Animadversiones in rectificationem ellipsis
(Observations on the rectification of an ellipse)
Leonard Euler

English translation of E154 by
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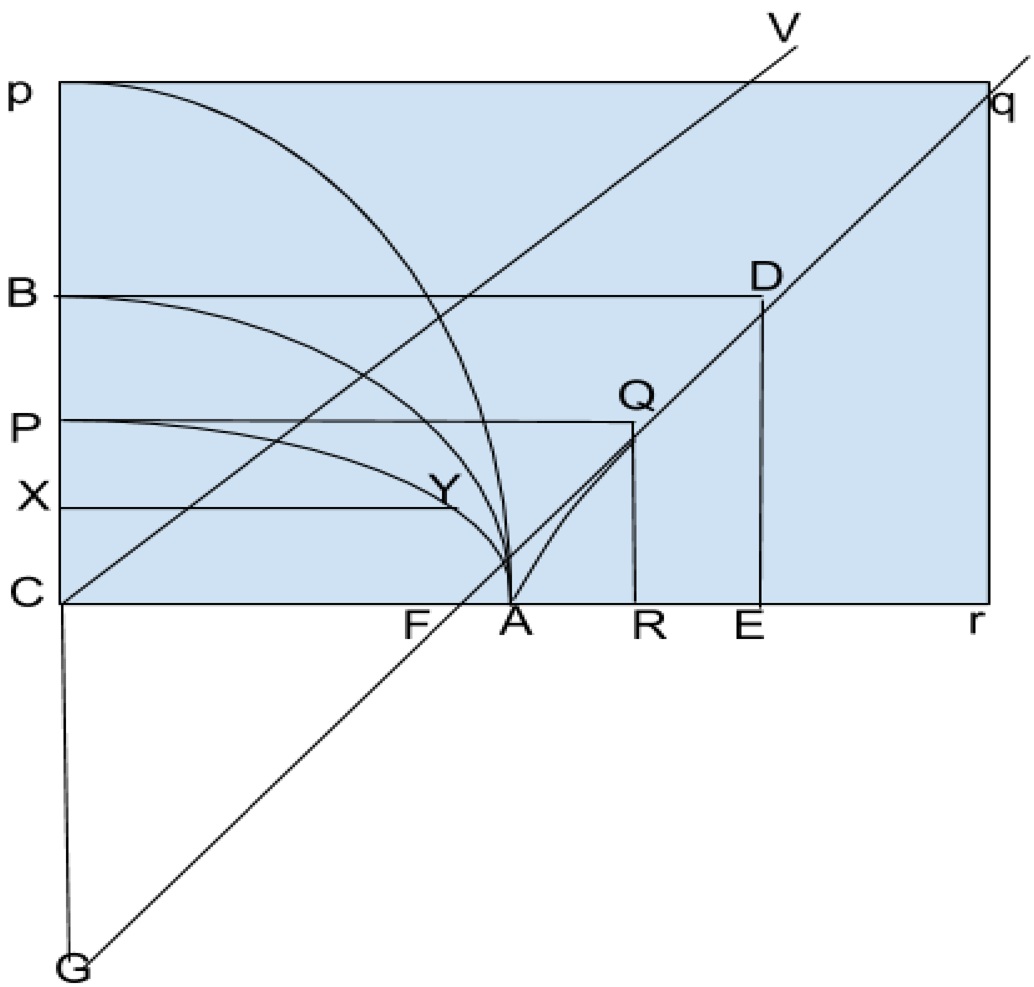
October 20, 2018

I. Now the rectification of an ellipse is handled in vain by so many various methods, so that not only are we unable to expect the comparison of the arcs of ellipses with straight lines, but likewise not even with circular or parabolic lines. When indeed that differential equation, whose indefinite integral of elliptic arcs expresses, is by no means able to be freed from irrationality; this is certain proof, that its integration, not only not algebraically, but is also not even to be able to be made by permitted quadratures of a hyperbola and a circle. Whatever must be held concerning the indefinite rectification of an ellipse follows this not at all thus far, that anything definite of an arc, just as the entire perimeter of an ellipse, thoroughly rejects every comparison with lines either straight or circular: because now innumerable rectifiable curves and an ellipse are not able to be assigned indefinitely, in which, nevertheless, definite arcs are able to be measured through straight lines.

II. Therefore, with the indefinite rectification of an ellipse disregarded, rather I attempted a definite rectification, I am going to expect, and whether the whole perimeter of which ellipse is not easily able for measured knowns, likewise to what end I propose that logarithms and circular arcs are recalled through finite expressions. However though, in this investigation, I certainly achieved nothing which satisfied my scope; nevertheless, except the expectation, some unique phenomena adequately presented themselves to me, by which the theory of lines and curves does not seem to be moved forward ordinarily. Then, likewise, difficulties which occurred in this calculation truly supply a certain opportunity for discovering conspicuous methods, which so much as in the integral calculation seem to be able to often assert this remarkable utility in the theory of infinite series. For what reason I thought that the value of the work would exist, if I will distinctly expose these speculations, and as though it were the whole nature of my calculations.

Proposition



III. Concerning the given line AC, just as I derive infinite infinite elliptic quandrants lines AP, AB, and Ap, drawn from one semiaxis, one-fourth of an ellipse, therefore of all the center is C, truly the other semi-axes CP, CB, and Cp are joined. Then from single points P, B, and p, the elliptic arcs PA, BA, and pA are extended into the perpendicular, so that thus anywhere PQ is parallel to line CA and equal the one-fourth of the ellipse PYA: because if it is derived to be joined anywhere, these points are centered on Q on a certain linear curve AQDp, whose nature I decided to investigate.

To the beginning of this curve, soon it will even slightly extend to those stretching towards, that the curve has the following properties, which I will explain, until I will more diligently investigate into the very nature of this curve; in order that its figure and it having at least been considered is examined in passing.

IV. Therefore first, if on the indefinite line, CBp, which is normal to the given CA, whichever is grasped by the divided CP, by the PQ situated close, which answers to it, will be equal to one-fourth the perimeter of the ellipse, whose semi-axes are joined together, the given line CA and CP itself divided. From here, if it is grasped by the divided, CB=CA, by this chance one-fourth ellipse is changed into one-fourth circular AB, answering, the segment BD will be equal to one-fourth circumference of a part of a circle, assigned by the radius AC. Whence if the ratio of the diameter to the circumference is given 1:$\boldsymbol{\pi}$, the such divided BD=$\boldsymbol{\pi}$/2, AC: or if on account of $\boldsymbol{\pi}$= 3.1415926535897932, BD= 1.5707963267948961(AC).

V. Second: If the divided CP passes, the ellipse avoids the infinitely narrow and is brought together with a straight line. Therefore, by chance, this one-fourth of the ellipse changes into the very line AC, therefore to which passing line the close line is equal. Whereby the answering line itself CA will be next to the point C, and the sought out curve goes through point A. Therefore now, we have two known points of this curve, A and D, of which one, A, is given geometrically, and the other, D, is truly defined through the ratio of the diameter to the the circumference.

VI. Third: From a known point of any curve, Q, situated between A and D, another certain point of the curve q, situated beyond D, is able to be defined. The third Cp is understood to be the thir proportion to CP and CA, with the result that Cp=(CA CA)/CP, because it is CP:CA=CA:Cp, the elliptic quadrant Ap is similar to the elliptic quadrant AP, since from both parts the ratio is the same between the joined semi-axes. From here the arc Ap will be to the arc AP as AC to CP, and for that reason pq:PQ=AC:CP or pq= (AC PQ)/CP. Consequently, if the arc AD of the sought out curve will be only assigned now, from this remainder a part of the curve, Dq, extended into infinity, will be defined.

VII. Fourth: Now from here a property of the distinguished equation, by which the nature of the curve AQDq is expressed, is realized. If in fact the given line AC is denoted by one, AC=1; however, anything divided by one, the minor arc CP=p, and answering to it, the situated near PQ=q; then truly that divided one is placed Cp=P and the adjacent connected pq=Q; P=1/p and Q=p/q. Hence, since the same equation, which is between p and q, ought to be the same between P and Q, it is well known that no equation would undergo a change, if in this place, p was considered 1/p and q/p in place of q. From where it is permitted to conclude what sort of the function of p itself is q.

VIII. Fifth: In consequence, it is well known that the those connected multiply the divided CP, since they are always greater than those divided. In truth if those divided are set to infinity, those connected by these very ones are equal: indeed a small difference will extend infinitely; from whence we deduce that the desired curve has an asymptote, and truly a line CV bisecting the straight angle ACB. Therefore, the form of this curve will be similar to the equilateral hyperbola centered on C, the axis CA and the asymptote of the holding CV. From the description, it is hereafter understood that the curve below the produced line CA would be similar of itself, and therefore that the line CA would likewise by the diameter of this hyperbola. Nevertheless, this easy thing is examined that our curve approaches its own asymptote CV much more slowly than the hyperbola. For on the equilateral hyperbola, to which we compare our curve, the sought out divided PQ is equal to the straight line AP; from whence since the division of our curve is equal to the arc AP, it is well known that the hyperbola of our curve would be circumscribed, thus still as in the beginning A, and they touch each other in mutual infinite space.

IX. Since these cuttings, observed in this method, were extended more widely, let us inquire more accurately into the very nature of this curve, and whatever proposed, divided CP=p, we find valor of the responding division that PQ=q; which is not able to be contained by a finite expression, it will ought to be presented through an infinite series. Therefore, it is necessary to resolve the following.

