrak{A}$ turns around the point P uniformly with angular speed $\frac{\mathfrak{r}\sqrt{\mathfrak{G}}}{bb}$ in the direction $\mathfrak{A}A$. Furthermore, the speed at which the arc AB nevertheless turns around the point A, by which the angle PAB decreases, is also constant:

$$\frac{\mathfrak{r}\sqrt{\mathfrak{G}}}{aab^4} - Bb^4\cos\mathfrak{l} = \frac{\mathfrak{r}\sqrt{\mathfrak{G}}\cos\mathfrak{l}}{aa}\left(\frac{aa}{bb} - 1\right).$$

Logically, the angular speed of the arc $PA = \mathfrak{l}$ around the fixed point P in the direction $\mathfrak{A}A$ is the angular speed by which the body nevertheless turns around the point A in a contrary direction to BP between 1 and $\left(1 - \frac{bb}{aa}\right)\cos\mathfrak{l}$. Here then this movement is represented in the same manner that we are accustomed to see from Earth in that the axis of Earth is mobile around the ecliptic poles.

17. One such movement could be represented by the motion of a Machine of the
Fig. 3† following manner. Let $PQRS$ be a circle freely moving around some diametrically opposed pivots P and R. In this circle is set, at A and D, an axis AD of a body $asdq$, around which the body can freely turn while the same circle turns around the pivots P and

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† This reference is missing in the original text.

R. Now to represent the movement from the preceding §, both rotational movements must be uniform, but in such a way that one is directed in a contrary direction in regard to the other. Assuming $aa > bb$ and the angular speed of the circle around the pivots P and R is like that of the body around the axis AD by the ratio 1 to $\left(1 - \frac{bb}{aa}\right) \cos PA$.

From this, it is seen that the movement of the body is much slower than that of the axis AD, and would completely disappear in the case where $aa = bb$. Now in the case that $aa < bb$, both movements would be directed in the same fashion. One such movement would be self-sustained and would not need outside forces.

18. Using a similar machine, the movement of a body, as determined above, could also be represented in general where all of the principal moments of inertia are inequivalent, but then the axis AD must be set in the circle such that the points A and D can be moved closer or farther from the pivots P and R. Furthermore, neither rotational movement will be uniform but must be determined by the above formulas. To this effect, the minimum and maximum of v must be considered. Assuming $aa > bb$ and $bb > cc$, since these three formulas must be real and not surpass unity,

$$\sqrt{(\cos \mathfrak{l}^2 + 2aa(bb - cc)v)}; \qquad \sqrt{(\cos \mathfrak{m}^2 - 2bb(aa - cc)v)}; \qquad \sqrt{(\cos \mathfrak{n}^2 + 2cc(aa - bb)v)},$$

the most positive value $+v$ is equal to the least of these three formulas

$$\frac{\sin \mathfrak{l}^2}{2aa(bb - cc)}; \qquad \frac{\cos \mathfrak{m}^2}{2bb(aa - cc)}; \qquad \frac{\sin \mathfrak{n}^2}{2cc(aa - bb)},$$

and the most negative value $-v$ equals the least of these three formulas

$$\frac{\cos \mathfrak{l}^2}{2aa(bb - cc)}; \qquad \frac{\sin \mathfrak{m}^2}{2bb(aa - cc)}; \qquad \frac{\cos \mathfrak{n}^2}{2cc(aa - bb)}.$$

19. It would therefore be possible that Earth had one such complicated rotational movement without which it would be necessary to find the cause in outside forces. But, although Earth's axis actually moves around the ecliptic poles, this movement is much different than that which I just described, because in Earth the movement of the axis is extremely slow with regard to the movement around the axis, whereas the described movement of the axis is much faster than that of the body around the axis. This observation suffices to assure us that the movement of Earth's axis, or its nutation, with the precession of the equinoxes, is the effect of an outside cause without which Earth's axis would stay absolutely stationary disregarding the annual movement. From this, it is also evident that the line we called Earth's axis is certainly one of its three principal axes. But maybe, in the planet Venus, it is completely different.

20. Let us see now how a celestial body's rotational movement will be disturbed by some outside force, which comes from the attraction of another celestial body, that I will call a center of force. Since here it is solely a matter of rotational movement, and I assume the inertial center of the proposed body is at rest, the center of force will describe around it a certain orbit, which is compared to a fixed sphere described around an inertial center of the body; let the directed line QFS follow the order of the celestial signs. The Fig. 4
center of force attracts proportionally by the reciprocal of the square of the distances, and at the distance e, the force by which a body is pushed is precisely equal to the weight that this body would have on Earth. Now, since gravity is not the same everywhere, it is

necessary to this effect to choose a certain place where the height by which a body undoubtedly falls in one second is exactly known. The letter g will constantly mark this height.

21. The point P and the circle PQR are considered fixed terms, the letter t corresponds to how many seconds have passed, the center of force corresponds to the point F, and let the arc $PF = p$ and the angle $QPF = q$. In addition, s expresses the distance from the center of forces to the inertial center of the proposed body. The quantities p, q, s can be considered functions of the time t. At the same instant, the principal axes of the body correspond to the points A, B, C, compared to which the body's moments of inertia are Maa, Mbb, Mcc, where M is the mass, and having derived from these three points the arcs of great circles to both point F and the fixed point P, let us call these arcs $FA = \zeta$, $FB = \eta$, $FC = \theta$, and $PA = l$, $PB = m$, $PC = n$. In addition, let us call the angles $QPA = \lambda$, $QPB = \mu$, $QPC = \nu$, which must be considered as negatives with regard to those that I had introduced in the general solution, where I had taken them from the opposite circle PSR. Finally, the body presently turns around the point O in the direction ABC with angular speed ϑ. Let us call the arcs $OA = \alpha$, $OB = \beta$, $OC = \gamma$ and call $\vartheta\cos\alpha = x$, $\vartheta\cos\beta = y$, and $\vartheta\cos\gamma = z$.

22. Now the attractive force of point F provides us with the following moments of forces.

I. The moment of force compared to the axis IA in the direction

$$BC = \frac{3Mee}{s^3}(cc - bb)\cos\eta\cos\theta = P.$$

II. The moment of force compared to the axis IB in the direction

$$CA = \frac{3Mee}{s^3}(aa - cc)\cos\zeta\cos\theta = Q.$$

III. The moment of force compared to the axis IC in the direction

$$AB = \frac{3Mee}{s^3}(bb - aa)\cos\zeta\cos\eta = R.$$

Therefore, if we call $\frac{bb - cc}{aa} = A$; $\frac{cc - aa}{bb} = B$; $\frac{aa - bb}{cc} = C$, we will have the following differential equations:

I. $\quad dx - Ayzdt + \frac{6Agee}{s^3}dt\cos\eta\cos\theta = 0.$

II. $\quad dy - Bxzdt + \frac{6Bgee}{s^3}dt\cos\zeta\cos\theta = 0.$

III. $\quad dz - Cxydt + \frac{6Cgee}{s^3}dt\cos\zeta\cos\eta = 0.$

IV. $\quad dl\sin l = dt(y\cos n - z\cos m).$

V. $\quad dm\sin m = dt(z\cos l - x\cos n).$

VI. $\quad dn\sin n = dt(x\cos m - y\cos l).$

VII. $\quad d\lambda\sin l^2 = dt(y\cos m + z\cos n).$

VIII. $\quad d\mu\sin m^2 = dt(z\cos n + x\cos l).$

IX. $\quad d\nu\sin n^2 = dt(x\cos l + y\cos m).$

23. The arcs ζ, η, θ can be expressed by the other quantities, of which the differentials are determined by these equations, because the principles of Spherical Trigonometry provide:

$$\cos\zeta = \cos(\lambda - q)\sin l \sin p + \cos l \cos p,$$
$$\cos\eta = \cos(\mu - q)\sin m \sin p + \cos m \cos p,$$
$$\cos\theta = \cos(\nu - q)\sin n \sin p + \cos n \cos p.$$

But it is easily understood that this substitution would not be of use and that the general solution of the formulas we just found is too difficult to be of practical use past here. The large number of variable quantities that enter here do not allow us any foresight into what methods would be necessary to follow. For this reason, I see myself obliged to confine my studies to some particular cases in which I can hope for some success. At the least, the case of Earth is not subject to such great difficulties that we can not surmount them.

24. To apply these derived formulas to the movement of Earth, I make the following assumptions:

I. I assume that the rotational axis O is very close to the principal axis A, so that the arc $OA = \alpha$ can be regarded as extremely small.

II. I assume that the moments of inertia compared to the two other principal axes B and C are equivalent so that $cc = bb$, and hence $A = 0$, and $B = 1 - \frac{aa}{bb}$, $C = \frac{aa}{bb} - 1$, therefore $C = -B$.

It seems certain that these two assumptions take place in Earth, having noted that if Earth were not subject to the action of the solar and lunar forces, then it would turn uniformly around its axis, which would stay fixed. Therefore, actually the rotational axis O never noticeably differs from the principal axis A that we call by excellence Earth's axis. Now, it would seem equally certain that the moments of inertia compared to the other two principal axes are equivalent because of the roundness of Earth around its axis A.

25. Since it is convenient to compare everything to pole A, let us call the angle $PAB = r$, and we will have $\cos m = \sin l \cos r$ and $\cos n = -\sin l \cos r$. Then, since the arc $AO = \alpha$ is almost infinitely small, having derived from O on the arcs AB and AC the perpendiculars Ob, Oc, let us call the angle $OAb = \rho$, and we will have $Ab = \alpha\cos\rho$ and $Ac = \alpha\sin\rho$, therefore $BO = \beta = 90° - \alpha\cos\rho$ and $CO = \gamma = 90° - \alpha\sin\rho$. From that, we derive $x =$ Ȣ, $y = \alpha$Ȣ$\cos\rho$, and $z = \alpha$Ȣ$\sin\rho$, neglecting the terms where α would have more than one dimension. Now the equations IV, V, VI will give

IV. $dl \sin l = -\alpha$Ȣ$dt \sin l \sin(r+\rho)$, or $dl = -\alpha$Ȣ$dt \sin(r+\rho)$,

V. $-dl \cos l \cos r + dr \sin l \sin r =$ Ȣ$dt\,(\alpha \cos l \sin\rho + \sin l \sin r)$,

VI. $dl \cos l \sin r + dr \sin l \cos r =$ Ȣ$dt\,(\sin l \cos r - \alpha \cos l \cos\rho)$,

from where the combination V $\sin r$ + VI $\cos r$ provides

$dr \sin l =$ Ȣ$dt\,(\sin l - \alpha \cos l \cos(r+\rho))$

or better yet $dr =$ Ȣ$dt - \dfrac{\alpha\text{Ȣ}dt \cos(r+\rho)}{\tan l}$,

and hence we will have almost exactly $dr =$ Ȣdt.

