Translation with notes of Euler's paper

Sur quelques proprietes des Sections coniques qui conviennent a un infinite d'autres lignes courbes

ON SOME PROPERTIES OF CONIC SECTIONS THAT ARE SHARED WITH INFINITELY MANY OTHER CURVED LINES (E083)

L. EULER

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Translated by
Edward Greve and Thomas J Osler
Mathematics Department
Rowan University
Glassboro, NJ 08028

Osler@rowan.edu

Introduction to the translation and notes:

This translation is the result of a fortunate collaboration between student and professor. Edward Greve was an undergraduate mathematics major when he made this translation. Tom Osler has been a mathematics professor for 46 years. Together we struggled to understand this brilliant work.

When translating Euler's words, we tried to imagine how he would have written had he been fluent in modern English and familiar with today's mathematical jargon. Often he used very long sentences, and we frequently converted these to several shorter ones. However, in almost all cases we kept his original notation, even though some is very dated. We thought this added to the charm of the paper. Also, in the original paper, almost none of the equations are "displayed", but appear written linearly in the sentences. We chose to display them here so that they are easier to read.

Euler was very careful in proof reading his work, and we found few typos. When we found an error, we called attention to it (***in parenthesis using bold type and underlining***) in the body of the translation. Other errors are probably ours. We also made a few comments of our own within the translation identifying our words in the same way.

The notes that follow this translation are a collection of material that we accumulated while trying to understand and appreciate Euler's ideas. In these notes we completed some steps that Euler omitted, introduced some modern notation and elaborated on a few of Euler's proofs that we found too brief. We also added a few simple ideas and comments of our own not found in the text.

It is not surprising that the paper has moments where the reader gets a "WOW", how did Euler think of that? In this regard, look at section 10 where he introduces a differential equation of infinite order, section 11 where he finds an unexpected geometric proof of a remarkable result, and section 13 where he applies the ODE of infinite order.

Section 1

Conic sections have several properties that are shared only among themselves; but they also have several properties in common with infinitely many other curves. For instance, the axis that cuts the plane in two, with the origin placed at the vertex of the curve, is shared with infinitely many curves, both algebraic and transcendental. This is obvious to anyone who looks at the nature of curves. But geometry has shown other properties, which at first glance seem to be unique to conic sections, that are also shared with other curves. It is evident that the properties, by which conic sections are defined, are really their own, and that they cannot be shared with any other curve. However beyond these we encounter other properties, some of which are not easily seen to be unique, or not, to conic sections. To remove this confusion, we must analytically find all curves that may have a certain property, and if we find that conic sections are the only curves satisfying that property, we will be certain that the property is unique to conic sections. Geometry has already given solutions to several questions of an applied nature; solutions that have considerably extended the art of mathematical analysis. We propose, thus, to add here some other similar questions, taken from the idea of oblique-angled diameters, that are shared principally with conic sections.

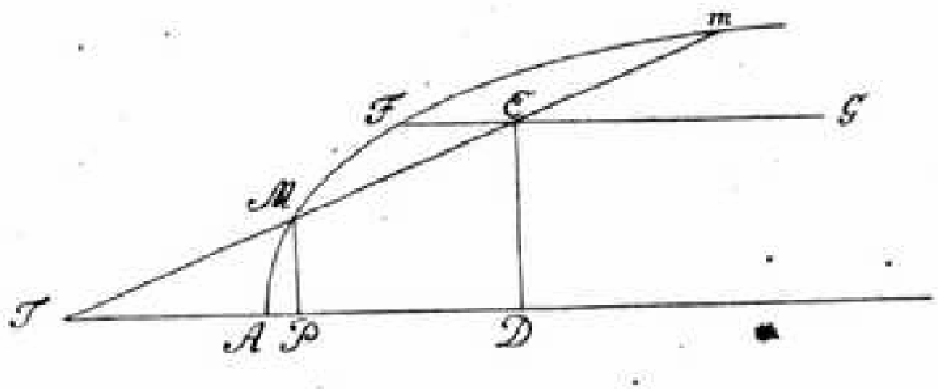


Fig. 1.

Section 2

We consider the following property of parabolas: any straight line parallel to the major axis is an oblique-angled diameter. These bisect all parallel lines drawn with a special angle inside of the parabola. Indeed, if *AMFm* is a parabola with major axis *AD*, and if we draw *FG* parallel to the major axis with *F* on the parabola, we know that *FG* bisects at *E* any chord *Mm* parallel to the tangent of the curve at *F*. We can define the orientation of these chords as "parallel to the tangent at *F* ", but this condition is true for all bisections. For, if in any curve whatsoever, the line FG meets all chords *Mm* making the given angle *GEm*, then the tangent to the curve at *F* is always parallel to these chords. Consider that the lines very near the point *T* **(error, should be F)** are parallel to the tangent. To see that this property is unique to the parabola, we turn to the following problem.

Section 3

To describe a curve AMFm over the axis AD, which has the diameter FEG parallel to the axis AD at the distance given DE from the axis, and which bisects all chords Mm which make ***(a fixed)*** *angle at T with the axis.*

Call the distance of the diameter to the axis *DE* = *a* and call the sine of the angle *MTA* *m*, while the cosine is called $n = \sqrt{1 - mm}$. If from any point *T* of the extended axis *AD*, we draw with the given angle the straight line *TMm*, it will cut the curve that we are looking for in two points *M* and *m*. This is why if we let *AT* = *t* and *TM* = *z*, the relation between *t* and *z* is given by an equation, with the property that for each value of *t* there are two values for *z*. Thus this equation will be a quadratic of the form $zz = 2Pz - Q$, with *P* and *Q* any functions of *t*. Given any value of *t* we can determine two values of *z* which identify the double intersection of the straight line *Tm* and the curve.

Section 4

The line segments *TM* and *Tm* represent the two roots of z from the equation

$$zz - 2Pz + Q = 0.$$

We have

$$TM + Tm = 2P$$

and consequently

$$\frac{TM + Tm}{2} = P.$$

Because *E* is the midpoint of the line segment *Mm*, we have

$$\frac{TM + Tm}{2} = TE \text{ and thus } P = TE.$$

But because $DE = a$ and $\sin(DTE) = m$, it follows that $\frac{a}{TE} = m$ or $TE = P = \frac{a}{m}$. Thus the equation between *z* and *t* becomes $zz = \frac{2az}{m} - Q$, where *Q* is any function of *t*.

Suppose the equation for our curve *AMm* uses coordinates with the abscissa *AP* = *x* and corresponding ordinate $PM = y$ Then $y : z$ will be = m and $\frac{t+x}{z} = n$, from which we get

$z = \frac{y}{m}$ and $t = nz - x = \frac{ny}{m} - x$. The curve *AMm* will have the property previously

required, if $\frac{yy - 2ay}{mm}$ is equal to any function whatsoever of $\frac{ny}{m} - x$.

Section 5

We set $ny - mx = X$ and $yy - 2ay = Y$ and then form the general equation between *X* and *Y*, that is to say

$$0 = \alpha + \beta X + \gamma Y + \delta X^2 + \varepsilon XY + \zeta Y^2 + \eta X^3 + \theta X^2 Y + \&c..$$

In this general equation we find all possible relations between *X* and *Y*
and thus we have *Y* = to any function of *X*, so that $yy - 2ay$ will be = to
any function of $ny - mx$ as required by our analysis. This is why to completely solve the proposed problem, we form any equation between the two variables *X* and *Y* and then we put $ny - mx$ in place of *X* and $yy - 2ay$ in place of Y. In this way we obtain an equation between *x* and *y* for the curve *AMm*. This equation has the property that the parallel *FG* to the axis *AD* which is the distance $DE = a$ from this axis, will be the oblique-angled diameter of the curve. This bisects all chords *Mm*, that make with it the angle *mEG*, whose sine is = *m*, the cosine = *n*.

Section 6

Thus there are an infinite number of curves, which have the property that was described previously in the problem. In other words, at a given distance from the axis *AD*, the parallel diameter to the axis bisects all curves parallel to the tangent at *F*. In addition to this property, in the parabola, all straight lines parallel to the axis are at the same time the diameter. In the curve we have found, only a single straight line parallel to the axis has this property. We now ask if there are curves other than the parabola, in which two or more straight lines parallel to the axis are diameters. To simplify our search, we ask if among the curves found, besides the parabola, there is any other, in which the axis *Ad* is at least the orthogonal diameter. For this purpose the following problem is proposed.