Problem

To define the length of the arc of the quadrant ATP from the given semi-axes CA and CP of an elliptic quadrant.

Solution

X. When a semi-axis is called AC=1, truly the other CP=p, and the arc AYP=q, first the desired indefinite arc PY, which =s, is sought. Now, led to the segment CP from the perpendicular XY, CX=x and XY=y, there will be from the nature of the ellipse $x = p\sqrt{1-y^2}$, and from here $dx = \frac{-pydy}{\sqrt{1-y^2}}$. Therefore, it happens on account of $ds = \sqrt{dx^2 + dy^2}$, $ds = \frac{dy\sqrt{1-y^2+p^2y^2}}{\sqrt{1-y^2}}$, whence it must be integrated $s = \int \frac{dy\sqrt{1-y^2+p^2y^2}}{\sqrt{1-y^2}}$, thus which integration ought to be made, if y=0, then *=0, because with the segment XY=y vanishing, at the same time PY=s vanishes. Therefore, I contrive this by an integral, if y=CA=1, the indefinite arc PY is changed into the length of the elliptic quadrant PYA=q, which we seek, is as thus: $q = \int \frac{dy\sqrt{1-y^2+p^2y^2}}{\sqrt{1-y^2}}$, if indeed after the integration is completed y=1.

XI. Therefore it is not necessary for our intention, as we seek the value of this indefinite integral, but only that which it puts on, after the integration the value, determined =1, is *divided/assigned* by the variable y: in which manner a much simpler series expressing the value q will be able to be obtained, in fact it is placed, for the sake of brevity, $1-p^2=n^2$, as it is $\sqrt{1-y^2+p^2y^2}=\sqrt{1-n^2y^2}$, and it will be this for evolving the formula into a series: $\sqrt{1-n^2y^2}=1-\frac{1}{2}n^2y^2-\frac{1\cdot1}{2\cdot4}n^4y^4-\frac{1\cdot1\cdot3}{2\cdot4\cdot6}n^6y^6$ etc. After which value has been substituted for $\sqrt{1-y^2+p^2y^2}$, thus arc q is expressed as it is:

$$q=\int\frac{dy}{\sqrt{1-y^2}}-\frac{1}{2}n^2\int\frac{y^2dy}{\sqrt{1-y^2}}-\frac{1\cdot1}{2\cdot4}n^4\int\frac{y^4dy}{\sqrt{1-y^2}}-\frac{1\cdot1\cdot3}{2\cdot4\cdot6}n^6\int\frac{y^6dy}{\sqrt{1-y^2}}\text{ etc.}$$

If indeed in each of these integrals after the integration it is placed y=1.

XII. Therefore we explain each of these integrals, and indeed first it is evident from a circle, that the formula $\int\frac{dy}{\sqrt{1-y^2}}$ expresses the curve of the arc of this circle=y, considering the radius=1: whence placed y=1, this formula will give ¼ part of the circumference, whose radius=1, therefore since the ratio of diameter to the circumference has been placed 1: π, $\int\frac{dy}{\sqrt{1-y^2}}=\frac{\pi}{2}$; thus now we have obtained the value of the first term of opur series before the discovery.

XII. The remaining terms are able to be expressed easily through the value π in a suitable way; in fact the integration of the any term is reduced to the integration of the preceding: because it is understood more easily by this we considered whatever formula $\int\frac{dy}{\sqrt{1-y^2}}$, following it will be a $\int\frac{y^{\mu+1}dy}{\sqrt{1-y^2}}$. Now we assume this algebraic formula $y^{\mu+1}\sqrt{1-y^2}$, whose differential, since it = $\frac{(\mu+1)y^{\mu}dy-(\mu+2)y^{\mu+2}dy}{\sqrt{1-y^2}}$, in turn will be $(\mu+1)\int\frac{y^{\mu}dy}{\sqrt{1-y^2}}-(\mu+2)\int\frac{y^{\mu+2}dy}{\sqrt{1-y^2}}=y^{\mu+1}\sqrt{1-y^2}$, from whence we deduce that it would be $\int\frac{y^{\mu+2}dy}{\sqrt{1-y^2}}=\frac{\mu+1}{\mu+2}\int\frac{y^{\mu}dy}{\sqrt{1-y^2}}-\frac{1}{\mu+2}y^{\mu+1}\sqrt{1-y^2}$. Therefore with the integral $\int\frac{ydy}{\sqrt{*-y^2}}$ contrived, from this the following integral $\int\frac{y^{\mu+2}dy}{\sqrt{1-y^2}}$ is easily elicited.

XIV. Because truly we desire only those values of these integrals, who appear if y=1; by this chance the algebraic quantity $\frac{1}{\mu+1}y^{\mu+1}\sqrt{1-y^2}$ vanishes, and on behalf of the chance y=1, will generally be $\int \frac{y^{\mu+2}dy}{\sqrt{1-y^2}} = \frac{\mu+1}{\mu+2}\int \frac{y^{\mu}dy}{\sqrt{1-y^2}}$. Now we substitute for μ the successive values 0, 2, 4, 6, 8,...

and because we saw that $\int \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2}$, it will be as follows:

If $\mu = 0: \int \frac{y^2}{\sqrt{1-y^2}} = \frac{1}{2}\int \frac{dy}{\sqrt{1-y^2}} = \frac{1\cdot\pi}{2\cdot2}$

If $\mu = 2: \int \frac{y^4}{\sqrt{1-y^2}} = \frac{3}{4}\int \frac{y^2dy}{\sqrt{1-y^2}} = \frac{1\cdot3\cdot\pi}{2\cdot4\cdot2}$

If $\mu = 4: \int \frac{y^6}{\sqrt{1-y^2}} = \frac{5}{6}\int \frac{y^4dy}{\sqrt{1-y^2}} = \frac{1\cdot3\cdot5\cdot\pi}{2\cdot4\cdot6\cdot2}$

If $\mu = 6: \int \frac{y^8}{\sqrt{1-y^2}} = \frac{7}{8}\int \frac{y^6dy}{\sqrt{1-y^2}} = \frac{1\cdot3\cdot5\cdot7\cdot\pi}{2\cdot4\cdot6\cdot8\cdot2}$

From whence the law, from which the sequences procede, is manifested by its own will.

XV. But if now those values for the integral formulas, from which the length of the elliptic quadrant q has been found to be composed, are substituted, it will be discovered that $q = \frac{\pi}{2} - \frac{1}{2}n^2(\frac{1\cdot\pi}{2\cdot2}) - \frac{1\cdot1}{2\cdot4}n^4(\frac{1\cdot3\cdot\pi}{2\cdot4\cdot2}) - \frac{1\cdot1\cdot3}{2\cdot4\cdot6}n^6(\frac{1\cdot3\cdot5\cdot\pi}{2\cdot4\cdot6\cdot2})$ etc. which is called back to the following series set in order sufficiently $q = \frac{\pi}{2}(1 - \frac{1\cdot1}{2\cdot2}n^2 - \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}n^4 - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6 -$ etc.) whose law of progression is manifested. Therefore its own value $1-p^2$ will be restored for n^2, and it will be $q = \frac{\pi}{2}(1 - \frac{1\cdot1}{2\cdot2}(1-p^2) - \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}(1-p^2)^2 - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}(1-p^2)^3 -$ etc.).

XVI. Since, concerning our curve AQDq, the letter p presents the segment CP and the letter q presents the connected PQ, now we arrive at the equation for that curve between these coordinates p and q, which although it is understood by an infinite series, still it not only includes the nature of this toward itself, but also next q presents the values of those near accurately enough, if the segment p differs a little from one, it is this, since CB=CA=1, if the point P was nearest to B itself; then in fact on account of $1-p^2 = n^2$, the invented series strongly converges to a very small quantity.

XVII. From here, therefore, we will be able to define the innate quality, the direction and curvature, of our curve near point D. First, in fact, it is well known as we saw, if $p = 1$, $q = \frac{\pi}{2}$, thus as a result CB=1, BD = $\frac{\pi}{2}$ =1.57079632679348961. Then in order to discover the position of a tangent, the ratio of differentials dq:dp is sought, which through differentiation is discovered: $\frac{dq}{dp} = \frac{\pi}{2}p(\frac{1}{2} + \frac{1\cdot1\cdot3}{2\cdot2\cdot4}(1-p^2) + \frac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot5}(1-p^2)^2 + \frac{1\cdot\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8}(1-p^2)^3 + etc.)$.

Now if p=1, $\frac{dq}{dp}=\frac{\pi}{4}$. Whence if DG is tangent to the curve at point D, since BD:BG=dq:dp, BG will equal $BD\frac{dp}{dq}=\frac{4}{\pi}BD$ and because BD= $\frac{\pi}{2}$, BH=2=2BC, and CG=BC. And thus by this chance the subtangent BG will be double the segment BC and since the tangent of the angle BGD $=\frac{dq}{dp}=\frac{pi}{4}$=0.78539816, angle BGD= $38^{0},\ 8^{I},\ 45^{II},\ 41^{III},\ 51^{IV}$.