26. From the last three equations, it suffices to take

VII. $d\lambda \sin l^2 = \alpha$Ȣ$dt \sin l \cos(r+\rho)$, or

$$d\lambda = \frac{\alpha\text{♉}dt\cos(r+\rho)}{\sin l},$$ because the angles μ and ν depend on λ:

$$\cos(\lambda-\mu) = -\frac{\cos l\cos m}{\sin l\sin m}; \qquad \cos(\lambda-\nu) = -\frac{\cos l\cos n}{\sin l\sin n},$$

$$\sin(\lambda-\mu) = -\frac{\cos n}{\sin l\sin m}; \qquad \sin(\lambda-\nu) = -\frac{\cos m}{\sin l\sin n}.$$

From this, since $\mu - q = (\lambda-q)-(\lambda-\mu)$ and $\nu - q = (\lambda-q)-(\lambda-\nu)$, we derive:

$$\cos(\mu-q) = -\frac{\cos l\cos m\cos(\lambda-q) - \cos n\sin(\lambda-q)}{\sin l\sin m}$$

$$= -\frac{\cos l\cos r\cos(\lambda-q) + \sin r\sin(\lambda-q)}{\sin m}$$

$$\cos(\nu-q) = -\frac{\cos l\cos n\cos(\lambda-q) + \cos m\sin(\lambda-q)}{\sin l\sin n}$$

$$= +\frac{\cos l\sin r\cos(\lambda-q) + \cos r\sin(\lambda-q)}{\sin n}$$

and as a result we obtain:

$$\cos\zeta = \sin l\sin p\cos(\lambda-q) + \cos l\cos p$$

$$\cos\eta = -\cos l\sin p\cos r\cos(\lambda-q) + \sin p\sin r\sin(\lambda-q) + \sin l\cos p\cos r$$

$$\cos\theta = +\cos l\sin p\sin r\cos(\lambda-q) + \sin p\cos r\sin(\lambda-q) - \sin l\cos p\sin r.$$

27. The first equation, because $A = 0$, first gives $dx = d\text{♉} = 0$, hence the angular speed will always be the same, which will be called ε, from which $x = \text{♉} = \varepsilon$, and hence $dl = -\varepsilon\alpha dt\sin(r+\rho)$; $d\lambda = \dfrac{\varepsilon\alpha dt\cos(r+\rho)}{\sin l}$; and $dr = \varepsilon dt - \dfrac{\varepsilon\alpha dt\cos(r+\rho)}{\tan l}$. It follows that since $y = \varepsilon\alpha\cos\rho$ and $z = \varepsilon\alpha\sin\rho$, equations II and III will become:

$$\text{II.}\quad \varepsilon d\alpha\cos\rho - \varepsilon\alpha d\rho\sin\rho - B\varepsilon\varepsilon\alpha dt\sin\rho + \frac{6Bgee}{s^3}dt\cos\zeta\cos\theta = 0.$$

$$\text{III.}\quad \varepsilon d\alpha\sin\rho + \varepsilon\alpha d\rho\cos\rho + B\varepsilon\varepsilon\alpha dt\cos\rho - \frac{6Bgee}{s^3}dt\cos\zeta\cos\eta = 0.$$

It is good to observe that the solution of these gives

$$\sin r\cos\eta + \cos r\cos\theta = \sin p\sin(\lambda-q)$$

$$\sin r\cos\theta - \cos r\cos\eta = \cos l\sin p\cos(\lambda-q) - \sin l\cos p$$

from which we derive these two other equations

$$\varepsilon d\alpha\cos(r+\rho) - \varepsilon\alpha d\rho\sin(r+\rho) - B\varepsilon\varepsilon\alpha dt\sin(r+\rho) + \frac{6Bgee}{s^3}dt\cos\zeta\sin p\sin(\lambda-q) = 0$$

$$\varepsilon d\alpha\sin(r+\rho) + \varepsilon\alpha d\rho\cos(r+\rho) + B\varepsilon\varepsilon\alpha dt\cos(r+\rho) + \frac{6Bgee}{s^3}dt\cos\zeta\cos l\sin p\cos(\lambda-q) - \sin l\cos p = 0$$

28. Let us call angle $PAO = r + \rho = \omega$ and $\lambda - q = \varphi$ to have

$dl = -\varepsilon\alpha dt \sin\omega$; $\quad d\lambda = \dfrac{\varepsilon\alpha dt\cos\omega}{\sin l}$; $\quad d\varphi = \dfrac{\varepsilon\alpha dt\cos\omega}{\sin l} - dq$,

$dr = \varepsilon dt - \dfrac{\varepsilon\alpha dt\cos\omega}{\tan l}$, and $d\rho = d\omega - \varepsilon dt + \dfrac{\varepsilon\alpha dt\cos\omega}{\tan l}$,

and our equations when reduced will be:

$$\varepsilon d\alpha\cos\omega - \varepsilon\alpha d\omega\sin\omega + (1-B)\varepsilon\varepsilon\alpha dt\sin\omega + \frac{6Bgee}{s^3}dt\sin p\sin\varphi(\sin l\sin p\cos\varphi + \cos l\cos p) = 0$$

$$\varepsilon d\alpha\sin\omega + \varepsilon\alpha d\omega\cos\omega - (1-B)\varepsilon\varepsilon\alpha dt\cos\omega + \frac{6Bgee}{s^3}dt(\sin l\sin p\cos\varphi + \cos l\cos p)(\cos l\sin p\cos\varphi - \sin l\cos p) = 0$$

Finally, let $\alpha\cos\omega = u$ and $\alpha\sin\omega = v$, so that $dl = -\varepsilon v dt$ and $d\varphi = \dfrac{\varepsilon u dt}{\sin l} - dq$, which will further reduce these equations to:

$$\varepsilon du + (1-B)\varepsilon\varepsilon u dt + \frac{6Bgee}{s^3}dt\sin p\sin\varphi(\sin l\sin p\cos\varphi + \cos l\cos p) = 0$$

$$\varepsilon dv - (1-B)\varepsilon\varepsilon u dt + \frac{6Bgee}{s^3}dt(\sin l\sin p\cos\varphi + \cos l\cos p)(\cos l\sin p\cos\varphi - \sin l\cos p) = 0$$

where it must be noted that $1 - B = \dfrac{aa}{bb}$.

29. Since the quantities u and v are nearly infinitely small, we could regard the arc l as a constant in these equations and assume $d\varphi = -dq$. Then, it will be permitted to regard the arc $PF = p$ as a constant or small variable depending on the choice of the point P. Also, the distance from the center of forces s does not ordinarily change so much that we could not regard it as constant, at least for finding the approaching integrals. Therefore, for brevity, let:

$\dfrac{6Bgee}{s^3} = N$; $\dfrac{aa}{bb} = \kappa$; and $dq = \delta dt$, or $d\varphi = -\delta dt$, and it is evident that we could satisfy our equations by letting:

$u = P + Q\cos\varphi + R\cos\varphi^2$, and $v = S\sin\varphi + T\sin\varphi\cos\varphi$, so that the letters P, Q, R, S are constants. Now, having substituted these values we will find that:

$Q = \dfrac{N\sin p\cos p(\varepsilon\kappa\cos 2l + \delta\cos l)}{\varepsilon(\varepsilon\varepsilon\kappa\kappa - \delta\delta)}$; $\quad S = -\dfrac{N\sin p\cos p(\varepsilon\kappa\cos l + \delta\cos 2l)}{\varepsilon(\varepsilon\varepsilon\kappa\kappa - \delta\delta)}$

$R = \dfrac{N\sin l\sin p^2(\varepsilon\kappa\cos l + 2\delta)}{\varepsilon(\varepsilon\varepsilon\kappa\kappa - 4\delta\delta)}$; $\quad T = -\dfrac{N\sin l\sin p^2(\varepsilon\kappa + 2\delta\cos l)}{\varepsilon(\varepsilon\varepsilon\kappa\kappa - 4\delta\delta)}$

$$P = -\frac{\delta N\sin l\sin p^2(\varepsilon\kappa + 2\delta\cos l)}{\varepsilon\varepsilon\kappa(\varepsilon\varepsilon\kappa\kappa - 4\delta\delta)} - \frac{N\sin l\cos l\cos p^2}{\varepsilon\varepsilon\kappa}$$

and $P + \tfrac{1}{2}R = \dfrac{N\sin l\cos l(\sin p^2 - 2\cos p^2)}{2\varepsilon\varepsilon\kappa}$.

30. Here the average values of l, p, N, and δ will be given. Having constant values for the letters P, Q, R, S, T, and then because $dt = -\frac{d\varphi}{\delta}$, we will have $dl = \frac{\varepsilon v d\varphi}{\delta}$ and $d\varphi = -\frac{\varepsilon u d\varphi}{\delta \sin l} - dq$, from where we derive by integration, setting l as the average value of l :

$$l = \mathit{l} - \frac{\varepsilon S}{\delta}\cos\varphi - \frac{\varepsilon}{4\delta} T\cos 2\varphi, \text{ and}$$

$$\lambda = \text{Const.} - \frac{\varepsilon P\varphi}{\delta\sin l} - \frac{\varepsilon Q\sin\varphi}{\delta\sin l} - \frac{\varepsilon R\varphi}{2\delta\sin l} - \frac{\varepsilon R\sin 2\varphi}{4\delta\sin l}.$$

Now furthering this, we will have $\varphi = \lambda - q$, and then $\alpha = \sqrt{(uu + vv)}$, and $\tan\omega = \frac{v}{u}$.

But since $dr = \varepsilon dt - \frac{\varepsilon u dt}{\tan l} = \varepsilon dt - d\lambda\cos l$, it follows that $r = \varepsilon t - \lambda\cos l$, and hence $\rho = \omega - r$. It is seen that if α has a very small value, it will always stay very small, so that our calculation remains, with the exception of the single case that follows.

31. To simplify the case, since taking into account the irregularity of the movement of the center of force F is too difficult for our purposes, let us assume that this point F moves uniformly in the circle QFS, around Earth at the constant distance s. In addition, let P be the pole of its orbit, so that the arc $PF = p = 90°$, and let us call $dq = \delta dt$, where δ is a constant quantity marking the angle described in a second by the point F. Since the arc $PA = l$ hardly changes, let l be its average value, and our equations reduce to the following, where $\frac{aa}{bb} = \kappa$

$$du + \varepsilon\kappa v dt + \frac{3Bgee}{\varepsilon s^3} dt\sin\mathit{l}\sin 2\varphi = 0,$$

$$dv + \varepsilon\kappa u dt + \frac{3Bgee}{\varepsilon s^3} dt\sin\mathit{l}\,\cos\mathit{l}\,(1 + \cos 2\varphi) = 0,$$

where $d\varphi = -\delta dt + \frac{\varepsilon u dt}{\sin\mathit{l}}$.

