

An English translation of portions of seven letters between Euler and Goldbach on Euler's complex exponential paradox and special values of cosine

Elizabeth Volz

August 20, 2008

The following is an English translation of portions of seven correspondences between Leonard Euler and Christian Goldbach dated between December, 1741 and June 1742. These letters discuss many mathematical topics; however, this translation focuses on portions which deal with a paradox discovered by Euler involving complex exponentials and special values taken on by the corresponding cosine function. These results are important since they represent some of Euler's earliest applications of his formula $e^{ix} = \cos x + i \sin x$, which he first published in 1748 in his pre-calculus textbook, *Introductio in analysin infinitorum* [E101]¹. These letters show that Euler certainly knew of this formula much earlier.

Acknowledgement: The author wishes to thank Reinert Schmidt for his help in the translation of these letters.

I. Euler to Goldbach: Berlin Dec 9, 1741 ([1], p. 91)

I have lately also found a remarkable paradox. Namely that the value of the expression $\frac{2^{+\sqrt{-1}} + 2^{-\sqrt{-1}}}{2}$ is approximately equal to 10/13 and that this fraction differs only in parts per million from the truth. The true value of this expression however is the cosine of the arc .6931471805599 or the arc of 39 degrees 42 min. 51 sec. 52 tenths of sec. and 9 hundredths of sec. in a circle of radius one.

I have also made several important discoveries over the integration of such formulas as $\frac{Pdx}{Q}$ all where P and Q are rational functions of x . At a different time the honor will be had to write this in a more noble and detailed way.

Notes:

Euler mentions that he has discovered the curiosity that the complex expression $\frac{2^{+\sqrt{-1}} + 2^{-\sqrt{-1}}}{2} = 0.769239\dots$ can be approximated by $10/13 = 0.769231\dots$ to five decimal places.

¹ Available at The Euler Archive: www.math.dartmouth.edu/~euler/

II. Goldbach to Euler: St. Petersburg Feb 13, 1742 ([1], p. 45)

With the observation as it was communicated to me that $\frac{2^{+\sqrt{-1}} + 2^{-\sqrt{-1}}}{2}$ is approximately

equal to 10/13 I have noticed that if you wanted to make it so that $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$ then p would have to be smaller than 3 and larger than 2. I confess that these limits are large but I do not have the curiosity to determine them any closer.

Notes:

Goldbach explores what value the exponent p must be in order to make the equation $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$ true, but only mentions that it must be between 2 and 3.

III. Euler to Goldbach: Berlin March 6, 1742 ([1], p. 96)

Now that I have the curiosity to investigate when $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$, it has given me the opportunity to remark that this can happen in infinitely many ways. First observed that p is between 2 and 3, namely 2.26618021. The true value is $p = \frac{\pi}{2l2}$ where $\pi = 3.14159265$ and $l2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{etc.} = .6931471803$. All following values are derived out of this in that you multiply these with 3,5,7,9 etc.

Notes:

Euler solves for p in the equation $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$ and finds its exact value to be

$p = \frac{\pi}{2 \ln 2}$. He then mentions the cyclic nature of the others solutions, namely

$p = \frac{(2n+1)\pi}{2 \ln 2}$, where $n = 1, 3, 5, 7, 9, \dots$. Observe that Euler writes $l2$ to denote $\ln 2$.

IV. Goldbach to Euler: St. Petersburg April 12, 1742 ([1], p. 100)

At the opportunity where you wrote $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$ I have observed that when the variable n is placed and $2^{np\sqrt{-1}} + 2^{-np\sqrt{-1}} = 2$ that as often as n is an even number they on the contrary equal -2 , as often as n is an odd number there is an integer q so that

$2^{(4n+q)p\sqrt{-1}} + 2^{-(4n+q)p\sqrt{-1}} = 2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}}$. It is in my opinion also remarkable that when you determine p through $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 3$ also then this will become

$2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}} = \frac{(1+\sqrt{5})^{2x+1} - (-1+\sqrt{5})^{2x+1}}{2^{2x+1}} - \frac{(1+\sqrt{5})^{2x-1} - (-1+\sqrt{5})^{2x-1}}{2^{2x-1}}$ as long as x is an integer.