XVIII. In order to define the radius of the mouth or of that pursued at the point D, since $\frac{dq}{dp}=\frac{\pi}{4}$, the element of the curve $\sqrt{(dp)^2+(dq)^2}=dp\sqrt{1+\frac{\pi^2}{16}}$, the radius of the mouth will be $(1+\frac{\pi^2}{16})^{3/2}\ dp^2 : d^2q$. But after the second derivative has been assumed, it will be

$$\frac{d^2q}{dp^2}=\frac{\pi}{2}(\frac{1}{2}+\frac{1\cdot1\cdot3}{2\cdot2\cdot4}(1-p^2)+\frac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6}(1-p^2)^2+etc.)-\frac{\pi}{2}p^2(\frac{1\cdot3}{2\cdot4}+\frac{1\cdot1\cdot3\cdot5}{2\cdot2\cdot4\cdot6}(1-p^2)+\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6}(1-p^2)^2+etc.)$$

Therefore if p=1, $\frac{d^2q}{dp^2}=\frac{\pi}{2}(\frac{1}{2}-\frac{3}{8})=\frac{\pi}{16}$: whence at the point of the curve D, the radius of the pursued will be equal to $\frac{16}{\pi}(1+\frac{\pi^2}{16})^{3/2}$, whose value is approximately 10.470672.

XIX. From here thus far another series is able to be invented, which expresses the value of PQ=q. Still, that one point of the curve, q, for which Cp=P and pq=Q, likewise it will be because P>1.

$$Q=\frac{\pi}{2}(1+\frac{1\cdot1}{2\cdot2}(P^2-1)-\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}(P^2-1)^2+\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}(P^2-1)^3-etc.)$$

Now as we noted above, if P= $\frac{1}{p}$, then Q= $\frac{q}{p}$. Therefore, after these values are substituted, we will obtain a new equation between p and q, by which the nature of the curve is equally expressed.

$q=\frac{\pi}{2}p(1+\frac{1\cdot1}{2\cdot2}(\frac{(1-p^2)}{p^2})-\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}(\frac{(1-p^2)^2}{p^4})+\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}(\frac{(1-p^2)^3}{p^6})-etc.)$, if when it is combined before the invention, innumerable other new equations are able to be obtained. Just as if through p, multiplied by this, it is subtracted, it produces

$q-pq=\frac{\pi}{2}p(\frac{1\cdot1\cdot(1-p^2)(1+p^2)}{2\cdot2\cdot p^2}-\frac{1\cdot1\cdot1\cdot3\cdot(1-p^2)^2(1-p^4)}{2\cdot2\cdot4\cdot4\cdot p^4}+etc.)$, which is reduced to this

$q=\frac{\pi}{4}(1+p)(\frac{1}{2}\frac{1+p^2}{p}-\frac{1\cdot1\cdot3(1-p^4)(1-p^2)}{2\cdot4\cdot4\cdot p^3}+\frac{1\cdot1\cdot3\cdot3\cdot5(1+p^6)(1-p^2)^2}{2\cdot4\cdot4\cdot6\cdot6\cdot p^5}-etc.)$ or since the sies is divisible by $\frac{1+p^2}{2p}$, it will be

$$q=\frac{\pi(1+p)(1+p^2)}{8\cdot p}(1-\frac{1\cdot3(1-p^2)}{4\cdot4\cdot p^2}(1-p^2)+\frac{1\cdot3\cdot3\cdot5(1-p^2+p^4)}{4\cdot4\cdot6\cdot6\cdot p^4}(1-p^2)^2-\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7(1-p^2+p^4-p^6)}{4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot p^6}(1-p^2)^3+etc.)$$

XX. It is unmistakable that these series are of produce too little of relief, if we wish to find segments, which are farther from BD, which answer to the segment p=1, they are remote, if in fact for p a number is placed either very large or very small, the invented series either certainly converge insufficiently or even diverge. If indeed then we wish to define the length of the first segment CA, which responds to the segment p=0, first with the series invented as soon as it

converges, because large terms/limits escape in the remaining infinitely. Therefore, concerning this chance p=0, we will have
$q = \frac{\pi}{2}(1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} - etc.)$, which so lightly converges, as even if very many terms are collected, still the true value of q, which we know to be 1, then is able to be discerned most difficulty.

XXI. However, although we know that $(1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} - etc.) = \frac{2}{\pi}$, nevertheless the invention of the sum of this series does not seem less difficult, if it is attempted by the previous. Indeed it is permitted to realize the truth from the formula, which Wallis formerly gave for the fourth of a circle, if the terms from the beginning are collected into one, it proceeds as thus:

$1 - \frac{1\cdot 1}{2\cdot 2} = \frac{1\cdot 3}{2\cdot 2}$

$\frac{1\cdot 3}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} = \frac{1\cdot 3(4\cdot 4-1\cdot 1)}{2\cdot 2\cdot 4\cdot 4} = \frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}$

$\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} = \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}$

Whence the value of the series will continue into infinity: $\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot 13}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot 14}$ *etc.* which expression, since it is the very Wallis formula, extends the sum of our series to be $\frac{2}{\pi}$. Nevertheless it is helpful to relate this series and to sum the other terms from the previous.

Problem

To find the sum of the infinite series $1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} - etc.$, whose law of progression is manifested from the first considered term.

Solution

XXII. Let the sum of this series be equal to s, so that
$s = 1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} - etc.$ Now the series is chosen, whose sum exists/is constant and whose coefficients are contained in these terms. Of what nature is this:

$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 + etc.$

Therefore, by multiplying and integrating through any differential dP, it will be:

$\int \frac{dP}{\sqrt{1-x^2}} = P + \frac{1}{2}\int x^2 dP + \frac{1\cdot 3}{2\cdot 4}\int x^4 dP + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int x^6 dP + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\int x^8 dP + etc.$

Now it is defined by the differential dP, so that after the integration x=1:

$* \int x dP =- \frac{1}{2}P *$

$\int x^4 dP = \frac{1}{4}\int x^2 =- \frac{1\cdot 1}{2\cdot 4}P$

$\int x^6 dP = \frac{3}{6}\int x^4 dP =- \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}P$

$\int x^8 dP = \frac{5}{8}\int x^6 dP = -\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}P$

By which fact, if these values are substituted, it will be held:

$\int \frac{dP}{\sqrt{1-x^2}} = P\left(1 - \frac{1\cdot1}{2\cdot2} - \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4} - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6} - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8} - etc\right)$

And for that reason $\int \frac{dP}{\sqrt{1-x^2}} = Ps$ if after the integration x=1.

XXIII. Therefore the thing resides here, in order to seek the formula of the differential dP, so that it satisfies the above conditions, whether it is $\int x^{\mu+2} dP = \frac{\mu-1}{\mu-2}\int x^{\mu} dP$ in type, if indeed after integration x=1. Therefore with this condition omitted, it is $\int x^{\mu+2} dP = \frac{\mu-1}{\mu+2}\int x^{\mu} dP + \frac{Qx^{\mu+2}}{\mu+2}$ where Q is a function of x, which vanishes when x=1. Therefore the differentials are seized, and will be divided through x^{μ}:

$x^2 dP = \frac{\mu-1}{\mu+2}dP + \frac{xdQ+(\mu+1)Qdx}{\mu+2}$

Or if $0 = (\mu-1)dP - (\mu+2)x^2 dP + xdQ + (\mu+1)Qdx$

Which equation, since it ought to have a place/locus for all values of μ, is resolved in these two:

$0 = dP - x^2 dP + Qdx$

$0 = -dP - 2x^2 dP + xdQ + Qdx$

Whence dP$= -\frac{Qdx}{1-x^2} = \frac{xdQ+Qdx}{1+2x^2}$ and $xdQ(1-x^2) = -Qdx(2+x^2)$ whereby since

$\frac{dQ}{Q} = \frac{dx(2+x^2)}{x(1-x^2)} = \frac{-2dx}{x} - \frac{3xdx}{1-x^2}$, it will be

$Q = \frac{-(1-x^2)^{3/2}}{x^2}$ and $dP = \frac{dx}{x^2}\sqrt{(1-x^2)}$

XXIV. Truly here it must be known, and if the value of Q vanishes when x=1; yet by chance $\mu = 0$, the algebraic quantity $\frac{Qx^{\mu+1}}{\mu+2}$ does not vanish, if x=0, which condition is equally necessary, thus so that by this chance it is not $\int x^2 dP = \frac{-1}{2}P$. When the remaining formulae have a place/locus from $\mu > 0$, from the formula, $\int x^2 dP$ must be undertaken and will be:

$\int x^4 dP = \frac{1}{4}\int x^2 dP$

$\int x^6 dP = \frac{3}{6}\int x^4 dP = \frac{1\cdot3}{4\cdot6}\int x^2 dP$

$\int x^8 dP = \frac{5}{8}\int x^6 dP = \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\int x^2 dP$ etc.