But since u is extremely small and we neglected the terms where u and v would be raised to more than one dimension, we could take $d\varphi = -\delta dt$, so that $dt = -\frac{d\varphi}{\delta}$.

32. Therefore, let us, as before, call $\frac{6Bgee}{s^3} = N$, to have

$$du - \frac{\varepsilon\kappa}{\delta} v d\varphi - \frac{N}{2\delta\varepsilon} d\varphi\sin\mathit{l}\,\sin 2\varphi = 0,$$

$$du + \frac{\varepsilon\kappa}{\delta} v d\varphi - \frac{N}{2\delta\varepsilon} d\varphi\sin\mathit{l}\,\cos\mathit{l}\,(1 + \cos 2\varphi) = 0,$$

which satisfies, like we just saw, these particular values:

$$u = P + \tfrac{1}{2}R + \tfrac{1}{2}R\cos 2\varphi\,; \qquad v = \tfrac{1}{2}T\sin 2\varphi\,.$$

But, to find the general integrals, let us form these two equations that can be integrated.

$$\left.\begin{aligned}&+du\sin\frac{\varepsilon\kappa}{\delta}\varphi+\frac{\varepsilon\kappa}{\delta}ud\varphi\cos\frac{\varepsilon\kappa}{\delta}\varphi-\frac{N}{2\delta\varepsilon}d\varphi\ \sin l\sin\frac{\varepsilon\kappa}{\delta}\varphi\sin 2\varphi\\&+dv\cos\frac{\varepsilon\kappa}{\delta}\varphi-\frac{\varepsilon\kappa}{\delta}vd\varphi\sin\frac{\varepsilon\kappa}{\delta}\varphi-\frac{N}{2\delta\varepsilon}d\varphi\ \sin l\ \cos l\cos\frac{\varepsilon\kappa}{\delta}\varphi(1+\cos 2\varphi)\end{aligned}\right\}=0$$

$$\left.\begin{aligned}&+du\cos\frac{\varepsilon\kappa}{\delta}\varphi-\frac{\varepsilon\kappa}{\delta}ud\varphi\cos\frac{\varepsilon\kappa}{\delta}\varphi-\frac{N}{2\delta\varepsilon}d\varphi\ \sin l\ \cos\frac{\varepsilon\kappa}{\delta}\varphi\sin 2\varphi\\&-dv\sin\frac{\varepsilon\kappa}{\delta}\varphi-\frac{\varepsilon\kappa}{\delta}ud\varphi\cos\frac{\varepsilon\kappa}{\delta}\varphi+\frac{N}{2\delta\varepsilon}d\varphi\ \sin l\ \cos l\sin\frac{\varepsilon\kappa}{\delta}\varphi(1+\cos 2\varphi)\end{aligned}\right\}=0$$

33. Let us say, for the sake of brevity, $\frac{\varepsilon\kappa}{\delta}=\frac{\varepsilon aa}{\delta bb}=m$ and $\frac{N}{2\varepsilon\delta}=\frac{3Bgee}{\delta\varepsilon s^3}=n$, and the integrals will be:

$$u\sin m\varphi+v\cos m\varphi-n\sin l\int d\varphi\sin m\varphi\sin 2\varphi-n\sin l\ \cos l\int d\varphi\cos m\varphi(1+\cos 2\varphi)=E$$
$$u\cos m\varphi-v\sin m\varphi-n\sin l\int d\varphi\cos m\varphi\sin 2\varphi+n\sin l\cos l\int d\varphi\sin m\varphi(1+\cos 2\varphi)=F\,.$$

Now $\sin m\varphi\sin 2\varphi=\tfrac{1}{2}\cos(m-2)\varphi-\tfrac{1}{2}\cos(m+2)\varphi$

$$\cos m\varphi\cos 2\varphi=\tfrac{1}{2}\cos(m-2)\varphi-\tfrac{1}{2}\cos(m+2)\varphi$$
$$\cos m\varphi\sin 2\varphi=-\tfrac{1}{2}\sin(m-2)\varphi+\tfrac{1}{2}\sin(m+2)\varphi$$
$$\sin m\varphi\cos 2\varphi=+\tfrac{1}{2}\sin(m-2)\varphi+\tfrac{1}{2}\sin(m+2)\varphi\,,$$

from which the integrals are formed.

$$u\sin m\varphi+v\cos m\varphi-\tfrac{1}{2}n\sin l\left(\frac{\sin(m-2)\varphi}{m-2}-\frac{\sin(m+2)\varphi}{m+2}\right)$$
$$-\tfrac{1}{2}n\ \sin l\ \cos l\left(\frac{2\sin m\varphi}{m}+\frac{\sin(m-2)\varphi}{m-2}+\frac{\sin(m+2)\varphi}{m+2}\right)=E\,,$$
$$v\cos m\varphi-v\sin m\varphi-\tfrac{1}{2}n\sin l\left(\frac{\cos(m-2)\varphi}{m-2}-\frac{\cos(m+2)\varphi}{m+2}\right)$$
$$-\tfrac{1}{2}n\ \sin l\cos l\left(\frac{2\cos m\varphi}{m}+\frac{\cos(m-2)\varphi}{m-2}+\frac{\cos(m+2)\varphi}{m+2}\right)=F\,.$$

Where it must be noted that in the case $m=2$, $\frac{\sin(m-2)\varphi}{m-2}=\varphi$ and $\frac{\cos(m-2)\varphi}{m-2}=\infty$ are constant, and hence contained in *F*, so that this term can be omitted.

34. Now, multiplying the first by $\sin m\varphi$ and the other by $\cos m\varphi$, their sum gives:

$$u-\tfrac{1}{2}n\sin l\left(\frac{\cos 2\varphi}{m-2}-\frac{\cos 2\varphi}{m+2}\right)-\tfrac{1}{2}n\sin l\ \cos l\left(\frac{2}{m}+\frac{\cos 2\varphi}{m-2}+\frac{\cos 2\varphi}{m+2}\right)$$
$$=E\sin m\varphi+F\cos m\varphi\,,$$

Then, multiplying the first by $\cos m\varphi$, and the other by $-\sin m\varphi$, we will obtain

$$v+\tfrac{1}{2}n\sin l\left(\frac{\sin 2\varphi}{m-2}+\frac{\sin 2\varphi}{m+2}\right)+\tfrac{1}{2}n\sin l\ \cos l\left(\frac{\sin 2\varphi}{m-2}-\frac{\sin 2\varphi}{m+2}\right)$$
$$=E\cos m\varphi-F\sin m\varphi\,.$$

Let us change the constants by calling $E = D\cos\xi$ and $F = D\sin\xi$ where ξ is a constant angle, and our equations reduce to this:

$$u - \frac{2n\sin l\cos 2\varphi}{mm-4} - \frac{n}{m}\sin l\cos l - \frac{mn\sin l\cos l\cos 2\varphi}{mm-4} = D\sin(m\varphi+\xi)$$

$$v + \frac{mn\sin l\sin 2\varphi}{mm-4} + \frac{2n\sin l\cos l\sin 2\varphi}{mm-4} = D\cos(m\varphi+\xi),$$

and hence we will have:

$$u = \frac{n}{m}\sin l\cos l + \frac{N\sin l\cos 2\varphi\,(2+m\cos l)}{mm-4} + D\sin(m\varphi+\xi)$$

$$v = -\frac{n\sin l\sin 2\varphi\,(m+2\cos l)}{mm-4} + D\cos(m\varphi+\xi).$$

35. These equations perfectly agree with those that we had found above, if we set the constant $D = 0$, but they are more general for the reason that they still contain the two constants D and ξ. Nevertheless, the case where $m = 2$ asks for a particular development in which it must be derived by the first integrals, which will be:

$$u\sin 2\varphi + v\cos 2\varphi - \tfrac{1}{2}n\sin l\,(\varphi\,\tfrac{1}{4}\sin 4\varphi) - \tfrac{1}{2}n\sin l\cos l\,(\sin 2\varphi + \varphi + \tfrac{1}{4}\sin 4\varphi) = E$$

$$u\cos 2\varphi - v\sin 2\varphi + \tfrac{1}{8}n\sin l\cos 4\varphi - \tfrac{1}{2}n\sin l\cos l\,(\cos 2\varphi + \tfrac{1}{4}\cos 4\varphi) = G\,.$$

From this, we derive these:

$$u - \tfrac{1}{2}n\sin l\,\varphi\sin 2\varphi + \tfrac{1}{8}n\sin l\cos 2\varphi - \tfrac{1}{2}n\sin l\cos l\,\varphi\sin 2\varphi - \tfrac{1}{2}n\sin l\cos l$$
$$-\tfrac{1}{8}n\sin l\cos l\cos 2\varphi = E\sin 2\varphi + G\cos 2\varphi$$

$$v - \tfrac{1}{2}n\sin l\,\varphi\cos 2\varphi + \tfrac{1}{8}n\sin l\sin 2\varphi - \tfrac{1}{2}n\sin l\cos l\,\varphi\cos 2\varphi$$
$$-\tfrac{1}{8}n\sin l\cos l\sin 2\varphi = E\cos 2\varphi - G\sin 2\varphi$$

and hence we will have:

$$u = \tfrac{1}{2}n\sin l\cos l + \tfrac{1}{2}n\sin l\,(1+\cos l)\varphi\sin 2\varphi + E\sin 2\varphi + (G\,\tfrac{1}{8}n\sin l\,(1-\cos l))\cos 2\varphi$$

$$v = \tfrac{1}{2}n\sin l\,(1+\cos l)\varphi\cos 2\varphi + E\cos 2\varphi - (G + \tfrac{1}{8}n\sin l\,(1-\cos l))\sin 2\varphi\,.$$

36. Comparing to this case $m = 2$ or $\frac{\varepsilon}{\delta} = \frac{2bb}{aa}$, I notice that the quantities u and v could grow to infinity since they contain terms multiplied by the arc φ that grows continually with time. Therefore, the arc $OA = \alpha = \sqrt{(uu + vv)}$ would soon be surpassing the limits of the smallness that I placed on it, and hence our solution absolutely excludes this case. It is therefore very remarkable that if the movement of the center of force F were to the diurnal movement of the planet like $aa : 2bb$, then the diurnal movement would soon be considerably disturbed, although it had started around a principal axis. Thus for Earth, where aa is very close to bb, if the Moon achieved its revolutions in two days, instead of 27, the rotational axis of Earth would suffer terrible perturbations, which there would hardly be a means to allocate.