Section 7

Among all curves AMm, with the line AD as an axis of symmetry, in other words, it divides the curve into two similar and equal parts, to determine those that at a given distance on both sides of the axis AD have two oblique-angled diameters, like FG, that cut in two all chords Mm making the (*fixed*) *angle with the axis AD.*

Because the axis *AD* divides the curve in two similar and equal parts, it is evident, if the straight line *FG* parallel to the axis *AD* is the diameter, that then in the other part of the curve at the same distance from the axis there must be a diameter parallel to the axis. But for the axis *AD* to be a similar orthogonal diameter, it is necessary that in the equation between *x* and *y* the variable *y* has only even powers and never an odd powers. Thus we must exclude from the general equation found for the solution of the preceding problem all cases in which the exponents of y are odd. But as *X* is $= ny - mx$ and $Y = yy - 2ay$, y has in both variables the exponent one, and consequently it is odd. We can form a new variable *Z* from the two variables *X* and *Y*, in which there will not be an odd power of *y*, and this selection is

$$Z = Y + \frac{2aX}{n} = yy - \frac{2m\,a\,x}{n}.$$

We will also satisfy the preceding problem by the general equation between *Y* and *Z*, which is:

$$0 = \alpha + \beta Y + \gamma Z + \delta Y^2 + \varepsilon YZ + \varsigma Z^2 + \eta Y^3 + \theta Y^2 Z + etc.$$

Setting

$$Y = yy - 2ay \text{ and } Z = yy - \frac{2max}{n},$$

this contains all curves that are suitable.

Section 8

Hence it appears that in all terms that do not contain Y, we do not find odd powers of y and that consequently these terms, α, γZ, ςZ^2, χZ^3 etc. must be distinguished because they are special. But the term Y must be excluded, since it contains y^1, this power can't be subtracted by any of the following terms, and for the same reason we exclude the terms YZ, YZ^2, YZ^3 etc. What is more, if we allow the term Y^2, because in contains the power y^3, we will be obliged to use at the same time Y^3 so that we can subtract y^3. But Y^3 contains y^5 that we can not subtract without Y^4. So it follows that any power of Y containing an odd power of y, that is not in the preceding terms, must be removed by the following terms, from which emerges a progression ad infinitum. The same must be said of the terms $Y^2 Z$, $Y^3 Z^2$, etc. of which none are to be used, without allowing an infinitely many terms. We will thus satisfy the requirements only by the equation

$$0 = \alpha + \gamma Z + \varsigma Z^2 + \chi Z^3 + \text{etc}$$

that contains no Y. And for this equation Z will be = to a constant, that is to say

$$yy - \frac{2max}{n} = C,$$

This is a parabola, and all other **(algebraic)** curves are excluded.

Section 9.

Beyond the parabola of Apollonius, there are thus other curves ***(which are transcendental)*** with an axis of symmetry, that have at least one diameter parallel to the axis, so that this property does not belong only to the parabola. But by virtue of the equation

$$yy - \frac{ma\,x}{n} = 0$$

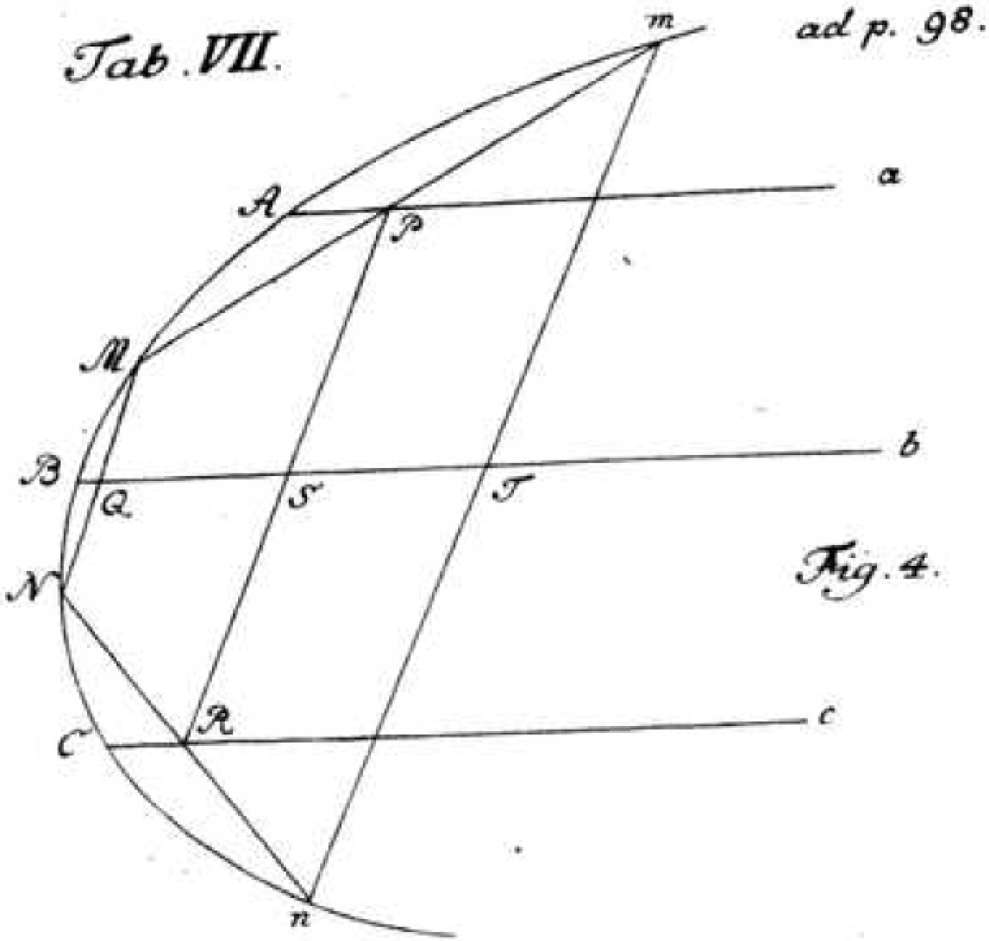
(we can make the constant C equal to zero) it seems that not only at the distance given by *a*, but at all distances from the axis, we find a diameter parallel to the axis. If we set $\frac{2ma}{n} = c$, for the equation $yy - cx = 0$, that is the equation for any parabola, if at any distance = *a*, we construct a line parallel to the axis, it will be the diameter and it will bisect all the chords, that make with the axis an angle, whose tangent is = $\frac{m}{n} = \frac{c}{2a}$. With the exception of the axis of symmetry, there need not be any oblique-angled diameters, and thus no straight lines parallel to the axis are at the same time diameters. But this analysis must be restricted only to algebraic curves, for the transcendents are not excluded by this progression of terms Y, Y^2, Y^3, etc. ad infinitum. We can produce many transcendent curves, that have many parallel diameters between them.

Section 10.

Our plan does not allow us to continue examining these transcendental curves, since in this memoir we have in mind algebraic curves. However, until we discover that similar transcendent curves that satisfy the present problem actually exist, we give a general equation, that incorporates in it all transcendent curves. Set $Y = yy - 2ay$, and we search for the value of the function T that satisfies this infinite differential equation,

$$0 = \frac{dT}{dY} + \frac{4a^2 Y d^3 T}{1.2.3.dY^3} + \frac{16a^4 Y^2 d^5 T}{1.2.3.4.5dY^5} + \frac{64a^6 Y^3 d^7 T}{1.2....7dY^7} + \&c\,.$$

Then T which is a function of Y is thereby a function of y, in which we find no odd powers of y. This is why, if we take W to be any function of T and $Z = y^2 - \frac{2ma\,x}{n}$, the equation $W = 0$ gives all curves that have the proposed property; that is to say, that beyond the axis AD which is an orthogonal diameter, we have on both sides at a given distance $= a$ from the axis the oblique-angled diameters parallel to the axis.