Whence it will be held

$$\int \frac{dP}{\sqrt{(1-x^2)}} = P + \int x^2 dP\left(\tfrac{1}{2} + \tfrac{1\cdot1\cdot3}{2\cdot4\cdot4} + \tfrac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot4\cdot4\cdot6\cdot6} + etc.\right)$$

But $\frac{1}{2} + \frac{1\cdot1\cdot3}{2\cdot4\cdot4} + \frac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot4\cdot4\cdot6\cdot6} + etc. = 2(1\text{-}s)$; and for that reason:

$\int \frac{dP}{\sqrt{(1-x^2)}} = P + 2(1-s)\int x^2 dP$, but because $dP = \frac{dx}{x^2}\sqrt{(1-x^2)}$, it will be:

$$P = C - \frac{\sqrt{(1-x^2)}}{x} - Asin(x);$$

$\int x^2 dP = \int dx\sqrt{1-x^2} = \frac{1}{2}Asin(x) + \frac{1}{2}x\sqrt{1-x^2}$, and $\int \frac{dP}{\sqrt{1-x^2}} = D - \frac{1}{x}$

Where the constants C and D ought to be accepted, so that these integrals vanish when x=0: although each *seorsim*is infinite, yet those connected will destroy themselves. In fact it will be:

$$\int \frac{dP}{\sqrt{1-x^2}} - P = D - \frac{1}{x} - C + \frac{\sqrt{1-x^2}}{x} + Asin(x)$$

Which, as it vanishes when x=0, ought to be D=C, and for that reason, when x= $\frac{\pi}{2}/1$ and if

$\int \frac{dP}{\sqrt{1-x^2}} - P = -1 + \frac{\pi}{2}$: and by this same chance $\int x^2 dP = \frac{\pi}{4}$, it will produce:

$-1 + \frac{\pi}{2} = 2(1-s)\frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi s}{2}$ and from here it is thought that $\frac{\pi s}{2} = 1$ and s= $\frac{2}{\pi}$ or
$1 - \frac{1\cdot1}{2\cdot2} - \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4} - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6} - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8} - etc. = \frac{2}{\pi}$ as we concluded from the nature of the thing.

XXV. Therefore because we believe in this beginning that CA=1, we seek out the quality of this curve near point A, or we inquire in the value of q, if the segment p was very small. To this end, we place, for a second time, $1 - p^2 = n^2$, and since
$q = \frac{\pi}{2}(1 - \frac{1\cdot1}{2\cdot2}n^2 - \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}n^4 - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6 - etc.)$ and because we know most nearly q=1, we add the equality invented in this manner, $0 = 1 - \frac{\pi}{2}(1 - \frac{1\cdot1}{2\cdot2}n^2 - \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}n^4 - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6 - etc.)$, we will have:

$$q = 1 + \frac{\pi}{2}\left(\frac{1\cdot1}{2\cdot2}(1-n^2) + \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}(1-n^4) + \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}(1-n^6) + etc.\right)$$

Since single terms of this series are divisible through $1 - n^2 = p^2$, this expression is reduced to this:

$$q = 1 + \frac{\pi}{8}p^2\left(1 + \frac{1\cdot3}{4\cdot4}(1+n^2) + \frac{1\cdot3\cdot3\cdot5}{4\cdot4\cdot6\cdot6}(1+n^2+n^4) + \frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{4\cdot4\cdot6\cdot6\cdot8\cdot8}(1+n^2+n^4+n^6) + etc.\right)$$

XXVI. Because if in this expression single terms are evolved to the powers of n, it is discovered:

(not sure how to transcribe this figure, working on it, found on page 19/139

$$q = 1 + \frac{\pi}{2} pp \begin{cases} + \frac{1.1}{2.2} + \frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \\ \frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8} + \&c. \\ + n^2 \left(\frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \right. \\ \frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8} + \&c. \\ + n^4 \left(\frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8} + \right. \\ \&c. \\ + n^6 \left(\frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8} + \&c. \right. \\ \&c. \end{cases}$$

But from the above inventions we have the sum of the first series: $\frac{1\cdot1}{2\cdot2} + \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4} + \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6} + etc. = 1 - \frac{2}{\pi}$, which if it is multiplied by the first term, will produce the following, which is the coefficient of n^2, thus it is: $\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4} + \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6} + etc. = \frac{1\cdot3}{2\cdot2} - \frac{2}{\pi}$. After this has once more been multiplied by the first term, it will give the coefficient of n^4: $\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6} + etc. = \frac{1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4} - \frac{2}{\pi}$, and at the same time the coefficient of n^6 is $\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6} - \frac{2}{\pi}$ and thus it will finally be obtained:
$q = 1 + \frac{\pi}{2}p^2[(1 - \frac{2}{\pi}) + (\frac{1\cdot3}{2\cdot2} - \frac{2}{\pi})n^2 + (\frac{1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4} - \frac{2}{\pi})n^4 + (\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6} - \frac{2}{\pi})n^6 + etc.]$ or it will be:
(Too fuzzy to tell)

XXVII. Now we place n=1, as a result we obtain of this form $q = 1 + Ap^2$, which is expressed by the nature of the curve near point A: since it is permitted to conjecture that the true equation would be of this form:
$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + etc.$
If segment p is assumed to be small, the remaining terms except the first two/pairs are able to be omitted, and from the equation $q = 1 + Ap^2$, to such an extent the tangent position, which the curvature will be able to collect/sum up. In fact when AR=x, RQ=y, it will be $q = 1 + x$ and $p = y$, and likewise if arc AQ will be the smallest, it is considered with a parabola, whose equation is $x = Ay^2$ or $y^2 = \frac{1}{A}x$, and therefore $\frac{1}{A}$ is a parameter. Whence it follows that the tangent of the curve at A would be perpendicular to the line AC, and the radius of curvature would be $\frac{1}{2A}$.

XXVIII. Therefore this coefficient A is discovered, if in the above series, through which the quantity p^2 is multiplied, it is placed $n = 1$, it is thus:
$A = (\frac{\pi}{2} - 1) + (\frac{1\cdot 3\cdot \pi}{2\cdot 2\cdot 2} - 1) + (\frac{1\cdot 3\cdot 3\cdot 5\cdot \pi}{2\cdot 2\cdot 4\cdot 4\cdot 2} - 1) + etc.$
Moreover if which summation is held, converging so in sufficiently, it is discovered/caught, so that we ought to suspect that its sum is infinite. In this suspicion we are more confirmed, if we evolve the invented series (15) first, the dimensions of p second, it is:
(photo)

$$\text{pi/2} \left\{ \begin{array}{l} \text{1-(1x1/2x2)-} \\ \\ \text{+p} \\ \\ \text{-p} \end{array} \right.$$

XXIX. From the coefficient of p^2 in the general equation for the curve
$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + etc.$ will be:
$A = \frac{\pi}{2}(\frac{1\cdot 1}{2\cdot 2}1 + \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}2 + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}3 + etc.)$ or $A = \frac{\pi}{4}(\frac{1}{2} + \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8} + etc.)$
And by a similar manner it will be permitted to elicit the remaining coefficients B, C, D, etc.from this series. Truly we will be able to to refrain from the labor, since it is evident/proven that not only coefficient A, but also the remaining coefficients would extend to infinity. And if this is clear from the solution of this problem.

Problem

To find the sum of this infinite series: $S = \frac{1}{2} + \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6} + etc.$

Solution

XXX. This formula is assumed to find the sum of the series “s”:
$\frac{1}{\sqrt{(1-x^2)}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + etc.$

$\int \frac{dP}{\sqrt{(1-x^2)}} = P + \frac{1}{2}\int x^2 dP + \frac{1\cdot3}{2\cdot4}\int x^4 dP + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\int x^6 dP + etc.$ and if after each integration x=1.

$\int x^2 dP = \frac{3}{4}P$

$\int x^4 dP = \frac{5}{6}\int x^2 dP = \frac{3\cdot5}{4\cdot6}P$

$\int x^6 dP = \frac{7}{8}\int x^4 dP = \frac{3\cdot5\cdot7}{4\cdot6\cdot8}P$ or from here

$\int \frac{dP}{\sqrt{1-x^2}} = P(1 + \frac{1\cdot3}{2\cdot4} + \frac{1\cdot3\cdot3\cdot5}{2\cdot4\cdot4\cdot6} + \frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot4\cdot4\cdot6\cdot6\cdot8} + etc.)$ or $\int \frac{dP}{\sqrt{1-x^2}} = 2P$

Whence after P is invented, s is discovered if after the integration x=1.

XXXI. Therefore it ought to be more general.

$\int x^{\mu+1} dP = \frac{\mu+3}{\mu+4}\int x^{\mu} dP + \frac{x^{\mu+1}Q}{\mu+4}$

Provided that Q is a function of this sort, which vanishes when x=1, it will be

$(\mu+4)\, x^2 dP = (\mu+3)\, dP + xdQ + (\mu+1)Qdx$

From which the two following equations are constructed:

$x^2 dP = dP + Qdx$

$4x^2 dP = 3dP + dQ + Qdx$

And $dP = \frac{-Qdx}{1-x^2} = \frac{-xdQ-Qdx}{3-4x^2}$

And from here having been elicited

$\frac{dQ}{Q} = \frac{2dx-3x^2dx}{x(1-x^2)} = \frac{2dx}{x}\ \frac{-xdx}{1-x^2}$ and $Q = -x^2\sqrt{1-x^2}$.

Therefore it will be held:

$dP = \frac{x^2 dP}{\sqrt{1-x^2}}$ and $\frac{dP}{\sqrt{1-x^2}} = \frac{x^2dx}{1-x^2} = -dx + \frac{dx}{1-x^2}$

Therefore P= $\frac{3}{4}\pi$, if after the integration x=1 and $\int \frac{dP}{\sqrt{1-x^2}} = -x + \frac{1}{2}(1)\frac{1+x}{1-x}$, whose value when x=1 is surely infinite. Therefore it will be s=∞, or the sum of the series is infinitely large.