37. But it is apparent that such a case exists nowhere in the Universe, or at least in our planetary system, which is the extent of our studies. I already noted that if the Moon were two or three times farther from Earth, which it is in fact not, its movement would be so irregular that it would be nearly impossible for us to acquire even a gross knowledge: because for a perfect understanding, there is still much required of us that we could never

acquire. If the Moon were much closer to Earth, we could more exactly determine its movement, but presently we perceive another inconvenience that renders the nutation of Earth's axis indeterminable. From this, it seems to result that Providence saw well to offer to our studies of such objects, which do not absolutely surpass the threshold of our spirit, although it was impossible for us to complete our work. Maybe such movements that would be inaccessible to us are found in other planetary systems where intelligent creatures are blessed with a higher degree of insight.

38. Let us therefore assume that the square of the number $m = \dfrac{\varepsilon aa}{\delta bb}$ differs considerably enough from 4 so that the quantities u and v always stay very small and the smallness hypothesis of the arc $OA = \alpha = \sqrt{(uu + vv)}$ stays unaltered. Then, from the solution that we just found, we will nearly exactly discover the phenomena of the rotational movement of the proposed bodies. Because having found

$$u = \frac{n}{2m}\sin 2\boldsymbol{l} + \frac{n\,(2 + m\cos\boldsymbol{l})\sin\boldsymbol{l}}{mm-4}\cos 2\varphi + D\sin(m\varphi + \xi)$$

$$v = -\frac{n\,(m + 2\cos\boldsymbol{l})\sin\boldsymbol{l}}{mm-4}\sin 2\varphi + D\cos(m\varphi + \xi)$$

we will have $\alpha = \sqrt{(uu + vv)}$ and $\tan\omega = \dfrac{v}{u}$. Then, because $dl = \dfrac{\varepsilon v\,d\varphi}{\delta}$ and

$$d\lambda = -\frac{\varepsilon u\,d\varphi \sin\boldsymbol{l}}{\delta\sin\boldsymbol{l}} :$$

$$l = \boldsymbol{l} + \frac{\varepsilon n\,(m + 2\cos\boldsymbol{l})\sin\boldsymbol{l}}{2\delta\,(mm-4)}\cos 2\varphi + \frac{\varepsilon D}{\delta m}\sin(m\varphi + \xi)$$

$$\lambda = \text{Const.} - \frac{\varepsilon n\cos\boldsymbol{l}}{\delta m}\varphi - \frac{\varepsilon n\,(m + 2\cos\boldsymbol{l})}{2\delta\,(mm-4)}\sin 2\varphi + \frac{\varepsilon D}{\delta m\sin\boldsymbol{l}}\cos(m\varphi + \xi)$$

and finally $r = \text{Const.} - \lambda\cos\boldsymbol{l} + \varepsilon t$ and $\rho = \omega - r$.

Application to the rotational movement of Earth.

39. To apply these formulas to Earth, it is advisable to take point P in the ecliptic
Fig. 5 pole, so that when F marks the Moon, the arc $PF = p$ is not a quarter circle, and hence it is necessary to resort to the general formulas from §28. Letting therefore the circle ♈☊L♎ be the ecliptic, ☊ the ascending node of the moon's orbit ☊FM, and the Moon is presently found at F at a distance from Earth s, which I regard as a constant. The attractive force of the Moon at distance e is equal to that of gravity. The fixed point ♈ is not the equinox, but rather the first star of Aries, from which the longitude of the ascending node is ♈☊ $= \zeta$, and the inclination of the lunar orbit at the ecliptic or the angle F☊$L = \gamma$, which I regard as constant, during that the arc ζ diminishes uniformly, for which the movement I propose is $d\zeta = -\beta dt$. Then, let the longitude of the Moon counted

since ♈, or the arc ♈☊L = q, which I also assume proportional to time, so that $dq = \delta dt$; since the inequality of the movement hardly influences the movement of Earth's axis.

40. Therefore, having the arc ☊$L = q - \zeta$ and the angle L☊$F = \gamma$, since γ does not exceed 5°, we will have approximately $\sin FL = \gamma \sin(q-\zeta)$ and $\cos FL = 1$, neglecting the terms where γ would have more that one dimension, so that $\sin p = 1$ and $\cos p = \gamma \sin(q-\zeta)$. Currently, let Earth's axis at A compared to that of the Earth's moment of inertia to be equal to Maa and compared to the other principal axes be equal to Mbb. Let us call the arc $PA = l$ and the angle ♈PA = λ. Moreover, let AB be the Prime Meridian derived on Earth and the angle $PAB = r$. In addition, Earth presently turns around pole O in the direction ♈L♎ following the order of the signs with angular speed ε. Let arc $AO = \alpha$, which I assume is extremely small, and the angle $BAO = \rho$. Now I have set $r + \rho = \omega$ and furthermore $\alpha \cos \omega = u$ and $\alpha \sin \omega = v$.

41. Let us subtract the longitude of the terrestrial pole A from the longitude of the Moon, and let angle $APF = q - \lambda = \varphi$, which, from before, would be $-\varphi$. We will have:

$$dl = -\varepsilon v dt\,; \quad d\lambda = \frac{\varepsilon u dt}{\sin l}\,; \quad d\varphi = \delta dt - \frac{\varepsilon u dt}{\sin l}, \ \text{ and } \ dr = \varepsilon dt - \frac{\varepsilon u dt \cos l}{\sin l}.$$

Now everything comes back to the solution of these two equations:

$$\varepsilon du + \frac{\varepsilon\varepsilon aa}{bb} v dt - \frac{6gee}{s^3}(1 - \frac{aa}{bb}) dt \sin\varphi(\sin l \cos\varphi + \gamma \cos l \sin(q-\zeta)) = 0$$

$$\varepsilon dv - \frac{\varepsilon\varepsilon aa}{bb} u dt + \frac{6gee}{s^3}(1 - \frac{aa}{bb}) dt (\sin l \cos\varphi + \gamma \cos l \sin(q-\zeta))(\cos l \cos\varphi - \gamma \sin l \sin(q-\zeta)) = 0$$

Let us, for brevity, call $\frac{\varepsilon aa}{bb} = \mu$ and $\frac{3gee}{\varepsilon s^3}(\frac{aa}{bb} - 1) = \nu$, and we will have:

$$du + \mu v dt + \nu dt \sin\varphi(\sin l \cos\varphi + \gamma \cos l \sin(q-\zeta)) = 0$$

$$dv - \mu u dt - 2\nu dt (\sin l \cos\varphi + \gamma \cos l \sin(q-\zeta))(\cos l \cos\varphi - \gamma \sin l \sin(q-\zeta)) = 0$$

or when reduced:

$$du + \mu v dt + \nu dt(\sin l \sin 2\varphi + \gamma \cos l \cos(\zeta - \lambda) - \gamma \cos l \cos(2q - \zeta - \lambda)) = 0$$

$$dv - \mu u dt - \nu dt(\sin l \cos l + \sin l \cos l \cos 2\varphi - \gamma \cos 2l \sin(\zeta - \lambda) + \gamma \cos 2l \sin(2q - \zeta - \lambda) - \gamma\gamma \sin l \cos l) = 0$$

42. Now, without repeating the general integral, since we know its form, let us say

$$u = A + B\cos 2\varphi + C\sin(\zeta - \lambda) + D\sin(2q - \zeta - \lambda) + \mathfrak{C}\sin(\mu t + \xi)$$

$$v = E\sin 2\varphi + F\cos(\zeta - \lambda) + G\cos(2q - \zeta - \lambda) - \mathfrak{C}\cos(\mu t + \xi)$$

and since $d\varphi = \delta dt$, $d\zeta = \beta dt$, $dq = \delta dt$, and $d\lambda = 0 dt$, because we neglect the terms where the small quantities would return, we will have

$$\frac{du}{dt} = -2B\delta \sin 2\varphi - C\beta\cos(\zeta - \lambda) + D(2\delta + \beta)\cos(2q - \zeta - \lambda) + \mathfrak{C}\mu\cos(\mu t + \xi)$$

$$\frac{dv}{dt} = 2E\delta\cos 2\varphi + F\beta\sin(\zeta - \lambda) - G(2\delta + \beta)\sin(2q - \zeta - \lambda) + \mathfrak{C}\mu\sin(\mu t + \xi)$$

Now in the differential equations, it is permitted to regard the arc $PA = l$ as a constant, and to put in place of l its average value, which is l like above. It therefore only rests to substitute these assumed values.

43. Now the first equation divided by dt gives

$$\left.\begin{array}{llll} -2B\delta\sin 2\varphi & -C\beta\cos(\zeta-\lambda) & +D(2\delta+\beta)\cos(2q-\zeta-\lambda) & +\mathcal{C}\mu\cos(\mu t+\xi) \\ +E\mu & +F\mu & +G\mu & -\mathcal{C}\mu \\ +v\sin\ell & +v\gamma\cos\ell & -v\gamma\cos\ell & \end{array}\right\} = 0$$

and the other gives:

$$\left.\begin{array}{lllll} & +2E\delta\cos 2\varphi & +F\beta\sin(\zeta-\lambda) & -G(2\delta+\beta)\sin(2q-\zeta-\lambda) & +\mathcal{C}\mu\sin(\mu t+\xi) \\ -A\mu & -B\mu & -C\mu & -D\mu & -\mathcal{C}\mu \\ -v\sin\ell\cos\ell & -v\sin\ell\cos\ell & +v\gamma\cos 2\ell & -v\gamma\cos 2\ell & \end{array}\right\} = 0$$

Equating all these separate members to zero, we first of all derive:

$$A = -\frac{v}{\mu}(1-\gamma\gamma)\ \sin\ell\cos\ell = -\frac{v}{2\mu}(1-\gamma\gamma)\ \sin 2\ell$$

and then:

$$-2B\delta + E\mu + v\sin\ell = 0; \qquad -2B\mu + 2E\delta - v\sin\ell\cos\ell = 0,$$

$$-C\beta + F\mu + v\gamma\cos\ell = 0; \qquad -C\mu + F\beta + v\gamma\cos 2\ell = 0,$$

$$D(2\delta+\beta) + G\mu - v\gamma\cos\ell = 0; \qquad -D\mu - G(2\delta+\beta) - v\gamma\cos 2\ell = 0,$$

44. The coefficients B, C, D, E, F, G will therefore have the following values:

$$B = -\frac{v\sin\ell(2\delta+\mu\cos\ell)}{\mu\mu-4\delta\delta}; \qquad E = -\frac{v\sin\ell(2\delta\cos\ell+\mu)}{\mu\mu-4\delta\delta}$$

$$C = -\frac{v\gamma(\beta\cos\ell-\mu\cos 2\ell)}{\mu\mu-\beta\beta}; \qquad F = +\frac{v\gamma(\beta\cos 2\ell-\mu\cos\ell)}{\mu\mu-\beta\beta}$$

$$D = -\frac{v\gamma((2\delta+\beta)\cos\ell+\mu\cos 2\ell)}{\mu\mu-(2\delta+\beta)^2}; \quad G = +\frac{v\gamma((2\delta+\beta)\cos 2\ell+\mu\cos\ell)}{\mu\mu-(2\delta+\beta)^2}$$

and like we had found $A = -\frac{v}{\mu}(1-\gamma\gamma)\ \sin\ell\cos\ell$, where instead of γ we can put $\sin\gamma$ and $\cos\gamma^2 = \tfrac{1}{2}+\tfrac{1}{2}\cos 2\gamma$ instead of $1-\gamma\gamma$; now γ marks the average inclination of the lunar orbit at the ecliptic. For the other two constants $\mathcal{C}$ and ξ, they stay arbitrary as the nature of complete integrals requires them. This would take place even when the force of the moon contained in the letter v would vanish.