Section 11

In the following we are concerned with a general law, that all curves which have two parallel diameters, have an infinity of similar diameters, equally distant from one another. Given that the curve *mABC* has two parallel diameters *Aa*, *Bb*, of which *A*a cuts in two all chords *Mm* parallel to the tangent at *A*, and *Bb* cuts likewise *MN*, *mn* parallel to the tangent at *B*. The terminal points *M* and *m* of any chord *Mm* are divided in two by the diameter *Aa*, and if we take the chords *MN* and *mn* parallel to the tangent at *B*, we have *MQ* = *NQ* and *mT* = *nT*. Now we draw the chord *Nn*, and it will make a **(*well defined*)** angle with the diameters *Aa* or *Bb*, because **(*it is one*)** of all angles from the given quadrangle *MNnm*. If we take *PSR* parallel to *MN*, *mn*, then this new chord *Nn* is bisected at *R* and the point *R* always lies on the straight line *Cc* parallel to *Aa* and *Bb* and its distance from the diameter *Bb* is equal to the distance from the diameter *Bb* to the diameter *Aa*. Thus this line *Cc* bisects all chords *Nn* and is consequently a diameter.

Section 12

Hence the angle *NRc* is completely determined by the given angles *MQb* and *mPa*, since

$$\cot mPa + \cot NRc \text{ is } = 2\cot MQb$$

and consequently

$$\cot NRc = 2\cot MQb - \cot mPa \,.$$

The cotangents of the angles *mPa*, *MQb*, *NRc* constitute an arithmetic progression. But as we have demonstrated that the ***(existence of)*** two diameters *Aa*, *Bb* ***(implies the existence of)*** a third *C*c, likewise if we have two adjacent diameters then it follows that we have an infinity of distant diameters with equal intervals between them. If the cotangent of the angle *mPa*, by which the first diameter cuts in two the chords is = p and the cotangent of the angle *MQb*, for the second diameter *Bb* is = q , then the cotangent of the angle *NRc*, by which the third diameter cuts in two the chords, will be $=2q-p$ and the following cotangent of the angle, under which the fourth diameter cuts in two all chords, $=3q-2p$, the cotangent for the fifth diameter $=4q-3p$ and so it goes. Thus the cotangents of all the angles, by which the diameters that follow in order cut the chords in two, constitute an arithmetic progression. The transcendent curves, in which we find three diameters, and where the one in the middle is orthogonal, have at the same time an infinity of diameters.

Section 13

From this we can now demonstrate with full mathematical rigor, that the parabola is the only curve, in which all the lines without exception, that are parallel to the axis, are at the same time diameters. To attribute this property to a curve, it suffices that it has two diameters that approach infinitely; then, by the preceding demonstrations, it must be that all lines which are parallel to them are diameters. Hence in (§10) we set the distance separating two adjacent diameters = *a*, and this is why this distance *a* must be vanishing. That being done, the equation of (§10) becomes

$$0 = \frac{dT}{dY};$$

and consequently

T = to a constant.

If thus $W = 0$ gives the general equation for all curves, of which all straight lines parallel to the axis are diameters, W will be any function of T or of a constant quantity, and of

$$Z = yy - \frac{2ma\,x}{n}$$

Thus from this equation Z = to a constant, and so

$$yy - \frac{2ma\,x}{n} = C\,.$$

This equation contains in itself no other curve than the parabola.

Section 14

After having easily completed the discussion of parallel diameters, those which are generalizations of the parabola, we now examine the diameters that converge at a point, to understand more completely the nature of the ellipse and the hyperbola, curves in which all straight lines through their center are diameters. Reasoning in the same manner, we will determine, if this property is not found in any other curves. There is truly not any doubt, that it is an attribute of conic sections, that all straight lines without exception, that pass through the center, are at the same time diameters. However, there may exist other curves, that do not have an infinity of diameters which meet at the same point, but have two or three. To discover this, we propose to solve the following problem.

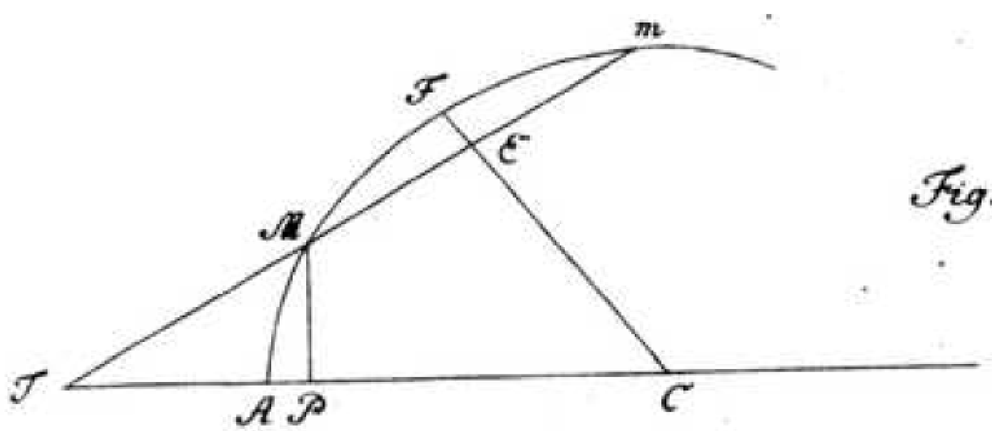


Fig. 2.

Section 15

Find all curves (Fig 3) **<u>(Should be Fig 2.)</u>** *AMm constructed above an axis AC with the following condition, that emerging from the point C the straight line CF, that makes the angle ACF with the axis, this line bisects at E all chords Mm parallel to the tangent at F.*

To begin it is apparent, that if all chords that the line *CF* cuts in two, are parallel to each other, the tangent to the point *F* must also be parallel to them. Thus the angle *ETC* is constant, so we put the sine of the angle *ETC* = *m*,

$$\text{the cosine } = n = \sqrt{1-mm}\,,$$

and also

the sine of the angle *ACF* is = *p*,

$$\text{the cosine } = q = \sqrt{1-pp}\,.$$

The sine of the angle *CEm*, at which the diameter *CF* cuts in two the chords *Mm*, will be = *mq*+*np* and the cosine = *nq*-*mp*. Because the point *T* is variable, we set *CT* = *t* and when moving the line *TMm* keeping the constant angle *CTE*, this line will cut the curve in two points *M* and *m*. Thus the variable which is = *z*, describing the intersections of the line *TM*, will have two values, one for *TM*, another for *Tm*. This is why *z*, a function of *t*, will be determined by a quadratic equation, which is

$$zz = 2Pz - Q.$$

Here *P* and *Q* are functions of *t* and therefore *TM* will be $= P - \sqrt{PP-Q}$ and $Tm = P + \sqrt{PP-Q}$.

Section 16

Thus *TM* + *Tm* will be = 2*P* and because *E* is the mid-point of the chord *Mm*, *TE* will be = *P*. But from the angles given in the triangle *CTE*, we will have

CT : *TE* = sin *A.CET* : sin *A. TCE*,

(Here Euler writes sin *A.CET* to mean sine of the angle *CET*.)

t : *P* = *mq* + *np* : *p*,

from which we get

$$P = \frac{pt}{mq + np}$$

and thus we have between *z* and *t* the equation:

$$zz = \frac{2ptz}{mq + np} - Q,$$

where Q is any function of *t*. Now to determine the curve, we set ($CP = x$)

$PM = y$, and we get

$$\frac{y}{z} = \frac{PM}{TM} = m,$$

and consequently

$$z = \frac{y}{m} \text{ and } PT = t - x = nz = \frac{ny}{m},$$

so that

$$t = \frac{mx + ny}{m}.$$ **(Euler does not write the "*t*".)**

From this the desired equation for the curve is

$$\frac{yy}{mm} = \frac{2p(mx + ny)y}{mm(mq + np)} - Q,$$

Q being any function of

$$t = \frac{mx - ny}{m}.$$ **(The minus sign should be plus.)**

This is why we will have

$$\frac{2pmxy + (np - mq)yy}{mm(mq + np)} \text{ or } yy + \frac{2mpxy}{np - mq}$$

as any function of $x - \frac{ny}{m}$. **(The minus sign should be plus.)**

Or if we set

$$x - \frac{ny}{m} = X$$ **<u>(The minus sign should be plus.)</u>**

$$\text{and} \quad yy + \frac{2mpxy}{np - mq} = Y\,,$$

and let *W* be any function of *X* and *Y* , then the equation *W* = 0 will express the family of all curves, that we desire.