XXXII. Therefore because coefficient A of p^2, in the equation $q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + etc.$ is infinite, the radius of curvature at point A will certainly be infinitely small. Truly besides this equation: in which all coefficients A, B, C, D, etc. are entirely infinite, nothing completely compares/brings to the investigation/knowledge of the curve. Because the radius of curvature at A is infinitely small, the nature of the curve around point A is expressed by an equation of this manner, $q = 1 + ap^m$, in which the exponent *m* is less than 2, but greater than 1: but from all, which thus far were handed over/related, it extends from no way, from which we are able to examine this (ex potentem). Since this number is no able to be an integer, no series, which we elicit for q, thus has been prepared, so that it is permitted to elicit the irrational power of p from it.

XXXIII. From here we know that the problems are exceedingly difficult, where an elementary equation is only required, which at least shows the nature of the proposed curve AQDq nearest around point A. It was known, if it is placed AR=x and RQ=y, wherever the curve AQ was/will have been, that the nature of the this smallest portion around A is able to be comprehended by an equation of this sort: $y^m = Ax$; if indeed the curve is algebraic; moreover it seems certain for transcendent curves that whatever smallest portions of these are able to be compared with the arcs of algebraic curves. Therefore on our curve, even if it is transcendent, it ought to seem more miraculous than that, because no formula of this type, $y^m = Ax$, is able to be shown, which at least declares the nature of the smallest portion centered around A.

XXXIV. In order to resolve this knot/node, it is necessary for us to investigate a finite equation between the coordinates p and q, which although it rises to a differential of the second order, granted that it is easy to foresee, nevertheless it will have been adjusted for a more accurate understanding of the curve. However we will elicit an equation of this kind, which is understood by a finite number of terms, if first we will recall the series derived for the sum. When $1-p^2=n^2$, it is:

$$\frac{2q}{\pi} = 1 - \frac{1\cdot 1}{2\cdot 2}n^2 - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^4 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 - etc.$$

After differentiation it will be:

$$\frac{2dq}{\pi dn} = -\frac{1\cdot 1}{2}n - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4}n^3 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6}n^5 - etc.$$

Which, multiplied by n and then differentiated, is:

$$\frac{2}{\pi dn}d\frac{ndq}{dn} = -1\cdot 1n - \frac{1\cdot 1}{2\cdot 2}1\cdot 3n^3 - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}3\cdot 5n^5 - etc.$$

This is multiplied by $\frac{dn}{n}$ and then integrated:

$$\frac{2}{\pi}\int \frac{1}{n}d\frac{ndq}{dn} = -1n - \frac{1\cdot 1}{2\cdot 2}1n^3 - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}3n^5 - etc.$$

It is multiplied by $\frac{dn}{n^3}$ and integrated it will produce:

$$\frac{2}{\pi}\int \frac{dn}{n^3}\int \frac{1}{n}d\frac{ndq}{dn} = \frac{1}{n} - \frac{1\cdot 1}{2\cdot 2}n - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^3 - etc.$$

Which series, since itself was proposed, divided by n will be:

$$\frac{2}{\pi}\int \frac{dn}{n^3}\int \frac{1}{n}d\frac{ndq}{dn} = \frac{2q}{\pi n} \text{ or } \int \frac{dn}{n^3}\int \frac{1}{n}d\frac{ndq}{dn} = \frac{q}{n}$$

XXXV. Now we sum the differentials, and it will be held:

$$\frac{dn}{n^3}\int \frac{1}{n}d\frac{ndq}{dn} = \frac{ndq-qdn}{n^2} \text{ or } \int \frac{1}{n}d\frac{ndq}{dn} = \frac{n^2dq}{dn} - nq$$

And differentiated again:
$\frac{1}{n}d\frac{ndq}{dn} = nd\frac{ndq}{dn} + ndq - ndq + qdn$ or $(1-n^2)d\frac{ndq}{dn} + qndn = 0$
Now because $1-n^2 = p^2$, it will be:
$ndn = -pdp$ and $\frac{dn}{n} = -\frac{pdp}{1-p^2}$
From which we get:
$-p^2 d\frac{(1-p^2)dq}{pdp} - pqdp = 0$ or $d\frac{(1-p^2)dq}{pdp} + \frac{qdp}{p} = 0$
Now constant dp is summed, it will be:
$\frac{(1-p^2)dq}{pdp} - \frac{dq(1+p^2)}{p^2} + \frac{qdp}{p} = 0$ or $p(1-p^2)d^2q - dpdq(1+p^2) + pqdp^2 = 0$

XXXVI. Behold! Therefore the position for pursuing the differential equation for the proposed curve

$$p(1-p^2)d^2q - dpdq(1+p^2) + pqdp^2 = 0$$

from which the power of p in the equation $q = 1 + Ap^m$ is ought to be elicited, if the segment p is placed very small. Since if $dq = mAp^{m-1}$ and $d^2q = m(m-1)Ap^{m-2}$, it will emerge
$m(m-1)Ap^{m-1} - mAp^{m-1} + p - m(m-1)Ap^{m+1} - mAp^{m+1} + Ap^{m+1} = \alpha$ or
$m(m-2)Ap^{m-1} - m^2Ap^{m+1} + p = \alpha$

Therefore it ought to be m=2, in order to terminate Ap^{m-1}, since p is able to be compared, but then for the second time $A = \infty$ is obtained: because it is perceived that the exponent m is able to be a fraction by no means, thus as a result the difficulty seems to be increased beyond the recount ore than to be destroyed.

XXXVII. And if after the rules/straight edges were joined/devised as we wish to evolve the derived equation into a series, which proceed according to the powers of p, because we know the first term of the series to be =1, then it is permitted to deduce no other form except this:
$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + etc.$
Then is:
$\frac{dq}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7 + etc.$ and
$\frac{d^2q}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + etc.$
Which values having been substituted in the equation will present:

$$\begin{array}{llllll} +2Ap & +12Bp^3 & +30Cp^5 & +56Dp^7 & +etc & \\ & -2A & -12B & -30C & -etc & \\ -2Ap & -4B & -6C & -8D & -etc. & =0 \\ +p & +A & +B & +C & +etc. & \end{array}$$

From which all coefficients A, B, C, etc proceed to infinity.

XXXVIII. Therefore from here we see the ordinary rules, according to which the form of the series has become accustomed to be judged, into which the differential equation must be transmuted, that it is not sufficient, since by this chance they convey no use: whence our equation deserves greater attention. Yet by the following means it will be able to be deduced from the nature of the curve near point A, from which at the same time it is understood that, likewise, just as in other cases, the defectives are supplied by such a use of regular * *, and they ought to be adjusted to the *praxin*. Because we hold the segment p for a part infinitely in the equation, it will be permitted to place 1 for $1-p^2$ and $1+p^2$, and because we know by this chance q is nearly 1, for the finite quantity q we wrote 1: by which fact the differential equation derived by this chance, since segment p is very small, will put on the following form of differentials:

$$pd^2q - dpdq + pdp^2 = 0$$

XXXIX. Now the resolution of this equation is easy, since dp is constant, it is placed dq=rdp, it will be $d^2p = drdp$, and it will be held:

$$pdr - rdp + pdp = 0 \quad \text{or} \quad \frac{pdr-rdp}{p^2} + \frac{dp}{p} = 0$$

The integration of which is:

$$\frac{r}{p} + ln(p) = C$$

Whence it is:

$$r = Cp - pln(p)$$

And then:

$$dq = Cpdp - pdqln(p)$$

And this equation integrated will give:

$$q = 1 + \tfrac{1}{2}Cp^2 - p^2ln(p) + \tfrac{1}{4}p^2$$

In which, since the term p^2 is incomparably less than $p^2ln(p)$, the beginning A for the curve will be:

$$q = 1 - \tfrac{1}{2}p^2ln(p)$$

XL. Therefore we are now able to simplify the nature of the curve near the beginning A by a definite equation, if we may call AR=x and RQ=y, because p=y and q=1+x, this will arise that $x = -\tfrac{1}{2}y^2ln(y)$, to which the general equation for the curve is recalled, if the coordinates x and y are as small as possible. Therefore it extends that not even the smallest little arc around A, just as a portion of the algebraic curve, is able to be seen, but that its nature implicates logarithms. And because the logarithmic equation is able to be transformed into an exponential form, the beginning of our curve A will be common/joint/related with a transcendent line, whose equation is $e^{-2x} = y^{(y^2)}$, with e accepted for the number whose hyperbolic logarithm (natural logarithm) is 1.

XLI. By this equation $x = -\frac{1}{2}y^2 ln(y)$, these are also confirmed, which we noted already above from the affections of this curve at point A. First it is known if y=0, that likewise it would be $y^2 ln(y)$ and so then x=0, although by this chance $ln(y) = -\infty$. Then, since $dy = -ln(y)^2 - \frac{*}{*}yln(y)$, because y is incomparably less than $yln(y)$, it will be $dx = -ydyln(y)$, and for this reason $\frac{dy}{dx} = \frac{-1}{yln(y)} = \infty$ if y=0; whence it is understood that the tangent to the curve at A is perpendicular to the segment AR. Further, since the subnormal is $\frac{ydy}{dx} = -\frac{1}{ln(y)}$, and by this chance the subnormal is equal to the radius of curvature, because $ln(y) = \infty$ if y=0, it is manifested that the radius of curvature of this curve at A is infinitely small.