45. Having found these coefficients, it will no longer be difficult to assign the other quantities that determine the movement. Firstly the differential $dl = -\varepsilon v dt$ gives

$$l = \ell + \frac{E\varepsilon}{2\delta}\cos 2\varphi + \frac{F\varepsilon}{\beta}\sin(\zeta-\lambda) - \frac{G\varepsilon}{2\delta+\beta}\sin(2q-\zeta-\lambda) + \frac{\mathcal{C}\varepsilon}{\mu}\sin(\mu t+\xi)$$

from where we would know for each time the distance of Earth's pole A to the ecliptic pole P. Then, for the longitude of the terrestrial pole A, or the angle ♈$PA = \lambda$, we will have

$$\lambda = \text{Const.} + \frac{A\varepsilon t}{\sin l} + \frac{B\varepsilon}{2\delta \sin l} \sin 2\varphi + \frac{D\varepsilon}{\beta \sin l} \cos(\zeta - \lambda) -$$

$$\frac{D\varepsilon}{(2\delta + \beta) \sin l} \cos(2q - \zeta - \lambda) - \frac{\mathcal{C}\varepsilon}{\mu \sin l} \cos(\mu t + \xi)$$

Then, for the angle $PAB = r$, or the movement of the Prime Meridian, we would have $r = \varepsilon t - \lambda \cos l$. Finally, for the rotational pole O, we would have the distance $AO = \alpha = \sqrt{(uu + vv)}$, and call the sum of the angles $OAB + BAO = r + \rho = \omega$; since $\tan \omega = \frac{v}{u}$, we will have the angle $BAO = \rho = \omega - r$. By this method we acquire a perfect knowledge of the diurnal movement of Earth.

46. Although these formulas properly regard the effect of the Moon, it is easy to apply them to those of the Sun when calling $\gamma = 0$, and then δ will mark the average movement of the Sun, or the arc described in one second. So as not to confuse these two effects, let the longitude of the Sun since the first star of Aries = Q and its average movement, or the angle traveled in its orbit during one second = Δ. Since λ marks the angle ♈PA, this same letter λ expresses the longitude of the Summer solstice from the same term ♈, therefore $\lambda - 90°$ signifies the longitude of the Spring equinox and $Q - \lambda$ is the solar longitude since the Summer solstice Φ. As a result, if we let the solar longitude since the Spring equinox = Ψ, we will have $\Psi = \Phi + 90°$, and hence $\Phi = \Psi - 90°$ and $2\Phi = 2\Psi - 180°$.

47. Now, if the force of the Sun at the distance e is equal to gravity, at the assumed distance from Earth s, it will be $\frac{ee}{ss}$ calling gravity 1. Now the speed of Earth in its orbit is Δs, and the height from where a falling body acquires the same speed will be $\frac{\Delta\Delta ss}{4g}$, which divided by half the distance $\frac{s}{2}$ gives the centrifugal force $\frac{\Delta\Delta s}{2g}$ that must be equal to the central force $\frac{ee}{ss}$ from where we derive $\frac{2gee}{s^3} = \Delta\Delta$. Let N be the value of the letter v for the Sun, and we will have $N = \frac{3\Delta\Delta}{2\varepsilon}(\frac{aa}{bb} - 1)$, where ε marks the angular speed of the diurnal movement of Earth and $\mu = \frac{\varepsilon aa}{bb}$. From this we will have:

$$A = -\frac{3\Delta\Delta}{2\varepsilon\varepsilon aa}(aa - bb)\sin l \cos l; \quad \text{or} \quad A = \frac{-N}{\mu}\sin l \cos l,$$

$$B = \frac{-N\,(2\Delta + \mu \cos l) \sin l}{\mu\mu - 4\Delta\Delta}; \qquad \mathrm{E} = \frac{-N\,(2\Delta \cos l + \mu) \sin l}{\mu\mu - 4\Delta\Delta},$$

and then $C = 0$, $F = 0$, $D = 0$, $G = 0$.

48. Therefore, for the distance of Earth's pole A to the ecliptic pole P, we will have:

$$PA = l = l + \frac{N\varepsilon\,(2\Delta \cos l + \mu) \sin l}{2\Delta\,(\mu\mu - 4\Delta\Delta)} \cos 2\Psi + \frac{\mathcal{C}\varepsilon}{\mu} \sin(\mu t + \xi)$$

and for its longitude, or the angle ♈$AA = \lambda$:

$$\lambda = \text{Const.} - \frac{N\varepsilon t \cos l}{\mu} + \frac{N\varepsilon\,(2\Delta + \mu \cos l)}{2\Delta\,(\mu\mu - 4\Delta\Delta)} \sin 2\Psi - \frac{\mathfrak{C}\varepsilon}{\mu \sin l} \cos(\mu t + \xi)$$

The rotational movement of the Prime Meridian AB around the pole A may be regarded as a constant with angular speed c. Now we would not know the true rotational pole O without having determined the effect of all the forces that act on Earth, because it is necessary to find the complete values of the two letters u and v that resulted from all the forces, and then we will have $AO = \sqrt{(uu + vv)}$, $\tan \omega = \frac{v}{u}$, and from that the angle $BAO = \omega - r = \omega - PAB$. But in Earth, the points A and O are indiscernible.

49. For the Moon, we are uncertain of the distance e where its attractive force would be equal to gravity, and hence also the value of the letter v. But we can conclude by comparing the letters v and N of the effects that the Moon and Sun produce in the tides that they are proportional. Nevertheless, it is uncertain how to carry this conclusion to a high degree of precision. Newton believed that the value of v referring to the Moon was about four times greater than that of N referring to the Sun. Now Mr. Daniel Bernoulli proved that this comparison is not much greater than double. Let us therefore say $v = mN$, providing m is a number greater than 2. Then let the lunar longitude since the Spring equinox be ψ so that $\varphi = \psi - 90°$ and $2\varphi = 2\psi - 180°$; δ is the angle described by the Moon in one second, and β the angle by which the lunar nodes decline in the same time. Let us say longitude of the ascending node since the Spring equinox = θ, and we will have $\theta = \zeta - \lambda + 90°$ and $\zeta - \lambda = \theta - 90°$. Therefore, since $q = \varphi + \lambda = \psi + \lambda - 90°$ and $\zeta = \theta + \lambda - 90°$, we will have $2q - \zeta - \lambda = 2\psi - \theta - 90°$.

50. Let us introduce these angles that Astronomy reveals for each time, and we will have the distance of Earth's pole A to the ecliptic pole P:

$$PA = l = l + \frac{mN\varepsilon\,(\mu - \delta \cos l) \sin l}{2\delta\,(\mu\mu - \delta\delta)} \cos 2\psi + \frac{mN\varepsilon\,(\mu \cos l - \beta \cos 2l) \sin\gamma}{\beta\,(\mu\mu - \beta\beta)} \cos\theta +$$

$$\frac{mN\varepsilon\,(\mu \cos l + (2\delta + \beta) \cos 2l) \sin\gamma}{(2\delta + \beta)(\mu\mu - (2\delta + \beta)^2)} \cos(2\psi - \theta) + \frac{\mathfrak{C}\varepsilon}{\mu} \sin(\mu t + \xi),$$

and its longitude counted since the first star of Aries

$$♈PA = \lambda = \text{Const.} - \frac{mN\varepsilon t \cos\gamma^2 \cos l}{\mu} + \frac{mN\varepsilon\,(2\delta + \mu \cos l)}{2\delta\,(\mu\mu - 4\delta\delta)} \sin 2\psi - \frac{\mathfrak{C}\varepsilon}{\mu \sin l} \cos(\mu t + \xi) -$$

$$\frac{mN\varepsilon\,(\beta \cos l - \mu \cos 2l) \sin\gamma}{\beta\,(\mu\mu - \beta\beta) \sin l} \sin\theta + \frac{mN\varepsilon\,(\mu \cos 2l + (2\delta + \beta) \cos l) \sin\gamma}{(2\delta + \beta)(\mu\mu - (2\delta + \beta)^2) \sin l} \sin(2\psi - \theta).$$

Now we only have to combine these anomalies with those that produce the solar force to have all of the perturbations that disturb Earth's pole A, both compared to its distance from the ecliptic pole and its longitude counted since the first star of Aries.

51. Before the effects of the solar and lunar forces are further developed, I would like to note that even if these forces had not existed, it may have been possible that Earth's axis A was still not stationary. This is because setting $N = 0$ still will give us

$$PA = l = \boldsymbol{l} + \frac{\mathcal{C}\varepsilon}{\mu}\sin(\mu t + \xi), \text{ and} \qquad \Upsilon PA = \lambda = \text{Const.} - \frac{\mathcal{C}\varepsilon}{\mu \sin \boldsymbol{l}}\cos(\mu t + \xi),$$

where the constant $\mathcal{C}$ does not depend on the solar and lunar forces, so that if it were not 0, Earth's axis would be perturbed by some nutation while Earth would turn uniformly
Fig. 6 around it. This is because, taking the arc $P\alpha = \boldsymbol{l}$, Earth's pole A would uniformly describe a circle 1, 2, 3, 4 around the fixed point α in the same direction as the diurnal movement, and the radius of the circle αA would be $\frac{\mathcal{C}\varepsilon}{\mu}$, or of an arbitrary size, the angular speed is $\mu = \frac{aa}{bb}\varepsilon$. This case would take place if Earth would have begun to turn around a different axis than its principal axes. Since we would not know for sure that the constant $\mathcal{C}$ is absolutely 0, it is important to expand on the phenomena of this axis's nutation.