Section 17

This analysis is incomplete, for it contains only the curves, that have but one oblique-angled diameter. Here the intersection *C* is fully arbitrary, and depends on the position of the axis *AC* which is also arbitrary. We continue our search by selecting from these many curves, those that the axis *AC* divides in two similar and equal parts. In other words, look for curves for which the axis *AC* is an orthogonal diameter **<u>(i.e. an axis of symmetry)</u>**. We thus require that in the equation found below, the powers of *y* are everywhere even exponents. This can occur if the odd powers of *y* are cancelled by later terms from the series. Thus in the general equation sought, we put

$$X = x + \frac{ny}{m} \quad \text{and} \quad Y = yy + \frac{2mpxy}{np - mq}\,,$$

and get

$$0 = \alpha + \beta X + \gamma Y + \delta X^2 + \varepsilon XY + \zeta Y^2 + \eta X^3 + etc.\,,$$

where the coefficients must be determined so that the odd powers of *y* vanish.

Section 18

At once we see that β must be = 0, because the term $\frac{ny}{m}$ cannot be cancelled by any of the following terms. On the other hand, we see that γ and δ can be determined such that the terms $\alpha + \gamma Y + \delta X^2$ have no odd powers of *y*: if we put

$$\gamma = np - mq \quad \text{and} \quad \delta = -\frac{mmp}{n},$$

we get

$$(np - mq)Y - \frac{mmp}{n} X^2 = -mqyy - \frac{mmpxx}{n}.$$

Therefore we let

$$Z = nqyy + mpxx = mpX^2 - \frac{n(np - mq)}{m} Y,$$

and observe that *Z* is a function, in which y has only even powers. This is why, if *W* is any arbitrary function of

$$Z = nqyy + mpxx \quad \text{and} \quad X = mx + ny,$$

then the equation *W* = 0 will contain solutions to the previous problem, and beyond that we will want to throw out the odd powers of *y*.

Section 19

Therefore we set

$$X = mx + ny \text{ and } Z = mpxx + nqyy,$$

in the equation for the curves, in which the straight line *CF* is a diameter, and get

$$0 = \alpha + \beta X + \gamma Z + \delta X^2 + \varepsilon XZ + \zeta Z^2 + \eta X^3 + etc.$$

If all the terms, in which *X* appears vanish, then every appearance of the variable *y* has even powers and the resulting curve is simultaneously divided by the axis *AC i*n two similar and equal parts. Then Z will be = C **(Euler means "is a constant")** or *aa* =

mpxx+nqyy. This equation contains conic sections with center *C* and with principal axis *AC*. Writing *bb* in place of $\frac{aa}{nq}$, we get

$$yy = bb - \frac{mp}{nq}xx \ .$$

At present we have the general equation for conic sections $yy = bb - kxx$ and it seems we should be able to determine more from this. Here lines *CF* passing through the center make an angle *FCA* with the axis, with tangent $= \frac{p}{q}$. This straight line bisects all chords *Mm*, that extended make the angle *MTC* with the axis *AC*, with tangent $\frac{m}{n} = k\frac{q}{p}$.

So the tangent of the angle *CEm*, under which the chords *Mm* are cut in two by the diameter *CF*, will be

$$= \frac{pp + kqq}{(1-k)pq} .$$

From this we see that the tangent $\frac{p}{q}$ is not a fixed number, but can be an arbitrary value, so that every straight line *CF* emerging from the center is a diameter. If *k = t* **(this should by *k* = 1)**, then all of these lines are orthogonal diameters and it is clear that the curve is a circle.

Section 20

We can find other curves, in which *AC* is the orthogonal diameter, if we determine the coefficients of the terms, in which *X* is found, such that no *y* term has an odd power. Initially we see that neither *X* nor X^2 can occur, because *y* and *xy* are not subtracted by any following terms. Otherwise, if *y* does not enter in by X, it must be that n is = 0 and so that the term xy is not in XX, mn should be = 0. But the terms X^3 and XZ generate homogeneous terms, from which the terms y^3 and xxy can be removed, if np is = $3mq$. In a similar way, from the terms X^5, X^3Z and XZ^2, which are homogeneous, we can remove the odd terms, if we take

$$\frac{np}{mq} \text{ is } = 3 \text{ or } \frac{np}{mq} = 5 + 2\sqrt{5}.$$

In the same manner there must always be a certain relation between the tangents $\frac{m}{n}$ and $\frac{p}{q}$, so that the powers occur as required. If these relations are not found, it is impossible to produce other curves that are satisfied besides the conic sections.

Section 21

To investigate special cases, in which $\frac{m}{n}$ has a certain relation with $\frac{p}{q}$, we write

$$\frac{m}{n} = g \text{ and } \frac{mp}{nq} = k,$$

so that

$$X \text{ is } = gx + y \text{ and } Z = kxx + yy.$$

Taking homogeneous terms of order three, that is $\alpha X^3 + \beta XZ$; and after substitution we get

$$\begin{array}{cccc} +\alpha g^3 & +3\alpha g^2 & +3\alpha g & +\alpha \\ x^3 & x^2 y & xy^2 & y^3 \\ +\beta gk & +\beta k & +\beta g & +\beta \end{array}$$

(The above notation is a convenient shorthand. For the expression

$a+b+(c+d)x+(e+f)x^2+\ldots$**, Euler will write** $\dfrac{a\ \ +c\ \ +e\ \ +\ \ldots}{b\ \ +d\ \ +f\ \ +\ \ldots}\ \ x\ \ x^2$ **.)**

in which the terms, that contain the odd dimensions of *y*, must vanish. Thus $\alpha+\beta$ will be = 0 and $3\alpha g^2+\beta k=0$. Therefore

$$\beta=-\alpha \text{ and } k = 3gg \text{ or } \frac{mp}{nq}=\frac{3mm}{nn},$$

and it follows that $\frac{p}{q}=\frac{3m}{n}$. Hence $\alpha X^3+\beta XZ$ changes into $2\alpha\left(gxyy-g^3x^3\right)$ or

$$2\alpha\left(\frac{m}{n}xyy-\frac{m^3x^3}{n^3}\right).$$

Section 22

Thus if the tangent of the angle *ACF* = $\frac{p}{q}$, is three times bigger than the tangent of the angle *CTE* = $\frac{m}{n}$, then we will be able to find infinitely many curves *AMm*, in which *AC* is the orthogonal diameter and *CF* is the oblique-angled diameter. Further, if we let the tangent of the angle *ACE* = θ, we will have

$$\frac{p}{q}=\theta \text{ and } \frac{m}{n}=\frac{1}{3}\theta,$$

and the tangent of the angle *CEM* will be = $\frac{4\theta}{3-\theta\theta}$. Then write

$$Z = yy+\frac{1}{3}\theta\theta xx \text{ and } V=\frac{1}{3}\theta xyy-\frac{1}{27}\theta^3x^3,$$

and if *W* denotes any function of *Z* and *V*, then the equation *W* = 0 will express the curve, that possesses the aforementioned property. Hence it is obvious, with

AC being the orthogonal diameter, then the straight line from *C* passing under *AC*, will make with *AC* an angle below the axis, of which the tangent is = θ. This will be an oblique-angled diameter, of the same type as *CF* at the top. Therefore the curves with this property are contained in the general equation:

$$0 = \alpha + \beta Z + \gamma V + \delta Z^2 + \varepsilon ZV + \zeta V^2 + \eta Z^3 + etc.$$

Section 23

From the infinity of curves of this nature we select the curves of third order, that are contained in the equation:

$$a^3 = byy + \frac{1}{3}\theta\theta\zeta xx + \frac{1}{3}\theta xyy - \frac{1}{27}\theta^3 x^3$$

(In the second term on the right ζ should be *b*.)