XLII. This curve most greatly differs from algebraic curves, which at the beginning A likewise have a vanishing radius of curvature. In fact the nature of algebraic around the beginning A, which rejoice in this innate quality, is expressed by a formula of of this type:

$$x = \alpha y^m$$

Proving to be $m < 2$ and $m > 1$. Therefore, $m = 2 - \sigma$, with σ proving to be a fraction less than 1, so that $x = \alpha y^{2-\sigma}$, it will be dx= $\alpha(2-\sigma)y^{1-\sigma}$ dy, and therefore $\frac{dy}{dx} = \frac{1}{\alpha(1-\sigma)y^{1-\sigma}} = \infty$, because $y^{1-\sigma} = 0$: but the radius of curvature, which is equal to the subnormal $\frac{ydy}{dx}$, will be $= \frac{y^\sigma}{\alpha(2-\sigma)} = 0$. Considering our curve, the derived radius of curvature is $\frac{-1}{ln(y)}$, whence the radius of curvature vanishing in whatever algebraic curve at point A as $-y^\sigma ln(y)$ is to $\alpha(2-\sigma)$ as 0 is to 1. Therefore on our curve indeed the radius of curvature at A is infinitely small, but yet the infinities are greater than the radius of curvature in every algebraic curvature.

XLIII. With the beginning of the series known, which the value of PQ=q is expressed by the segment CP=p, certainly:

$$q = 1 - \frac{1}{2}p^2 ln(p) + Ap^2$$

From here it will not be difficult to deduce the form of the entire series. Since from the equation of differentials it is understood that the powers of the following terms of p come forth from two, the value of q generally is expressed by twin infinite series and will be:

$$\begin{aligned} q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + etc. \\ -\alpha p^2 ln(p) - \beta p^4 ln(p) - \gamma p^6 ln(p) - \delta p^8 ln(p) - etc \end{aligned}$$

In which we know that $\alpha = \frac{1}{2}$.

XLIV. Therefore, since the true value of q is contained by the double/duplicate series, in order to elicit each *seorsim*/ series, we place:

$$q = r - sln(p)$$

And through differentiation it will be:

$$dq = dr - \frac{sdp}{p} - dsln(p)$$

$$d^2q = d^2r - \frac{2dpds}{p} + \frac{sdp^2}{p^2} - d^2sln(p)$$

These values in our equation of differentials are substituted:

$$p(1-p^2)d^2q - dpdq(1+p^2) + pqdp = 0$$

And the terms affected by ln(p) are equal to nothing *seorsim*, by this means two equations are obtained:

I. $p(1-p^2)d^2s - (1+p^2)dpds + psdp^2 = 0$

II. $p(1-p^2)d^2r - (1+p^2)dpdr + prdp^2 - 2(1-p^2)dpds + \frac{2sdp}{p} = 0$

XLV. To resolve these equations, it is placed:

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + etc.$$

$$s = \alpha p^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + etc.$$

And with the differentials summed, it will be:

$$\frac{dr}{dp} = 2Ap + 4Bp^3 + 6Cp^3 + 8Dp^7 + etc.$$

$$\frac{d^2r}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + etc.$$

$$\frac{ds}{dp} = 2\alpha p + 4\beta p^3 + 6\gamma p^5 + 8\delta p^7 + etc.$$

$$\frac{d^2s}{dp^2} = 2\alpha + 12\beta p^2 + 30\gamma p^4 + 56\delta p^6 + etc.$$

After the values are substituted, the first equation will be changed into this:

$$\begin{array}{llllll} 2\alpha p & +\,12\beta p^3 & +\,30\gamma p^5 & +\,56\delta p^7 & +\,90\varepsilon p^9 & +\,etc. \\ & -\,2\alpha & -\,12\beta & -\,30\gamma & -\,56\delta & -\,etc. \\ -\,2\alpha p & -\,4\beta & -\,6\gamma & -\,8\delta & -\,10\varepsilon & -\,etc. \\ & -\,2\alpha & -\,4\beta & -\,6\gamma & -\,8\delta & -\,etc. \\ & +\,\alpha & +\,\beta & +\,\gamma & +\,\delta & +\,etc. \end{array} \quad =0$$

XLVI. After the series, handed over earlier in the other equation, are substituted, it will come forth:

$$\begin{array}{llllll} 2Ap & +\,12Bp^3 & +\,30Cp^5 & +\,56Dp^7 & +\,90Ep^9 & +\,etc. \\ & -\,2A & -\,12B & -\,30C & -\,56D & -\,etc. \\ -\,2A & -\,4B & -\,6C & -\,8D & -\,10E & -\,etc. \end{array}$$

$$\begin{array}{llllllllllll} & -\,2A & - & 4B & - & 6C & - & 8D & - & etc. & & \\ +\,1 & +\;A & + & B & + & C & + & D & + & etc. & & =0 \\ -\,4\alpha & -\;8\beta & - & 12\gamma & - & 16\delta & - & 20\varepsilon & - & etc. & & \\ & +\,4\alpha & + & 8\beta & + & 12\gamma & + & 16\delta & - & etc. & & \\ +\,2\alpha & +\;2\beta & + & 2\gamma & + & 2\delta & + & 2\varepsilon & + & etc. & & \end{array}$$

Whence by the same means it is elicited:

$$2A - 2A + 1 - 2\alpha = 0\,;\text{ from here } \alpha = \tfrac{1}{2}$$
$$8B - 3A \cdot 6\beta + 4\alpha = 0\,; 2 \cdot 4B - 1 \cdot 3A + 2(2 - \tfrac{1\cdot3\cdot3}{2\cdot4})\alpha = 0$$
$$24C - 15B \cdot 10\gamma + 8\beta = 0\,; 4 \cdot 6C - 3 \cdot 5B + 2(4 - \tfrac{3\cdot5\cdot5}{4\cdot6})\alpha = 0$$
$$48D - 35C \cdot 14\delta + 12\gamma = 0\,; 6 \cdot 8D - 5 \cdot 7C + 2(6 - \tfrac{5\cdot7\cdot7}{6\cdot8})\alpha = 0$$
$$80E - 63D \cdot 18\varepsilon + 16\delta = 0\,; 8 \cdot 10E - 7 \cdot 9D + 2(8 - \tfrac{7\cdot9\cdot9}{8\cdot10})\alpha = 0$$

XLVIII. Therefore I know the series “s” by the value of $\alpha = \frac{1}{2}$, which involves the logarithm of p, the entirety becomes known, it will be:

$$\alpha = \tfrac{1}{2}$$
$$\beta = \tfrac{1\cdot1\cdot3}{2\cdot2\cdot4}$$
$$\gamma = \tfrac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6}$$
$$\delta = \tfrac{1\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8}$$
$$\varepsilon = \tfrac{1\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10}$$

And from here it is made:

$$s = \alpha p^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + etc.$$

XLIX. Moreover because it holds on to the other series:

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + etc.,$$

the first coefficient A remains indeterminate, of which thing the ratio is, because we elicited these series from the differential equation of the second position, which requires duplicate/dual determination in order to be suited to our case. Because it is necessary to define the value of the coefficient A from the nature of the curve, with it (the value) derived, the remaining become known from these formulae, they return to those above.

$$B = \tfrac{1\cdot3}{2\cdot4}A - \tfrac{1}{8}\alpha(\tfrac{3}{2\cdot2} + \tfrac{1}{1\cdot1})$$
$$C = \tfrac{3\cdot5}{2\cdot4}B - \tfrac{1}{8}\beta(\tfrac{3}{3\cdot3} + \tfrac{1}{2\cdot2})$$
$$D = \tfrac{5\cdot7}{6\cdot8}C - \tfrac{1}{8}\gamma(\tfrac{3}{4\cdot4} + \tfrac{1}{4\cdot4})$$
$$E = \tfrac{7\cdot9}{8\cdot10}D - \tfrac{1}{8}\delta(\tfrac{3}{5\cdot5} + \tfrac{1}{4\cdot4})$$

L. After all the coefficients were derived to give whatever segment CP=p, the value of the responding segment/arc is thus defined, so that

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + etc.$$
$$- \alpha p^2 lnp - \beta p^4 lnp - \gamma p^6 lnp - \delta p^8 lnp - etc.$$

Which series, if segment p will have been much less than 1, it converges sufficiently (promte), so that then the value of q is able to be known. From here, the segments (applicata), which respond to the segments (scissis) much less than 1, are able to be defined, because the applicata $\frac{p}{q}$ responds to the abscissa $\frac{1}{p}$. Because if the segment is much greater than 1, it is placed equal to P and the segment responding to it equal to Q, because $p = \frac{1}{P}$ and $q = pQ$, it will be:

$$Q = P + \frac{A}{P} + \frac{B}{P^3} + \frac{C}{P^5} + \frac{D}{P^7} + etc.$$
$$+ \frac{\alpha lnP}{P} + \frac{\beta lnP}{P^3} + \frac{\gamma lnP}{P^5} + \frac{\delta lnP}{P^7} + etc.$$

From here, if segment P is infinite, it will be:

$$Q = P + \frac{\alpha lnP}{P} \text{ or } Q - P = \frac{\alpha lnP}{P}$$

Whence the nature of the *branch/bough* (curve?) Pq, extended to infinity and approaching the asymptote CV, is deduced.