52. Earth would therefore uniformly turn around its principal axis A, with angular speed ε, while the axis A would describe around a fixed point α a circle with angular speed $\frac{aa}{bb}\varepsilon$. Let T be the time of one revolution of Earth around the axis, and the time of one revolution of this axis about a fixed point α will be $\frac{bb}{aa}T$. If $bb = aa$, these two times would be equal and the point in Earth that would have once corresponded to the fixed point α would always correspond to it. Hence we would take this point α rather than A for Earth's pole, and in effect, in this case all the moments of inertia of Earth would be equivalent. Yet, if the moments of inertia Maa and Mbb are not equal, there is no point on Earth that would stay at rest, and the movement of the pole A will be the least complicated, so that in this case we have no reason to rather regard some other point in Earth as its pole.

53. Let us call the radius of the small circle 1, 2, 3, 4 that describes Earth's axis A around a fixed point α, or the arc $\alpha A = \sigma$, and since the distance of the pole A to the ecliptic pole P is equal to the obliquity of the ecliptic, and the longitude of the pole A corresponds to the Summer solstice, it follows that in the interval of time $\frac{bb}{aa}T$, the obliquity of the ecliptic varies by the quantity 2σ, and the equinoctial points undergo a change in their longitude of $\frac{2\sigma}{\sin \boldsymbol{l}} = 5\sigma$. Now, during each interval of time $\frac{bb}{aa}T$, the same inequalities come back. Since Earth is an elliptic spheroid, with the diameter of the equator at the axis between approximately 201 and 200, if Earth were homogenous, $\frac{bb}{aa} = 1 - \frac{1}{101}$ and the period of these inequalities would be $\frac{200}{201} * 24$ hours, or $23^{\mathrm{h}}, 53'$. In this interval of time, the variations in the obliquity of the ecliptic and the longitude of the equinoctial points will be all the larger: at its largest, the radius of the circle σ will be

much larger, and at its smallest, this radius does not vanish entirely, but it is certainly extremely small. To discover these inequalities would be a huge problem for Astronomers.

54. Some may object me for in this case we did not take the extremity of the axis A for Earth's pole, but rather the point α, which would effectively be the rotational pole if aa were equal to bb. But the lesser inequality between aa and bb completely reverses this idea, because although the point α does not noticeably change place during some revolutions, it will describe a type of spiral in which the turns become larger and larger, and if $\frac{bb}{aa} = 1 - \frac{1}{101}$, then after 50 revolutions or days, the point α will be found in the same circle 1, 2, 3, 4, and after 100 days it will describe a circle whose diameter is two times larger. Then, its turns will retract so that after 200 days, it returns to the center of the circle 1, 2, 3, 4, from where we see that this point would not at all be proper for comparing the diurnal movement, so it would be necessary to absolutely hold to Earth's true axis A, which describes the circle 1, 2, 3, 4.

55. The inequalities caused by the solar and lunar forces are independent of this, which would result from the nature of Earth, provided that it was not considerable, as the original calculation assumed it. We could therefore regard the circle 1, 2, 3, 4 as completely vanishing, and this all the more likely since there are no observation from which we may conclude the contrary. Let us therefore examine more carefully the inequalities that are produced by the solar and lunar forces and perpetuated in the variations of the arc $PA = l$ and the angle $♈PA = \lambda$. Now the arc l expresses the obliquity, and λ the longitude of the Summer solstitial point, counted since the first star of Aries. Therefore $\lambda - 90°$ will be the longitude of the Spring equinoctial point, and hence reciprocally $90° - \lambda$ is the longitude of the first star of Aries since the Spring equinoctial point. It is therefore a matter of determining from each proposed time both the longitude of the first star of Aries since the equinoctial point and the obliquity of the ecliptic.

56. Now the formulas that we just found contain two elements for which we do not know the exact value. The one is the number m, which marks by how much the lunar force is greater than that of the Sun in the production of the tides, and we know by the judicious reflections of Mr. Bernoulli that this number m is about $2\frac{1}{2}$. The other element is the fraction $\frac{aa}{bb}$, for which we do not absolutely know the value, because the knowledge of the exterior figure of Earth is not at all determined, just that Earth is not composed of a homogenous material, in which case we will have approximately $\frac{aa}{bb} = \frac{201}{200}$. But, since it is very probable that the material of Earth is not at all less than homogenous, and we have no knowledge of its distribution, I will call $\frac{aa}{bb} = n$, so that $\mu = \varepsilon n$, and I will regard the number n as unknown, although we can be assured that it does not noticeably differ from unity. From this we will have $N = \frac{3(n-1)\Delta\Delta}{2\varepsilon}$, and it will be necessary to conclude by the phenomena the exact values of the two numbers m and n.

57. For the angular speeds ε, Δ, δ, and β, it suffices in knowing the comparisons, which only enter in our formulas. Let us therefore take their values for a day, where Earth makes one complete revolution around its axis, so that $\varepsilon = 360° = 1296000''$. Then the Astronomical Tables give us, according to the average movements:

the everyday movement of the Sun	$\Delta = \quad 59',\ 8'' = 3548''$
the everyday movement of the Moon	$\delta = 13°,\ 10',\ 35'' = 47435''$
the everyday movement of the descending node	$\beta = \quad 3',\ 11'' = 191''$

hence, we will have:

$$\varepsilon = 1296000\,; \qquad \mu = 1296000n$$

$$\Delta = 3548\,; \qquad \delta = 47435$$

$$\beta = 191\,; \qquad 2\delta + \beta = 95061$$

or better yet, these proportional fractions

$$\varepsilon = 1.0000000\,; \qquad \varepsilon\varepsilon = 1.0000000$$

$$\Delta = 0.0027376\,; \qquad \Delta\Delta = 0.0000075$$

$$\delta = 0.0366011\,; \qquad \delta\delta = 0.0013396$$

$$\beta = 0.0001474\,; \qquad \beta\beta = 0.0000000$$

$$2\delta + \beta = 0.0733495\,; \quad (2\delta + \beta)^2 = 0.0053801$$

The arc PA = l marking the average obliquity of the ecliptic will be l = 23°, 28′, 30″, and the average inclination of the lunar orbit may be called γ = 5°, 9′.

Of the Variation in the obliquity of the ecliptic.

58. Calling the average obliquity of the ecliptic l, which is for the beginning of this century† = 23°, 28′, 43″ and for the end = 23°, 27′, 55″, the first correction depends on the longitude of the Sun, which is called = Ψ; the correction will be

$$+\ \frac{3(n-1)\,\Delta\,(\varepsilon n + 2\Delta\cos l)\sin l}{4(\varepsilon\varepsilon nn - 4\Delta\Delta)}\ \cos 2\Psi,$$

which in substituting the marked values reduces to

$$+\ \frac{0.0020532\,(n-1)(n + 0.0054752\cos l)\sin l}{nn - 0.0000300}\ \cos 2\Psi.$$

If the coefficient were = 1, then it would have the value 57°, 17′, 45″ = 206265″; therefore, reducing this coefficient in minutes and seconds, this connection will be thus expressed in seconds.

$$+\,423.503\ \frac{(n-1)(n + 0.0054752\cos l)\sin l}{nn - 0.0000300}\ \cos 2\Psi,$$

or using the value for l:

$$+\,168.70\ \frac{(n-1)(n + 0.00502)}{nn - 0.00003}\ \cos 2\Psi \text{ seconds}$$

59. The second correction depends on the lunar longitude, which is called = ψ, and will be

$$+\ \frac{3m\,(n-1)\,\Delta\Delta\,(\varepsilon n + 2\delta\cos l)\sin l}{4\delta\,(\varepsilon\varepsilon nn - 4\delta\delta)}\ \cos 2\psi,$$

which reduces to

† *this century* is in reference to the 18th century.

$$+\ \frac{0.0001538\,(n-1)\,m\,(n+0.0732022\cos l)\sin l}{nn\ -0.0253584}\ \cos 2\psi.$$

This formula is reducible to minutes and seconds and the value of the arc l can be used to give

$$+\ 12.618\ \frac{m\,(n-1)(n+0.06710)}{nn\ -0.00536}\ \cos 2\psi \text{ seconds},$$

from where we see that this correction is much smaller than the preceding, since we know that the number m is certainly less than 4. For that matter, it is also certain that the number n differs very slightly from unity, so that $n-1$ is an extremely small fraction.

60. The third correction depends on the longitude of the ascending node, which is called θ, this correction will be

$$+\ \frac{3m\,(n-1)\,\Delta\Delta\,(\varepsilon n\cos l-\beta\cos 2l)\sin\gamma}{2\beta\,(\varepsilon\varepsilon nn\ -\beta\beta)}\ \cos\theta,$$

which when reduced to numbers, and then in minutes and seconds, becomes

$$+\ 1295.45\ \frac{m\,(n-1)(n-0.00011)}{nn}\ \cos\theta \text{ minutes and seconds.}$$

Finally, the fourth correction is proportional to the cosine of the angle $2\psi-\theta$, and expressed as such:

$$+\ \frac{3m\,(n-1)\,\Delta\Delta\,(\varepsilon n\cos l+(2\delta+\beta)\cos 2l)\sin\gamma}{2\,(2\delta+\beta)(\varepsilon\varepsilon nn\ -(2\delta+\beta)^2)}\ \cos(2\psi-\theta),$$

which when reduced to numbers, and then in minutes and seconds, becomes

$$+\ \frac{2.603\,m\,(n-1)(n+0.05459)}{nn\ -0.00538}\ \cos(2\psi-\theta) \text{ minutes and seconds,}$$

so that this correction would nearly vanish compared to the preceding.

61. All these corrections become greatest positively if $2\Psi=0$ and $2\psi=\theta=180°$, and then they combine together in neglecting the small fractions attached to the numbers n and nn, $\frac{n-1}{n}(161.70+1310.671m)$ seconds. Now, if the same angles 2Ψ, 2ψ, and θ are $180°$, then it will result in the greatest negative correction: $\frac{n-1}{n}(168.70+1305.465m)$ seconds. Therefore, the greatest change in the obliquity of the ecliptic will be able to rise to $\frac{n-1}{n}(337.40+2616.136m)$ seconds. Now, by the observations of Mr. Bradley, we know that this change is about $18''$, or maybe a little bigger, since all circumstances of his observations had not attempted to show the greatest change.