or

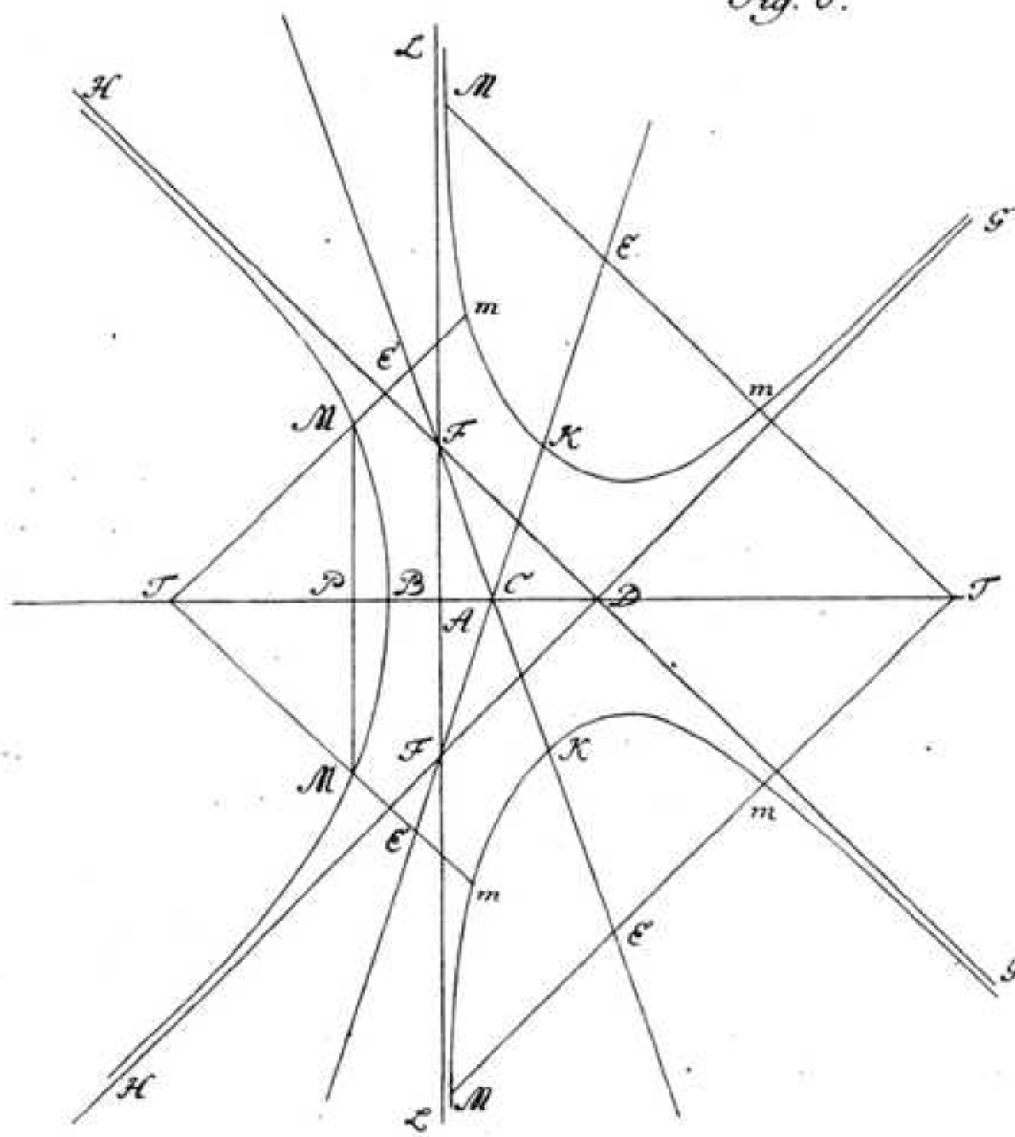
$$yy = \frac{a^3 - \frac{1}{3}\theta\theta bxx + \frac{1}{27}\theta^3 x^3}{b + \frac{1}{3}\theta x}.$$

These curves are examples of the redundant hyperbolas of Newton, that have only one orthogonal diameter. The general equation for these is

$$yy = \frac{Av^3 + Bv^2 + Cv + D}{v}.$$

Here the origin of the abscissa is the point of the axis, where the asymptote is parallel to the direction of *y*. **(This asymptote is the *y*-axis.)** Among these curves, those that will satisfy our requirements have *C* **(this is C from the above equation)** equal to $\frac{BB}{4A}$. So then, $v = -\frac{B}{6A}$, is the point *C*, **(on the axis)** from which the straight line *CF*, makes with the axis the angle *FCA*, for which the tangent is = $3\sqrt{A}$. This straight line will be the oblique-angled diameter, cutting in two the chords Mm, that make with the axis *CA* an angle, whose tangent = $\sqrt{A}$. The tangent of the angle, made where these chords meet the diameter is = $\frac{4\sqrt{A}}{1-3A}$.

Fig. 6.



Section 24

Therefore, (Fig. 6) *MMmm* is a similar redundant hyperbola, having the axis *AP*, that is at the same time an orthogonal diameter, such that in taking the abscissa $AP = v$ and in letting the ordinate $PM = y$, yy is

$$= \frac{Av^3 + Bv^2 + Cv + D}{v} = \frac{(2Av + B)^2}{4A} + \frac{D}{v}.$$

Here C is = $\frac{B^2}{4A}$. The straight line *LAL* normal to the axis will be an asymptote of the curve and the two other asymptotes *HDG* will cross at the point *D* of the axis, such that *AD* is = $\frac{B}{2}$. The tangent of the angle *HDA* will be = $\sqrt{A}$ and the whole curve will be composed of three parts in hyperbola like form *MBM*, *mKm*, and *mKm*. Now take AC = $\frac{B}{6A}$ and construct above and below the lines *CF*, *CF*, such that the tangent of the angle *FCA* is $= 3\sqrt{A}$, these two diameters will bisect all chords *Mm*, that were extend to make with the axis the angle *MTA*, with tangent = $\sqrt{A}$. Those bisected straight lines *Mm* will therefore be parallel to one of these diameters. As for the rest this curve, it can take many different shapes, depending on the value of *B*, seeing that *A* is an affirmative **(positive)** number. The one intersection of the axis at *B* shown in the figure is not the only possible case. It can occur that the curve cuts the axis in three points and this happens if we let $AB = a$, and we take

$$v = \frac{-a}{2} - \frac{B}{24} \pm \sqrt{\left(-\frac{Ba}{2A} - \frac{3aa}{4}\right)}.$$

Section 25

The same values (§22)

$$Z = yy + \frac{1}{3}\theta\theta xx \quad \text{and} \quad V = \frac{1}{3}\theta xyy - \frac{1}{27}\theta^3 x^3,$$

can be made to find infinitely many curves of higher order, that in addition to the orthogonal diameter, have two or more oblique-angled diameters. But like the formula V found by $\alpha X^3 + \beta XZ$ (§21) when making the odd powers of y vanish, seemingly we can do the same thing with greater powers. For example, let

$$V = \alpha X^4 + \beta X^2 Z$$

and setting

$$\frac{m}{n} = g \text{ and } \frac{mp}{nq} = k,$$

we get

$$\begin{array}{ccccc}
+\alpha g^4 & +4\alpha g^3 & +\zeta\alpha g^2 & +4\alpha g & +\alpha \\
x^4 & x^3 y & & xy^3 & \\
+\beta g^2 k & +2\beta gk & +\beta k\, x^2 y^2 & +2\beta g & +\zeta\, y^4 \\
 & & +\beta gg & &
\end{array}$$

Thus $2\alpha g^2 + \beta k$ must be = 0 and $2\alpha + \beta = 0$: from which we get

$$\beta = -2\alpha \text{ and } k = gg \text{ or } \frac{mp}{nq} = \frac{mm}{nn}.$$

Thus $\frac{m}{n} = \frac{p}{q}$ and if we set as before $\frac{p}{q} = \theta$, this will make

$$\frac{m}{n} = \theta\,,\ g = \theta\,,\ k = \theta\theta\,;$$

and like this *V* will be

$$= -\alpha\theta^4 x^4 + 2\alpha\theta^2 x^2 y^2 - \alpha y^4 \text{ or } V = -\alpha\left(\theta^2 xx - yy\right)^2.$$

This is why if *W* is taken as any function of

$$Z = \theta^2 xx - yy \text{ and } V = \left(\theta^2 xx - yy\right)^2$$

and we set $W = 0$, the curve, besides the orthogonal diameter *CA*, will have oblique-angled diameters emerging from *C*, that make with *CA* an angle, whose tangent $= \theta$, and these diameters will cut in two the chords *Mm* inclined at the axis *CA* under the angle, of which the tangent $= \theta$. Among these curves, the simplest is that which is given by the equation

$$\alpha^4 = \theta^4 x^4 y^4.$$

In addion to all these curves, besides the orthogonal diameter *CA*, many also have an orthogonal diameter that is vertical at *C*; for these the equations have not only *y*, but also *x*, with only even powers.