LI. Because we knew if p=1, then q= $\frac{\pi}{2}$, considering this chance, the derived equation puts on this form because ln(1)=0:

$$\frac{\pi}{2} = 1 + A + B + C + D + E + etc.$$

Therefore, since the value of A is not yet defined, the remaining B, C, D, etc. depend on A, this equation contains a condition, what value of A is determined. Thus of course it is necessary that the value of A be provided, so that the sum of the infinite series $1 + A + B + C + D + E + etc. = \frac{\pi}{2}$. If the values of the remaining letters B, C, D, etc., which depend on A, are evolved, to such an extent the expressions resound/rebound to those involved, so that from here the value of A is able to be elicited by *no* means.

LII. To determine the constant A, another way is known, the perimeter of the given ellipse will have been derived from another formula in numbers. Since which requires methods to evolve all coefficients in decimal fractions, with the computation finished, it is discovered:

$\alpha = 0.5000000000;$ *A is sought*
$\beta = 0.1875000000;$ $B = 0.3750000000$ $A - 0.1093750000$
$\gamma = 0.1171875000;$ $C = 0.2343750000$ $A - 0.0820312500$
$\delta = 0.0854492188;$ $D = 0.1708984375$ $A - 0.0641886393$
$\varepsilon = 0.0672912598;$ $E = 0.1345825195$ $A - 0.0524978638$
$\zeta = 0.0555152893;$ $F = 0.1110305786$ $A - 0.0443481445$
$\eta = 0.0472540855;$ $G = 0.0945081711$ $A - 0.0383663416$
$\theta = 0.0411363691;$ $H = 0.0822727382$ $A - 0.0337966961$
$\iota = 0.0364228268;$ $I = 0.0728456536$ $A - 0.0301949487$
$\kappa = 0.0326793696;$ $K = 0.0653587392$ $A - 0.0272843726$

With these values derived, if the segment is CP=p, the value of segment q is thus defined:

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} + Gp^{14} + Hp^{16} + Ip^{18} + Kp^{20} + etc.$$
$$- p^2 lnp(\alpha + \beta p^2 + \gamma p^4 + \delta p^6 + \varepsilon p^8 + \zeta p^{10} + \eta p^{12} + \theta p^{14} + \iota p^{16} + \kappa p^{18} + etc.)$$

LIII. Then we thus derive the value of the segment q expressed above, so that:

$$q = \tfrac{\pi}{2}(1 - \tfrac{1\cdot 1}{2\cdot 2}(1 - p^2) - \tfrac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}(1 - p^2)^2 - \tfrac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}(1 - p^2)^3 - etc.)$$

Therefore, now from either formula we elicit the value of q considering any value of p, with the result that hereafter we seek to elicit the value of the coefficient A from the equality of these two. It is appropriate that a not exceedingly small fraction be substituted, in order that the following expression not converge very lightly; we assume one so small that coefficients for the above form of the computed value of q are sufficient by derivation to 10 figures.

LIV. Therefore, we place p= $\frac{1}{5}$ for the sake of calculating; in the natural log it will be:

$$- lnp = 1.60943791243$$

Now it is:

$$\alpha p^{2} = 0.02000000000$$
$$\beta p^{4} = 0.00030000000$$
$$\gamma p^{6} = 0.00000750000$$
$$\delta p^{8} = 0.00000021875$$
$$\varepsilon p^{10} = 0.00000000689$$
$$\zeta p^{12} = 0.00000000023$$
$$\eta p^{14} = 0.00000000001$$

0.02030772588 The coefficient of $-lnp$

1.60943791243

0.03268402394 The product

Then it is:

$$Ap^{2} = 0.04000000000\ A$$
$$Bp^{4} = 0.00060000000\ A - 0.00017500000$$
$$Cp^{6} = 0.00001500000\ A - 0.00000525000$$
$$Dp^{8} = 0.00000043750\ A - 0.00000016432$$
$$Ep^{10} = 0.00000001378\ A - 0.00000000538$$
$$Fp^{12} = 0.00000000045\ A - 0.00000000018$$
$$Gp^{14} = 0.00000000002\ A - 0.00000000001$$

$$0.04061545175\ A - 0.00018041989$$

From these it is completed:

$$q = 0.04061545175A + 1.03250360409$$

LV. Now we seek the same value of q from the other equation, and since p= $\frac{1}{5}$, it will be $1-p^{2}=\frac{24}{25}$; then $n^{2}=\frac{24}{25}$, it will be:

$$q = \tfrac{\pi}{2}(1 - \tfrac{1\cdot 1}{2\cdot 2}n^{2} - \tfrac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^{4} - \tfrac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^{6} - etc.)$$

To abbreviate, it is placed:

$$q = \frac{\pi}{2} - (scriptA)n^2 - \mathfrak{B}n^4 - (scriptC)n^6 - (scriptD)n^8 - \mathfrak{E}n^{10} - etc.$$

Truly by this chance because $n^2 = \frac{24}{25}$, that series converges lightly very much, as in that from here the value of q may be able to be elicited sufficiently accurately; therefore, in order that we may obtain equal convergence on both sides, we place $p = \frac{1}{\sqrt{2}}$, so that all the same $p^2 = \frac{1}{2}$ as $n^2 = \frac{1}{2}$. Let us procure so great a calculation to 6 figures and it will be:

$Ap^2 = 0.500000\ A$
$Bp^4 = 0.093750\ A - 0.027344$
$Cp^6 = 0.029297\ A - 0.010254$
$Dp^8 = 0.010681\ A - 0.004012$
$Ep^{10} = 0.004206\ A - 0.001640$
$Fp^{12} = 0.001735\ A - 0.000693$
$Gp^{14} = 0.000738\ A - 0.000300$
$Hp^{16} = 0.000321\ A - 0.000132$
$Ip^{18} = 0.000142\ A - 0.000059$
$Kp^{20} = 0.000064\ A - 0.000026$
Sum of the remaining $60\ A - \quad 24$

Sum of all $0.640994\ A - 0.044484 + 0.320497 ln(\frac{1}{p}) + 1$

Therefore: $q = 1.066592 + 0.640994A$

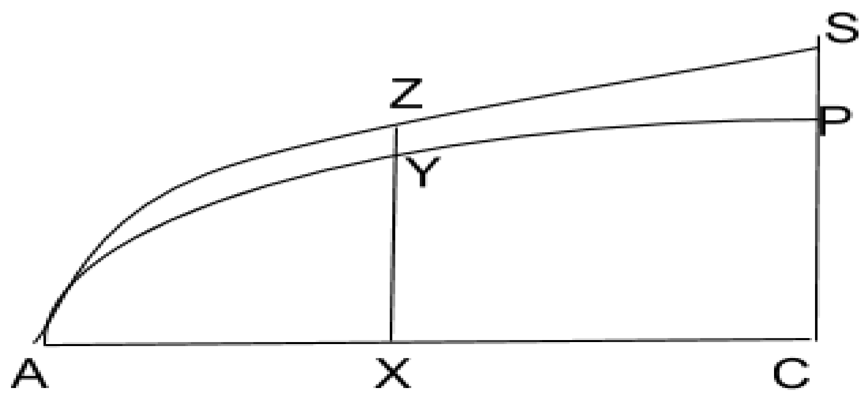
But the other expression gives $q = 1.350647$, whence it is:

$$A = \frac{284055}{640994} = 0.443147$$

LVI. Although this value does not extend beyond 6 figures, nevertheless it does not seem that it must be granted to chance, because that number, 0.443147, derived from the (natural) logarithm of 2, 0.6931478, may falter from precisely one-fourth, 0.25. Which conjecture, if it is agreeable to the truth, might be permitted to exhibit the value of letter A to 6 figures.

Since $ln(2) = 0.693147180559945309417232$1
And likewise since $A = ln(2) - \frac{1}{4}$
Then $A = 0.4431471805599453094172321$

Because the value of the coefficient A is in fact $= ln(2) - \frac{1}{4}$, I demonstrate the following means and confirm this conjecture.



LVII. Of course I compare the elliptic arc AYP, whose semiaxes AC=1 and CP=p, with the parabolic arc AZS above the drawn axis AC, which has at A common curvature with the ellipse. With a common segment assumed AX=x and the segment of the ellipse XY=y and of the parabola XZ=z, it will be:

$$y = p\sqrt{2x - x^2} \quad \text{and} \quad z = p\sqrt{2x}$$
$$dy = \frac{pdx(1-x)}{\sqrt{2x-x^2}} \quad \text{and} \quad dz = \frac{pdx}{\sqrt{2x}}$$

Whence the elliptic arc is:

$$AY = \int dx\sqrt{1 + \frac{p^2(1-x)^2}{2x-x^2}}$$

The parabolic arc:

$$AZ = \int dx\sqrt{1 + \frac{p^2}{2x}}$$

It stands that:

$$AZ = x\sqrt{1 + \frac{p^2}{2x}} + \frac{1}{4}p^2 ln(\frac{\sqrt{1+\frac{p^2}{2x}}+1}{\sqrt{1+\frac{p^2}{2x}}-1})$$

From here, if x=1, the parabolic ar will be:

$$AZS = \sqrt{1 + \tfrac{1}{2}p^2} + \tfrac{1}{4}p^2 ln(\frac{\sqrt{1+\frac{1}{2}p^2}+1}{\sqrt{1+\frac{1}{2}p^2}-1})$$