Of the Precession of Equinoxes.

62. Here it is necessary to firstly consider the average movement of the equinoctial points, contained in the proportional terms at the time t, which is:

$$-\frac{3(n-1)\Delta\Delta}{2\varepsilon n}(1+m\cos\gamma^2)t\cos l,$$

from where we see that the longitude of the equinoctial points counted since the first star of Aries start to diminish, assuming that $n > 1$, or better yet, the longitude of this star counted since the equinox increases with the time. Let us therefore find this increase for the time of one year, and then Δt will have the value 360°, hence the annual precession of the equinoxes will be

$$\frac{3(n-1)\Delta}{2\varepsilon n}(1+m\cos\gamma^2)360°\cos l,$$

which reduces to $0.0037666(1+0.991943m)\frac{(n-1)}{n}360°$, or better yet to the form $\frac{n-1}{n}(1+0.991943m)4881\frac{1}{2}$ seconds. Now we know by the observations and the remarks that I made on the planetary action that this precession is $50\frac{1}{3}$ seconds.

63. Now, if we assume the greatest variation in the obliquity of the ecliptic is $18''$ and this difference had been observed at the same season of the year, then the angle Ψ, or the longitude of the Sun, has no influence here, so we will have these two equations for determining the two unknown numbers m and n,

$$22616m\frac{n-1}{n}=18 \text{ and } 4881\tfrac{1}{2}(1+0.991943m)\frac{n-1}{n}=50\tfrac{1}{3},$$

therefore $\frac{4881\frac{1}{2}(1+0.991943m)}{2616m}={}^{151}\!/_{154}$, from where we derive $4881\frac{1}{2}=2472.9m$ and $m=1.974$. If instead of $18''$, we would have taken it to be a little bigger, we would have found $m = 2$ and, in this case, it would result that: $\frac{n-1}{n}={}^{1}\!/_{289}$ and $n={}^{289}\!/_{288}$. Now it seems that it cannot be the case that $m < 2$, since Newton found $m = 4$ and Mr. Bernoulli, after having better examined the same observations, concluded $m = 2\frac{1}{2}$.

64. But, since we are not so very assured of the number $18''$, which marks the greatest variation in the obliquity, which we sum from the average precession, let us consider the number m as given, and we will firstly have $\frac{n-1}{n}=\frac{1}{96.98+96.20m}$, from where the greatest change in the obliquity of the ecliptic instead of $18''$ will be $\frac{2616m}{96.98+96.20m}$ that is $= \alpha$ for the same season of the year; therefore the following hypothesis are given:

if we will have

$m = 2$; $\quad \dfrac{n-1}{n} = \dfrac{1}{289.38}$; $\quad n = {}^{289}\!/_{288}$; $\quad \alpha = 18\,{}^{1}\!/_{10}$ seconds,

$m = 2\,{}^{1}\!/_{4}$; $\quad \dfrac{n-1}{n} = \dfrac{1}{313.43}$; $\quad n = {}^{313}\!/_{312}$; $\quad \alpha = 18\,{}^{8}\!/_{10}$ seconds,

$m = 2\,{}^{1}\!/_{2}$; $\quad \dfrac{n-1}{n} = \dfrac{1}{337.48}$; $\quad n = {}^{337}\!/_{336}$; $\quad \alpha = 19\,{}^{4}\!/_{10}$ seconds,

$m = 2\,{}^{3}\!/_{4}$; $\quad \dfrac{n-1}{n} = \dfrac{1}{361.53}$; $\quad n = {}^{362}\!/_{361}$; $\quad \alpha = 19\,{}^{9}\!/_{10}$ seconds,

$m = 3$; $\quad \dfrac{n-1}{n} = \dfrac{1}{385.58}$; $\quad n = {}^{386}\!/_{385}$; $\quad \alpha = 20\,{}^{3}\!/_{10}$ seconds,

$m = 3\,{}^{1}\!/_{4}$; $\quad \dfrac{n-1}{n} = \dfrac{1}{409.63}$; $\quad n = {}^{410}\!/_{409}$; $\quad \alpha = 20\,{}^{7}\!/_{10}$ seconds,

$m = 3\,{}^{1}\!/_{2}$; $\quad \dfrac{n-1}{n} = \dfrac{1}{433.68}$; $\quad n = {}^{434}\!/_{434}$; $\quad \alpha = 21\,{}^{1}\!/_{10}$ seconds.

All the same, m could equal ∞, in which case n would equal 1, the largest change of the obliquity of the ecliptic would not surpass $27\,{}^{2}\!/_{10}$ seconds.

Some inequalities in the Precession of the Equinoxes.

65. Since λ marks the longitude of the Summer solstice since the first star of Aries, $\lambda - 90°$ will mark that of the Spring equinox, hence $90° - \lambda$ is the longitude of the first star of Aries counted since the Spring equinox. Therefore, having found by the average movement the average longitude of the first star of Aries, which the Astronomical Tables show under the heading of the precession of the equinoxes, in counting $50\,{}^{1}\!/_{3}{}''$ per year, the other terms that enter in the expression of λ taken negatively will give the periodical inequalities that must either be added or subtracted from the average longitude. In this manner, we will find the true longitude of the first star of Aries since the equinoctial point for each proposed time. But, if we want to go back several centuries, it is necessary to take into account the planetary actions of Jupiter and Venus, from where both the average obliquity and the average precession of equinoxes is changed, as I had made seen in our Memoirs Vol. X, to which I refer to myself here.

66. The first correction therefore depends on the solar longitude Ψ and is proportional to the sine of twice this longitude. This correction is contained in this formula

$$-\ \frac{3(n-1)\,\Delta\,(\varepsilon n \cos l + 2\Delta)}{4(\varepsilon\varepsilon nn - 4\Delta\Delta)}\ \sin 2\Psi,$$

which in substituting for ε, Δ, and l, their values change as such,

$$-\ 0.0018833\,(n-1)\ \frac{(n + 0.0059693)}{nn - 0.0000300}\ \sin 2\Psi,$$

which upon reducing to minutes and seconds gives

$$-\,396.60\,(n-1)\,\frac{n+0.00597}{nn-0.00003}\,\sin 2\Psi \text{ seconds.}$$

Since $n = {}^{289}\!/_{288}$, this correction would never surpass $1\frac{1}{3}$ seconds, which is hardly noticeable. Nevertheless, if we want exact calculations, then this small correction must not be neglected.

67. The second correction depends on the longitude of the Moon ψ and is proportional to $\sin 2\psi$.

$$-\,\frac{3(n-1)\,m\Delta\Delta\,(\varepsilon n\cos l + 2\delta)}{4\delta\,(\varepsilon\varepsilon nn - 4\delta\delta)}\,\sin 2\psi,$$

which by substitution will thus be expressed in minutes and seconds.

$$-\,29.06m\,(n-1)\,\frac{n+0.079807}{nn-0.0053584}\,\sin 2\psi \text{ seconds.}$$

The third equation depends on the longitude of the ascending node θ and is proportional to its sine,

$$-\,\frac{3(n-1)\,m\Delta\Delta\,(\varepsilon n\cos 2l - \beta\cos l)\sin\gamma}{2\beta\,(\varepsilon\varepsilon nn - \beta\beta)\sin l}\,\sin\theta,$$

which when reduced to minutes and seconds will be

$$-\,2420.4m\,(n-1)\,\frac{n-0.000198}{nn-0.000000}\,\sin\theta \text{ seconds.}$$

Finally, the fourth depends on the angle $2\psi-\theta$

$$-\,\frac{3(n-1)\,m\Delta\Delta\,(\varepsilon n\cos 2l + (2\delta+\beta)\cos l)\sin\gamma}{2(2\delta+\beta)\,(\varepsilon\varepsilon nn - (2\delta+\beta)^2)\sin l}\,\sin(2\psi-\theta),$$

and gives in minutes and seconds

$$-\,4.86m\,(n-1)\,\frac{n+0.098557}{nn-0.005380}\,\sin(2\psi-\theta) \text{ seconds.}$$

68. Let us now consider the three hypotheses $m=2$, $m=2\frac{1}{2}$, and $m=3$, and the corrections for the obliquity of the ecliptic will be:

	if $m=2$	if $m=2\frac{1}{2}$	if $m=3$
$+\,A\cos 2\Psi$	$A=0''.58$	$A=0''.50$	$A=0''.44$
$+\,B\cos 2\psi$	$B=0.08$	$B=0.09$	$B=0.10$
$+\,C\cos\theta$	$C=8.96$	$C=9.61$	$C=10.07$
$+\,D\cos(2\psi-\theta)$	$D=0.02$	$D=0.02$	$D=0.02$

Now the corrections for the longitude of 1 + ♈ will be

	if $m = 2$	if $m = 2\frac{1}{2}$	if $m = 3$
$-\mathfrak{A}\cos 2\Psi$	$\mathfrak{A} = 1''.37$	$\mathfrak{A} = 1''.18$	$\mathfrak{A} = 1''.03$
$-\mathfrak{B}\cos 2\psi$	$\mathfrak{B} = 0.20$	$\mathfrak{B} = 0.21$	$\mathfrak{B} = 0.22$
$-\mathfrak{C}\cos\theta$	$\mathfrak{C} = 16.75$	$\mathfrak{C} = 17.95$	$\mathfrak{C} = 18.81$
$-\mathfrak{D}\cos(2\psi-\theta)$	$\mathfrak{D} = 0.03$	$\mathfrak{D} = 0.03$	$\mathfrak{D} = 0.04$

69. These formulas are perfectly in agreement with those that I had found in our Memoirs Vol. V, and hence I will no longer keep myself here at their application. I will only remark that Earth is not a homogenous mass, since then the value of the number n would need to be $^{201}\!/_{200}$, which is nevertheless according to all observations less than $^{301}\!/_{300}$. From this, it follows that the inequality between the principal moments of inertia is not as great as if it were homogenous, or better yet it more approaches the nature of a globe by the distribution of its material than by its figure. It is therefore necessary that Earth contains inside it a heavier material and is more equally distributed around the inertial center. Now this is also all that we can conclude. To the rest, if Earth did not turn very closely around a principal axis and its moments of inertia compared to the two other principal axes were not equivalent, then it would have been near impossible to determine its rotational movement.

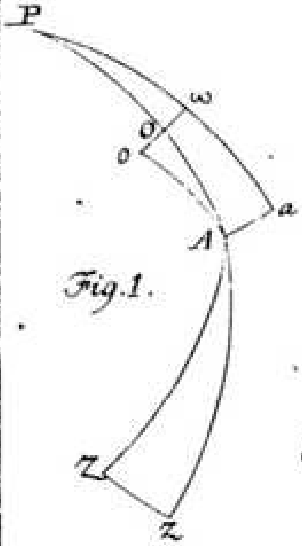

Fig. 1.