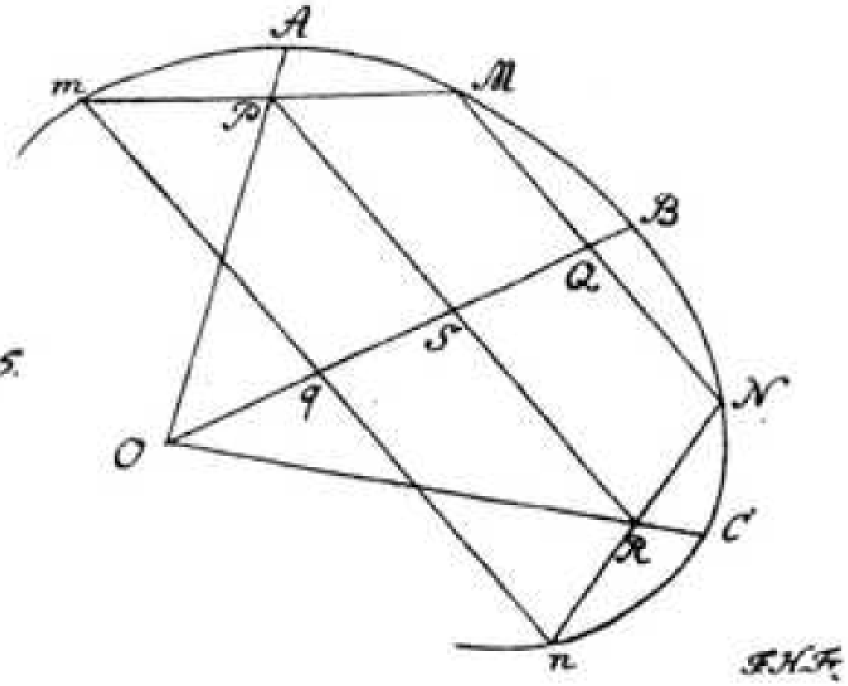


Fig. 5.

Section 26

Using the same method we can go further by studying homogeneous expressions of higher powers to eliminate odd powers that must be thrown away. We find other functions for V , that require other relations between

$$\frac{m}{n} \text{ and } \frac{p}{q}.$$

We will not stop there however, but we bring back a property of very grand importance regarding diameters, that can be accommodated by all the curves that we find. Here is the question. If the curve *ABC* (Fig. 5) has two diameters *AO, BO,* that intersect at *O*, this same curve will have many more diameters, that meet at the point *O*. Sometimes an infinite number occur, if new diameters do not coincide with previous ones. To see this, consider that the curve has two diameters *AO* and *BO*, where *AO* bisects all chords *Mm* making angle *mPO* and *BO* intersects likewise the chord *MN* with the angle *MQO*. From points *M* and *m* of any chord *Mm* bisected by the diameter

AO, we extend cords through the other diameter *BO*, that are *MN* and *Mn*. These are bisected by the diameter *BO* at *Q* and *q*. After connecting the chord *Nn*, all the angles will be known in the quadrilateral *MNnm.* If from *P* we extend *PR* parallel to *MN*, *mn*, this line will intersect the chord *Nn* at *R*. Now construct *ORC*, the angles *BOC* and *NRC* will also be defined, from which it follows that the line *OC* will be again a diameter, that bisects the chord *Nn* making given angle *NRO*.

Section 27

To understand more fully the above phenomenon, let the tangent of the angle $mPO = \alpha$, the tangent of the angle $AOB = B$ and the tangent of the angle MQO = β. It follows from this that the cotangent of the angle *BOC*

$$= \frac{1}{B} + \frac{2}{\beta},$$

and consequently the tangent of the angle *BOC,* that is = $C = \dfrac{\beta B}{\beta + 2B}$. If we let the tangent of the angle *NRO* = γ, γ will be

$$= \frac{\alpha\beta^2(1+BB)}{2\alpha\beta + 4\alpha B - 2\alpha\beta B^2 - \beta^2 - 4\beta B - 4BB - \beta\beta BB}.$$

Now it follows that if the tangents of the next angles in the same order are called *D* and δ,

$$\text{D will be} = \frac{\gamma C}{\gamma + 2C}$$

and

$$\delta = \frac{\beta\gamma^2(1+CC)}{2\beta\gamma + 2\zeta C - 2\zeta\,\gamma C^2 - \gamma^2 - 4\gamma C - 4CC - \gamma\gamma CC}.$$

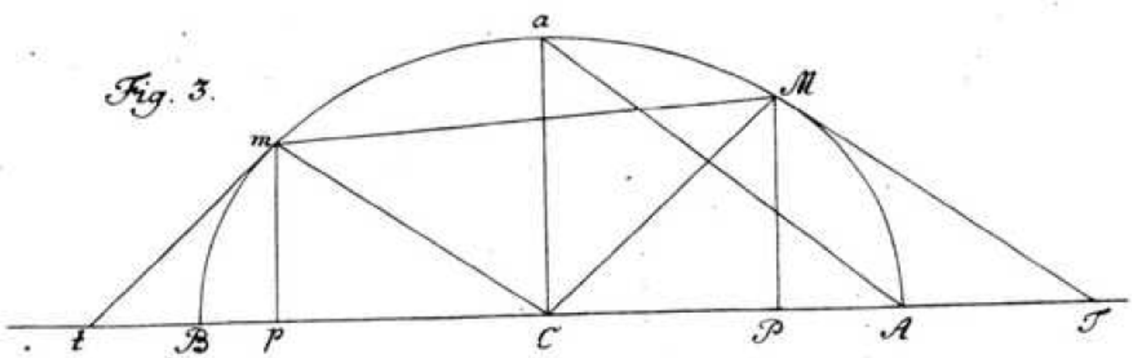
(In the numerator the original has $\underline{\beta\gamma(1+CC)}$. Also the second term in the denominator should be $\underline{4\zeta C}$.))

In some cases we can continue to infinity, at least for those **(diameters)** that do not coincide exactly with the first.

Section 28

To these problems on parallel diameters, or those that intersect at a given point, I will add another that is closely related and which the skilled Mr. Clairaut mentions in one of the letters he has done me the honor of writing. The origin of this problem comes from the property of the ellipse, in which the inscribed parallelograms around the two conjugate diameters, encompass everywhere the same area. Since all the lines in the other curves intersecting at a fixed point, as from a center, are not diameters, we will not pay attention in this research to the conditions, that define diameters and we propose only the following problem.

Fig. 3.



We seek a curve AMamB that has two orthogonal diameters ACB and aC that are *perpendicular to each other. Thus the center of this curve is at C. Like the ellipse, it* *must have the following property: that extending from the cente*r *C any ray CM and at the same time another ray Cm parallel to the tangent MT at the point M*, the area of the triangle *MCm is always constant , and is equal to the area of the triangle ACa*.

Section 29

To solve this problem, let, after having dropped a perpendicular *MP* from the point *M* on the axis *AC*, we let the abscissa $CP = x$ and the ordinate $PM = y$. First the equation for the curve, that is $W = 0$, will necessarily be such that *x* and *y* have on both sides even power. Letting *x* or *y* or both be negative, the equation remains always the same. Thus *W* will be any function of *xx* and *yy*. This condition follows from the above property, in virtue of which, the lines *AC* and *aC* must be orthogonal diameters of the curve. Now from the point *m* we construct on the axis *AB* the perpendicular *mp.* We say that $Cp = t$ and $pm = u$. It follows from the continuity of the curve that the same equation is found between *t* and *u*, that is between *x* and *y*. If in the equation $W = 0$ in place of *xx* we put *tt*, the value of *yy* changes into *uu.*

Section 30

We introduce a new variable **(parameter)** *z*, from which we can determines the values *xx* and *yy*, such that when *z* is eliminated, we arrive at the equation for the curve $W = 0$. We conceives some quantity *z*, which when made negative, *xx* changes into *tt* and *yy* to *uu*. Thus it is obvious that in eliminating *z*, the equation between *tt* and *uu* must be the same as that between *xx* and *yy*, as the law of continuity requires. Suppose we are given *P* and *R* as even functions of *z*, that remain the same in setting $-z$ in the place of $+z$, and given *Q* and *S* as functions of odd powers of *z*, that change into their negatives, if we set $-z$ in the place of $+z$. Thus if we set $xx = P + Q$ and $yy = R + S$, then when making z negative, we will have $tt = P - Q$ and $uu = R - S$. By these manipulations we arrive at a curve that continuous connects the parts *AMa* and *amB* and where both *AC* and *aC* are orthogonal diameters.