Notes:

Goldbach introduces an additional variable in the exponent to explore solutions to $2^{np\sqrt{-1}} + 2^{-np\sqrt{-1}} = 2$ and notes that when n is an even integer $2^{np\sqrt{-1}} + 2^{-np\sqrt{-1}} = -2$. For n odd he claims there exists an integer q so that $2^{(4n+q)p\sqrt{-1}} + 2^{-(4n+q)p\sqrt{-1}} = 2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}}$, which turns out to not be entirely true as Euler points out in his next letter to Goldbach (May 8, 1742). Goldbach then claims that if $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 3$, then

$$2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}} = \frac{(1+\sqrt{5})^{2x+1} - (-1+\sqrt{5})^{2x+1}}{2^{2x+1}} - \frac{(1+\sqrt{5})^{2x-1} - (-1+\sqrt{5})^{2x-1}}{2^{2x-1}}.$$

This can be shown as follows: Let $y = 2^{p\sqrt{-1}}$. Then $y + \frac{1}{y} = 3$. It follows from the

quadratic formula that $y = 2^{p\sqrt{-1}} = \frac{3 \pm \sqrt{5}}{2} = \frac{3 \pm \sqrt{5}}{2}$ and so

$$\begin{aligned} 2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}} &= \left(\frac{3+\sqrt{5}}{2}\right)^x + \left(\frac{3-\sqrt{5}}{2}\right)^x \\ &= \frac{(1+\sqrt{5})^{2x+1} - (-1+\sqrt{5})^{2x+1}}{2^{2x+1}} - \frac{(1+\sqrt{5})^{2x-1} - (-1+\sqrt{5})^{2x-1}}{2^{2x-1}}. \end{aligned}$$

Observe that this expression in turn equals $(F_{2n+1} + F_{2n-1})$, where $\{F_n\}$ is the Fibonacci sequence. It is unclear whether Goldbach or Euler noticed this connection.

V. Euler to Goldbach: Berlin May 8, 1742 ([1], p. 101)

Generalizing $a^{p\sqrt{-1}} + a^{-p\sqrt{-1}} = b$ so that

$$a^{xp\sqrt{-1}} + a^{-xp\sqrt{-1}} = \left(\frac{b + \sqrt{bb-4}}{2}\right) + \left(\frac{b - \sqrt{bb-4}}{2}\right)^x$$

And when $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 3$ so will

$$2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}} = \left(\frac{3 + \sqrt{5}}{2}\right)^x + \left(\frac{3 - \sqrt{5}}{2}\right)^x = \left(\frac{\sqrt{5} + 1}{2}\right)^{\sqrt{2}x} + \left(\frac{\sqrt{5} - 1}{2}\right)^{\sqrt{2}x}$$

Otherwise I observe that with my general theorem $a^{p\sqrt{-1}} + a^{-p\sqrt{-1}} = 2 \cos(ap \ln 2)$ in most part agree only that $2^{(4n+q)p\sqrt{-1}} + 2^{-(4n+q)p\sqrt{-1}}$ is not equal to $2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}}$ when neither $(2n+q)p \ln 2$ nor $2np \ln 2$ is equal to $m\pi$ denoted 1: π

Notes:

There is an error in the translation involving the exponent in the second formula, which should be $2x$ instead of $\sqrt{2}x$:

$$2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}} = \left(\frac{3 + \sqrt{5}}{2}\right)^x + \left(\frac{3 - \sqrt{5}}{2}\right)^x = \left(\frac{\sqrt{5} + 1}{2}\right)^{2x} + \left(\frac{\sqrt{5} - 1}{2}\right)^{2x}$$