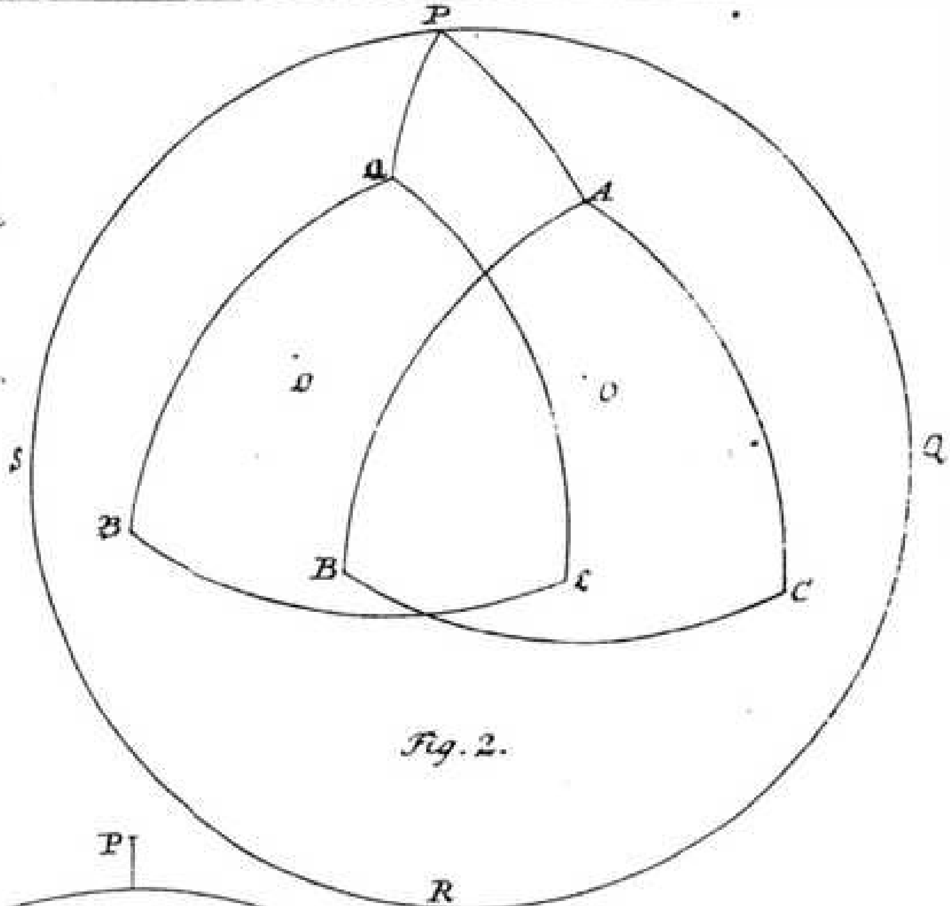

Fig. 2.

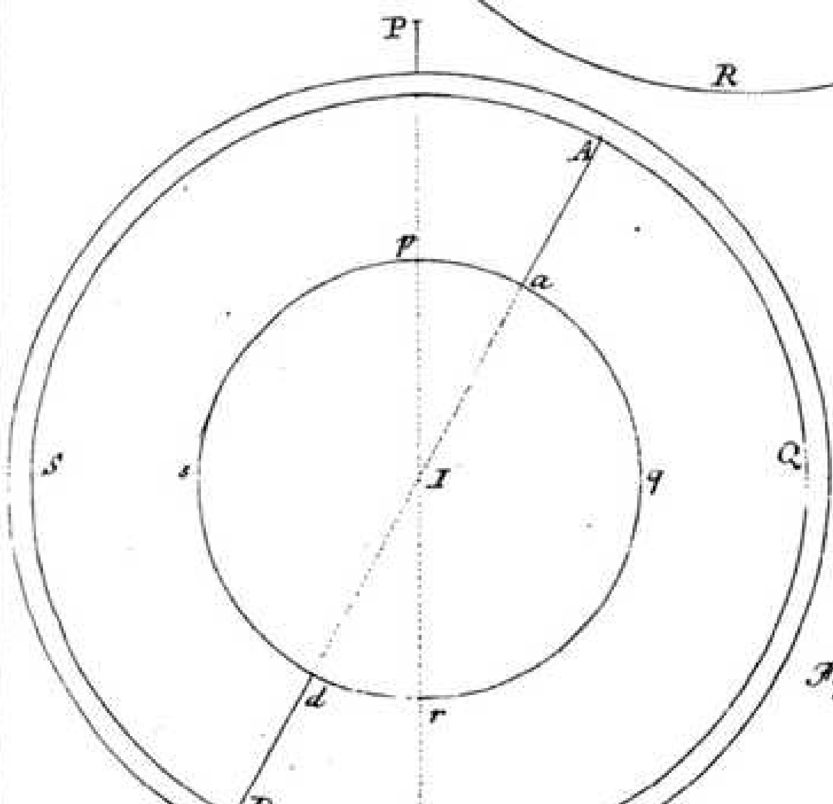

Fig. 3.

Mem. de l'Acad. 1759. Pl. 9. pag. 309.

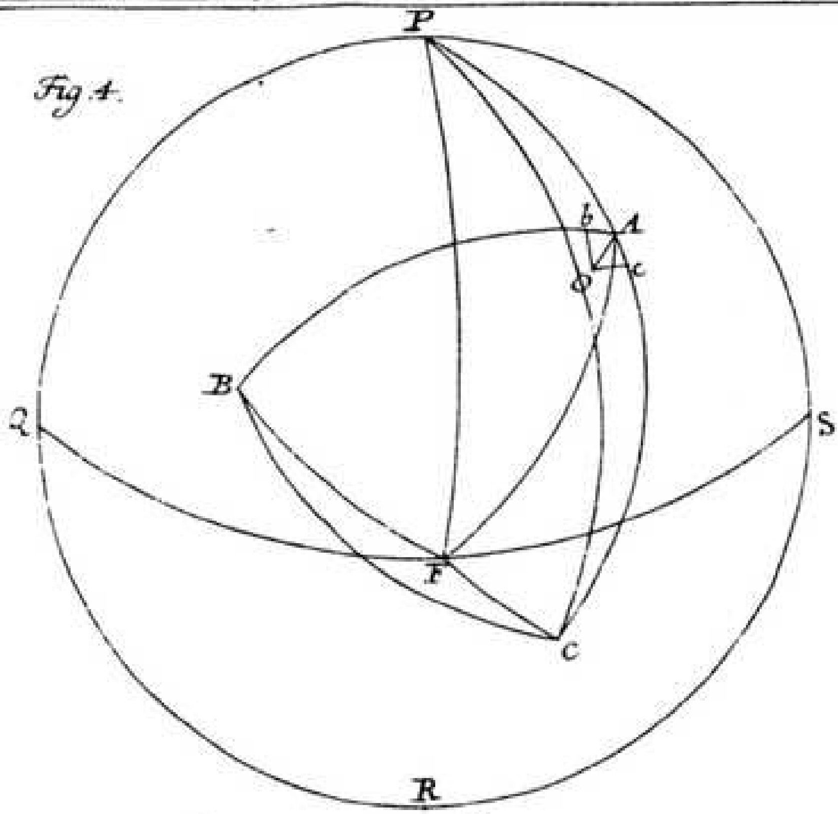

Fig. 4.

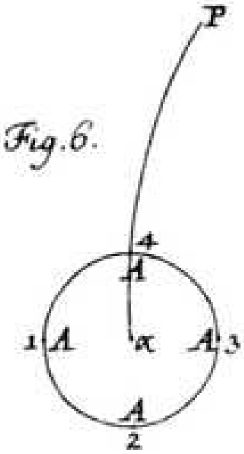

Fig. 6.

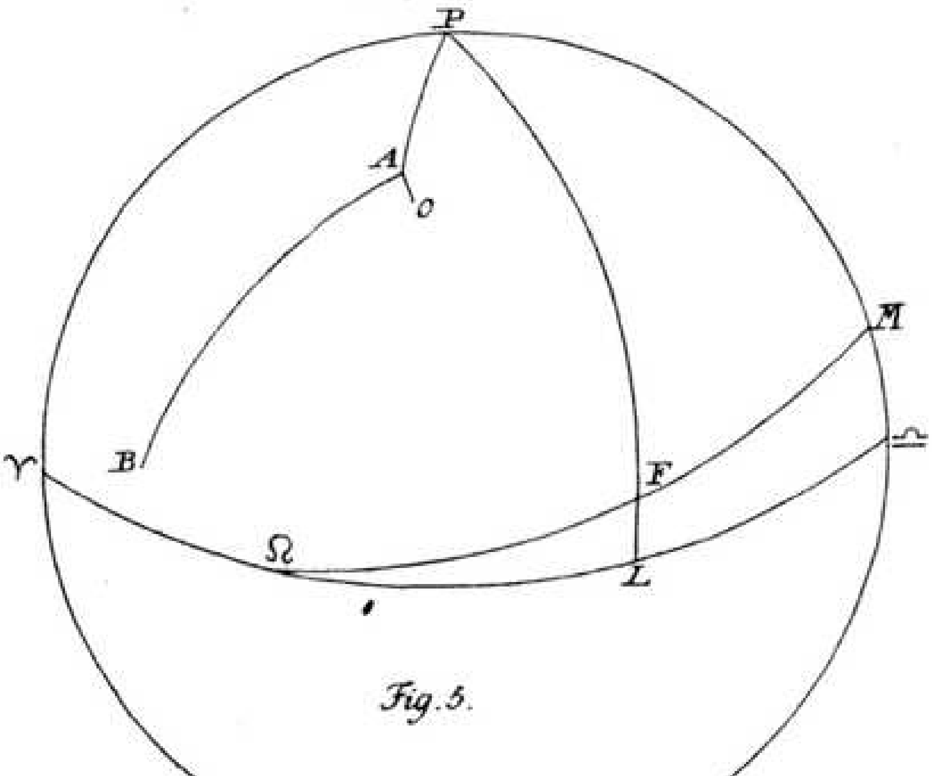

Fig. 5.

Commentary

Notation: ☊ - ascending node, the point in the lunar orbit where the Moon crosses from below to above the ecliptic plane.

♈ - Aries

♎ - Libra

ȣ - small *ou* ligature, I used this symbol because it most closely matched a variable used in the original work, in which I was unsure of its true nature.

The Greek used in the original work is an alternate alphabet, in which I used their Modern Greek equivalents.