Section 31

What is more, because the straight line *Cm* must be parallel to the tangent *MT*, the undertangent $PT = -\frac{ydx}{dy}$. This will make $PT : PM = Cp : pm$ or $-dx : dy = t : u$, from which we get

$$udx + tdy = 0 .$$

Finally since the area of the triangle *MCm* must be constant, we calculate this area and get

$$= \frac{1}{2} CM \cdot Cm \cdot \sin A \cdot MCm .$$

<u>(Recall that Euler writes "sin *A.MCm*" to mean sine of angle *MCm*.)</u>

But

$$\sin A \cdot MCm = \sin A \cdot (MCP + mCp) = \frac{PM \cdot Cp + CP \cdot pm}{CM \cdot Cm},$$

from which we find that the area of the triangle *MCm* is $= \frac{ty + ux}{2}$. Consequently the value of $\frac{ty + ux}{2}$ must be constant and so its differential is equal to zero. Therefore

$$ydt + tdy + udx + xdu = 0 .$$

Since $udx + tdy$ is $= 0$, this makes

$$ydt + xdu = 0 .$$

From this equation we read that the tangent *mt* is parallel to the ray *CM* and that the rays *CM* and *Cm* are reciprocally parallel to their tangents *MT* and *mt*.

Section 32

We let $ty + ux = 2cc$ and since

x is $= \sqrt{P+Q}$, $y = \sqrt{R+S}$, $t = \sqrt{P-Q}$ and $u = \sqrt{R-S}$,

we get after making substitutions

$$\sqrt{(P+Q)(R-S)} + \sqrt{(P-Q)(R+S)} = 2cc.$$

Suppose V denotes any odd functions of z and that we let

$$\sqrt{(P+Q)(R-S)} = cc + V .$$

If we make z negative, $\sqrt{(P+Q)(R-S)}$ changes to $\sqrt{(P-Q)(R+S)}$ so that

$$\sqrt{(P-Q)(R+S)} \text{ is } = cc - V,$$

as it is required by the nature of the given conditions. One can thus infer that

$$R+S = \frac{(cc-V)^2}{P-Q} \text{ and } R-S = \frac{(cc+V)^2}{P+Q}.$$

So

$$x \text{ is } = \sqrt{P+Q},\ t = \sqrt{P-Q},\ y = \frac{cc-V}{\sqrt{P-Q}} \text{ and } u = \frac{cc+V}{\sqrt{P+Q}}.$$

From these we have

$$dx = \frac{dP+dQ}{2\sqrt{P+Q}} \text{ and } dy = -\frac{dV}{\sqrt{P-Q}} - \frac{(cc-V)(dP-dQ)}{2(P-Q)\sqrt{P-Q}},$$

and thus $udx + tdy = \dfrac{(cc+V)(dP+dQ)}{2(P+Q)} - dV - \dfrac{(cc-V)(dP-dQ)}{2(P+Q}.$

<u>(This last expression should be</u>

$$\underline{udx + tdy = \frac{(cc+V)(dP+dQ)}{2(P+Q)} - dV - \frac{(cc-V)(dP-dQ)}{2(P-Q)}} \textbf{.)}$$

Since *udx* + *tdy* must be = 0,

0 will be = (*PP* - *QQ*) *dV* - *V* (*PdP* - *QdQ*) - *cc* (*PdQ* - *QdP*) .

Section 33

We divide this equation by $(PP-QQ)^{3/2}$ and we get

$$\frac{dV}{\sqrt{PP-QQ}} - \frac{V(PdP-QdQ)}{(P^2-Q^2)^{3/2}} = \frac{cc(PdQ-QdP)}{(P^2-Q^2)^{3/2}}.$$

After integration have

$$\frac{V}{\sqrt{PP-QQ}} = \int \frac{cc(PdQ-QdP)}{(PP-QQ)^{3/2}}.$$

Now let *Q* = *Pz* , and because *P* is an even function, *Pz* is an odd function,

as is *Q* . Because *dQ* = *Pdz* + *zdP* , we have

$$\frac{V}{P\sqrt{1-zz}} = \int \frac{ccdz}{P(1-zz)^{3/2}}.$$

This must be an integrable formula, if we want to discover algebraic curves. We write

$$\int \frac{ccdz}{P(1-zz)^{3/2}} = \frac{Z}{\sqrt{1-zz}},$$

so that *V* becomes = *PZ*. We see that *Z* is an odd function of *z*, because *V* was defined as

an odd function. From this we get

$$\frac{ccdz}{P} = (1-zz)dZ + Zzdz \text{ and } P = \frac{ccdz}{(1-zz)dZ + Zzdz}.$$

Section 34

Now let Z be an arbitrary odd function of z, then

$$P = \frac{ccdz}{(1-zz)dZ + Zzdz},$$

is an even function. Thus

$$Q = Pz = \frac{cczdz}{(1-zz)dZ + Zzdz} \text{ and } V = PZ = \frac{ccZdz}{(1-zz)dZ + Zzdz}.$$

Thus we arrive at the complete solution of the problem by this method in the following way. Let x and y be determined by taking Z as an arbitrary odd function of z, and get **(the parametric equations)**

$$xx \text{ is } = \frac{cc(1+z)dz}{(1-zz)dZ + Zzdz} \quad \text{and} \quad yy = \frac{cc(1-z)((1+z)dZ - Zdz)^2 dz}{((1-zz)dZ + Zzdz)dz}.$$

Hence virtue of the previous argument, by making z and Z negative we get tt and uu

$$tt = \frac{cc(1-z)dz}{(1-zz)dZ + Zzdz} \text{ and } uu = \frac{cc(1+z)((1-z)dZ + Zdz)^2 dz}{((1-zz)dZ + Zzdz)dz}.$$

Thus we get from these an infinite number of curves endowed with the proposed properties, First they have symmetry around the principal axes aC and AC. Next, extending from the center C the two rays CM and Cm, to the tangents of the curve at M and m reciprocally parallel, we get for the area of the triangle $MCm = cc$.

Section 35

Thus we find the equation for the curve in x and y, if we eliminate the variable Z from the two equations :

$$xx = \frac{cc(1+z)dz}{(1-zz)dZ + Zzdz}$$

and

$$yy = \frac{cc(1-z)((1+z)dZ - Zdz)^2\,dz}{((1-zz)dZ + Zzdz)dz}\,.$$

Dividing the one by the other we get

$$\frac{yy}{xx} = \frac{(1-z)((1+z)dZ - Zdz)^2\,dz}{(1+z)dz^2}$$

and

$$\frac{y}{x} = \frac{((1+z)dZ - Zdz)\sqrt{1-zz}}{(1+z)dz}\,,$$

and the product is

$$yx = \frac{cc((1+z)dZ - Zdz)\sqrt{1-zz}}{(1-zz)dZ + Zzdz}\,.$$

But if we do not desire the equation between *x* and *y*, the same discovered formulas give a convenient construction. Taking any value for *Z*, by which *z* is at the same time determined, we find the values for *xx* and *yy* and they determine one point of the curve. We are also able to create a geometric construction, if select a curve, with the coordinates *z* and *Z* that has this property, that in making *z* negative, the other *Z* becomes negative as well. The relation between *dZ* and *dz* is defined by the tangent of this curve.

Section 36

Because *Z* must be some function of z that changes to -*Z*, when using -*z* in place of *z*, we take the simplest case $Z = \alpha z$ and get

$$xx = \frac{cc(1+z)}{\alpha} \text{ and } yy = \alpha cc(1-z);$$

from which we have

$$1+z = \frac{\alpha xx}{cc} \text{ and } 1-z = 2 - \frac{\alpha xx}{cc}.$$

Thus

$$yy = 2\alpha cc - \alpha\alpha xx,$$

the equation for the ellipse, which obviously satisfies our problem.