And in the integral formulae in will be:

$$\sqrt{1 + \frac{p^2(1-x)^2}{2x-x^2}} = \sqrt{1 + \frac{p^2}{2x} - \frac{p^2(3-2x)}{4-2x}}$$

Because it is not necessary to extend the comparison to higher powers of p in order to seek anything, we defined the the coefficients of the higher powers of p from the smaller powers, after the terms were rejected, which contain p^4 and higher powers, it will be:

$$\sqrt{1+\frac{p^2(1-x)^2}{2x-x^2}} = \sqrt{1+\frac{p^2}{2x}} - \frac{p^2(3-2x)}{*4*(2-x)}$$

And likewise:

$$AY = \int dx\sqrt{1+\frac{p^2}{2x}} - \frac{1}{4}p^2\int\frac{dx(3-2x)}{2-x}$$

After the integrals are taken:

$$AY = x\sqrt{1+\frac{p^2}{2x}} + \frac{1}{4}p^2 ln\left(\frac{\sqrt{1+\frac{p^2}{2x}}+1}{\sqrt{1+\frac{p^2}{2x}}-1}\right) - \frac{1}{2}p^2 x - \frac{1}{4}p^2 ln\left(\frac{2-x}{2}\right)$$

Now it is placed x=1, so that arc AYP produces q, it will be:

$$q = x\sqrt{1+\frac{1}{2}p^2} + \frac{1}{4}p^2 ln\left(\sqrt{1+\frac{1}{2}p^2}+1\right) - \frac{1}{4}p^2 ln\left(\sqrt{1+\frac{1}{2}p^2}-1\right) - \frac{1}{2}p^2 + \frac{1}{4}p^2 ln(2)$$

LVIII. Now because we did not provide for the higher powers of p, it will be:

$$\sqrt{1+\frac{1}{2}p^2} = 1+\frac{1}{4}p^2$$

Whence it will be:

$$q = 1+\frac{1}{4}p^2 + \frac{1}{4}p^2 ln\left(2+\frac{1}{4}p^2\right) - \frac{1}{4}p^2 ln\left(\frac{1}{4}p^2\right) - \frac{1}{2}p^2 + \frac{1}{4}p^2 ln(2)$$

Where for $ln(2+\frac{1}{4}p^2) = ln(2)+\frac{1}{8}p^2$ it is permitted to write ln(2), so that it is:

$$q = 1-\frac{1}{4}p^2 + \frac{1}{2}p^2 ln(2) - \frac{1}{2}p^2 ln(p) + \frac{1}{2}p^2 ln(2)$$

Or

$$q = 1-\frac{1}{2}p^2 ln(p) + p^2\left(ln(2)-\frac{1}{4}\right)$$

Whence it is perceived that the coefficient of p^2 , which we indicated before with the letter A, is $= ln(2)-\frac{1}{4}$, in order that we might have been entirely pursued before the conjecture has been completed.

LIX. Therefore the beginning for the proposed curve AQDq (Fig.1), if the segment CP=p and PQ=q, it will be:

$$q = 1 + Ap^{2} + Bp^{4} + Cp^{6} + Dp^{8} + Ep^{10} + etc.$$
$$- (\alpha p^{2} + \beta p^{4} + \gamma p^{6} + \delta p^{8} + \varepsilon p^{10} + etc.)ln(p)$$

Where the coefficients are thus defined:

$A = ln(2) - \frac{1}{4}$ $\qquad \alpha = \frac{1}{2}$

$B = \frac{1\cdot3}{2\cdot4}\ A - \frac{1}{2}(\alpha - \beta) + \frac{1}{2}\cdot\frac{\beta}{2};$ $\qquad \beta = \frac{1\cdot3}{2\cdot4}\alpha$

$C = \frac{3\cdot5}{4\cdot6}\ B - \frac{1}{3}(\beta - \gamma) + \frac{1}{4}\cdot\frac{\gamma}{3};$ $\qquad \gamma = \frac{3\cdot5}{4\cdot6}\beta$

$D = \frac{5\cdot7}{6\cdot8}\ C - \frac{1}{4}(\gamma - \delta) + \frac{1}{6}\cdot\frac{\delta}{4};$ $\qquad \delta = \frac{5\cdot7}{6\cdot8}\gamma$

$E = \frac{7\cdot9}{8\cdot10}\ D - \frac{1}{5}(\delta - \varepsilon) + \frac{1}{8}\cdot\frac{\varepsilon}{5};$ $\qquad \varepsilon = \frac{7\cdot9}{8\cdot10}\delta$

$F = \frac{9\cdot11}{10\cdot12}\ E - \frac{1}{6}(\varepsilon - \zeta) + \frac{1}{10}\cdot\frac{\zeta}{6};$ $\qquad \zeta = \frac{9\cdot11}{10\cdot12}\varepsilon$

This series strongly converges, if segment p will have a very small fraction, on the contrary if it is much greater than 1, with the same coefficients remaining it will be:

$$q = p + \frac{A}{p} + \frac{B}{p^{3}} + \frac{C}{p^{5}} + \frac{D}{p^{7}} + \frac{E}{p^{9}} + etc.$$
$$+ (\frac{\alpha}{p} + \frac{\beta}{p^{3}} + \frac{\gamma}{p^{5}} + \frac{\delta}{p^{7}} + \frac{\varepsilon}{p^{9}} + etc.)ln(p)$$

LX. Truly if segment p does not differ much from 1, so that it agrees with the series derived above (Section 26), then:

$$q = 1 + p^{2}[(\frac{\pi}{2} - 1) + (\frac{1\cdot3}{2\cdot2}\cdot\frac{\pi}{2} - 1)(1 - p^{2}) + (\frac{1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4}\cdot\frac{\pi}{2} - 1)(1 - p^{2})^{2} + (\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6}\cdot\frac{\pi}{2} - 1)(1 - p^{2})^{3} + etc]$$

Which is converted into this form by the nature of the ellipse:

$$q = p + \frac{1}{p}[(\frac{\pi}{2} - 1) - (\frac{1\cdot3}{2\cdot2}\cdot\frac{\pi}{2} - 1)\frac{(1-p^{2})}{p^{2}} + (\frac{1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4}\cdot\frac{\pi}{2} - 1)\frac{(1-p^{2})^{2}}{p^{4}} - (\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6}\cdot\frac{\pi}{2} - 1)\frac{(1-p^{2})^{3}}{p^{6}} + etc]$$

Whence, exactly as it will have been, whether $p > 1$ or $p < 1$, it is permitted to choose this thing, whether whose terms proceed from the same signs or alternating signs. Generally, it is better to choose alternating signs to most closely define the sum.

Problem

LXI. With the given axes of the ellipse conjoined, to most closely show its perimeter in numbers.

Solution

The semiaxes of the ellipse are 1 and p, and one-fourth of the perimeter is q, and through the derived formulae the value of q is able to be defined in numbers, provided that this is chosen ,
whose terms converge most greatly. We obtained 4 formulae, which are:

I. $$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} + etc \\ - (\alpha p^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \zeta p^{12} + etc.)ln(p)$$

II. $$q = p + A\frac{1}{p} + B\frac{1}{p^3} + C\frac{1}{p^5} + D\frac{1}{p^7} + E\frac{1}{p^9} + F\frac{1}{p^{11}} + etc.$$

III. $$q = 1 + p^2[script\ A + \mathfrak{B}(1-p^2) + \mathbb{C}(1-p^2)^2 + script\ D(1-p^2)^3 + \mathfrak{E}(1-p^2)^4 + etc.]$$

IV. $$q = p + \frac{1}{p}(script\ A - \mathfrak{B}\frac{(1-p^2)}{p^2} + \mathbb{C}\frac{(1-p^2)^2}{p^4} - script\ D\frac{(1-p^2)^3}{p^6} + \mathfrak{E}\frac{(1-p^2)^4}{p^8} - etc.]$$

The values of these threefold coefficients in numbers are:

$A = 0.44314718056$	$\alpha = 0.50000000000$	$\mathfrak{A} = 0.57079632679$
$B = 0.05680519271$	$\beta = 0.18750000000$	$\mathfrak{B} = 0.17809724510$
$C = 0.02183137044$	$\gamma = 0.11718750000$	$script\ C = 0.10446616728$
$D = 0.01154452144$	$\delta = 0.08544921875$	$script\ D = 0.07378655152$
$E = 0.00714200029$	$\varepsilon = 0.06729125977$	$\mathfrak{E} = 0.05700863665$
$F = 0.00485474337$	$\zeta = 0.05551528931$	$\mathfrak{F} = 0.04643855029$
$G = 0.00351468795$	$\eta = 0.04725408554$	$\mathfrak{G} = 0.03917161591$
$H = 0.00266223578$	$\theta = 0.04113636911$	$\mathfrak{H} = 0.03386971991$
$I = 0.00208639732$	$\iota = 0.03642282682$	$\mathfrak{I} = 0.02983116632$
$K = 0.00167916842$	$\kappa = 0.03267936962$	$script\ K = 0.02665267507$
		$\mathfrak{L} = 0.02408604338$

From here, the converging series will be held for the type of whatever ellipse, whence its perimeter will be defined; just as if it is placed:

$$if\ p = \frac{1}{10},\ then\ q = 1.015993545021$$
$$if\ p = \frac{1}{5},\ then\ q = 1.050502227000$$
$$if\ p = \frac{1}{\sqrt{2}},\ then\ q = 1.3506429$$