In this letter Euler considers the generalization $a^{p\sqrt{-1}} + a^{-p\sqrt{-1}} = 2$ and mentions that he only agrees with Goldbach' claim (stated in his previous letter) to the extent that $2^{(4n+q)p\sqrt{-1}} + 2^{-(4n+q)p\sqrt{-1}} \neq 2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}}$ when neither $(2n+q)p \ln 2$ nor $2np \ln 2$ are equal to $m\pi$. This is because if $(2n+q)p \ln 2 = 2np \ln 2 = m\pi$, then $(4n+q)p \ln 2 = 2m\pi$ and $qp \ln 2 = 0$. It follows that $\cos[(4n+q)p \ln 2] = \cos[qp \ln 2]$. Thus,

$$2^{(4n+q)p\sqrt{-1}} + 2^{-(4n+q)p\sqrt{-1}} = 2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}}$$

VI. Goldbach to Euler: Moscow June 7, 1742 ([1], p. 103)

As much as I can remember the formula $2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}}$ where $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$ has application as a serpentine curve whose abscissa y (introduced) and which the axis is intersected as often as the formula equals 0 so that when the formula equals 2 then the maximum application below or above comes out. It follows that countless other must be equal to each other. None the less, an error crept in to my original expression which can easily be corrected in that it should be that when q is a number such that $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$ then $2^{(8n-4-q)p\sqrt{-1}} + 2^{-(8n-4-q)p\sqrt{-1}} = 2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}}$.

Notes:

Goldbach mentions that the expression $2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}}$ has application as a serpentine curve. He then admits an error in his original expression and provides a new one. This is true because if we choose $q = 8n - 4 - 2m$, where m is any integer, then

$$\begin{aligned} (8n-4-q)\frac{\pi}{2} &= q\frac{\pi}{2} + 2m\pi \\ \Rightarrow (8n-4-q)p \ln 2 &= qp \ln 2 + 2m\pi \\ \Rightarrow \cos[(8n-4-q)p \ln 2] &= \cos[qp \ln 2] \\ \therefore 2^{(8n-4-q)p\sqrt{-1}} + 2^{-(8n-4-q)p\sqrt{-1}} &= 2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}} \end{aligned}$$

VII. Euler to Goldbach: Berlin June 30, 1742 ([1], p. 109)

In general $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 2 \cos(Ap \ln 2)$. When $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$ also there must be $p \ln 2$ in such a circular arc the cosine of which equals 0. All the arcs have the formula $\frac{(2n+1)\pi}{2}$ and consequently will $p = \frac{(2n+1)\pi}{2l2}$. If $p = \frac{(2n+1)\pi}{2l2}$ or $2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}} = 0$ so will $2^{xp\sqrt{-1}} + 2^{-xp\sqrt{-1}} = 2 \cos Axp \ln 2 = 2 \cos A \frac{2(n+1)x\pi}{2}$. When therefore it should be that $2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}} = 2^{rp\sqrt{-1}} + 2^{-rp\sqrt{-1}}$, so must $\cos A \frac{(2n+1)q\pi}{2} = \cos A \frac{(2n+1)r\pi}{2}$ but the cosine from two different arcs are equal to each other when either the sum or the difference of the arcs are equal to a multiple of the entire perimeter 2π . Therefore it will be $\frac{(2n+1)q\pi}{2} \pm \frac{(2n+1)r\pi}{2} = 2m\pi$ and from that it follows that $q \pm r = \frac{4m}{2n+1}$ so that it will become $2^{\frac{(4m}{2n+1}-q)p\sqrt{-1}} + 2^{-\frac{(4m}{2n+1}-q)p\sqrt{-1}} = 2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}}$.

Note:

Euler appears to further his generalization by using the symmetry of the cosine function to find a relation between the quantities q and r when $2^{qp\sqrt{-1}} + 2^{-qp\sqrt{-1}} = 2^{rp\sqrt{-1}} + 2^{-rp\sqrt{-1}}$.

References

[1] Leonhard Euler und Christian Goldbach: briefwechsel 1729-1764. Berlin: Akademie-Verlag, 1965.

[2] The Euler Archive: <http://www.math.dartmouth.edu/~euler/>

Department of Mathematics
Rowan University
Glassboro, NJ 08028