Section 37

Now we let $Z = \alpha z^n$, where n is an odd number, so that Z becomes an odd function of z. Also

$$\frac{dZ}{dz} \text{ will be } = \alpha n z^{n-1}$$

and

$$(1+z)dZ - Zdz = \alpha z^{n-1} dz(n+(n-1)z)$$

and

$$(1-zz)dZ + Zzdz = \left(\alpha n z^{n-1} - \alpha(n-1)z^{n+1}\right)dz = \alpha n z^{n-1} dz(n-(n-1)zz)$$

from which we find

$$xx = \frac{cc(1+z)}{\alpha z^{n-1}(n-(n-1)zz)},$$

and

$$yy = \frac{\alpha cc(1-z)z^{n-1}(n+(n-1)z)^2}{n-(n-1)zz}.$$

Thus

$$xxyy = c^4(1-zz)\frac{(n+(n-1)z)^2}{(n-(n-1)zz)^2};$$

from which by eliminating z there results an algebraic equation of higher degree.

If we let $Z = \dfrac{\alpha z}{1-zz}$,

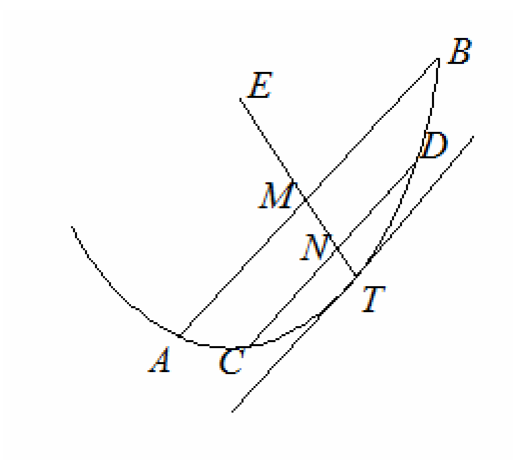
$$xx \text{ is } = \frac{cc(1+z)(1-zz)}{\alpha(1+2zz)} \text{ and } yy = \frac{\alpha cc(1-z+2zz)^2}{(1+2zz)(1-zz)(1-z)}.$$

We can in the same manner substitute infinitely many functions of z in place of Z, that will always generate equations for the curves, that satisfy that our requirements. I have found no selections among them, that lead to a simpler equation between x and y, however they are all easy to construct.

Translators Notes

The following notes are keyed to E083 by section number.

Notes for Section 1



ET is an oblique-angled diameter

Figure 1.1

Euler wrote of “oblique-angled diameters” and “orthogonal diameters” without explanation.

Definition: Given a curve *ACTDB* shown in figure 1.1. The line *ET* , which intersects the curve at *T*, is called an *oblique-angled diameter* if it bisects all chords (such as *AB* and *CD*), that are parallel to the tangent line at *T.*

In other words $AM = MB$ *and* $CN = ND$ since the chords *AB* and *CD* are parallel to the tangent line at the point *T* where the *oblique-angled diameter ET* intersects the curve *ACTDB.*

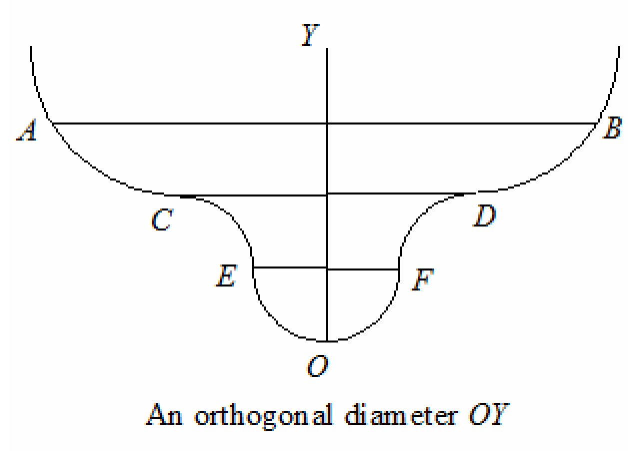


Figure 1.2

Many curves possess an oblique-angled diameter. For example, any curve that is symmetric about an axis has this property. In Figure 1.2 we see a curve that is symmetric about the line *OY*. Clearly this line bisects all horizontal chords. In this special case we call *OY* and *orthogonal diameter.* It is also clear that every diameter of a circle is an orthogonal diameter.

While many curves possess one oblique-angled diameter, in the case of the conic sections **all** appropriately defined diameters have this property. For the parabola, we define a "diameter" as any line parallel to the axis of the parabola. In this case, all diameters are oblique-angled diameters. We will prove this below. For the ellipse and the hyperbola, we define a "diameter" as any line that passes through the center of the curve. **All** such diameters are oblique-angled diameters.

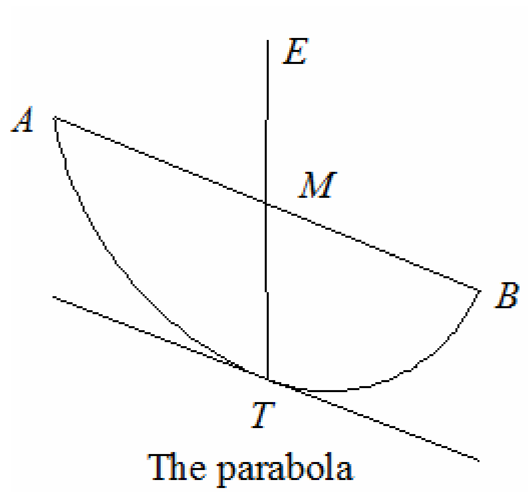


Figure 1.3

Theorem 1. Let *ATB* be a parabolic curve and let *ET* be any line parallel to the axis of the parabola. Let *AB* be any chord parallel to the tangent line at *T*. Then *ET* bisects the chord *AB* at the point of intersection *M*.

Proof Since all parabolas are similar, without loss of generality, we can call the equation of the parabola $y = x^2$. Call the coordinates of the point *T*, $\left(x_T, y_T\right)$ and the slope at this point is $\frac{dy}{dx} = m = 2x_T$. The line *AB* has the equation $y = mx + b$ and intersects the parabola at the points where $x^2 - mx - b = 0$. The solutions of this equation are $\frac{m}{2} \pm \frac{\sqrt{m^2 + 4b}}{2} = x_T \pm \sqrt{x_T^2 + b}$. Thus the *x* coordinate of the point *A* is $x_T - \sqrt{x_T^2 + b}$, and the *x* coordinate of the point *B* is $x_T + \sqrt{x_T^2 + b}$. Since the point *M* has *x* coordinate x_T it is clear from this result that the point *M* bisects the line segment *AB*.

Notes for Section 10

While Euler will not pursue transcendental curves further, he indicates a method for finding them. With $Y = yy - 2ay$ and $Z = y^2 - \frac{2ma\,x}{n}$ we seek a function of *Y* and *Z* with no odd powers of y, since the curve is symmetric about the *x* axis. Euler asks us, without explanation, to find a function $T(Y)$ satisfying the infinite differential equation

$$0 = \frac{dT}{dY} + \frac{4a^2Yd^3T}{1.2.3.dY^3} + \frac{16a^4Y^2d^5T}{1.2.3.4.5dY^5} + \frac{64a^6Y^3d^7T}{1.2....7dY^7} + \cdots \quad . \quad . \qquad (1)$$

He claims that this function *T* has no odd powers of y. With $W(T,Z)$ an arbitrary function, the equation $W(T,Z) = 0$ is a transcendental curve with axis of symmetry and oblique-angled diameter(s).

***Derivation of the differential equation of infinite order*:**

We give our derivation of (1). Using Taylor's theorem we have

$$T(Y) = T(y^2 - 2ay) = \sum_{n=0}^{\infty} \frac{d^nT(y^2)}{dY^n}\frac{(-2ay)^n}{n!} .$$

Separating even and odd powers we have

$$T(Y) = \sum_{n=0}^{\infty} \frac{d^{2n}T(y^2)}{dY^{2n}}\frac{(-2ay)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{d^{2n+1}T(y^2)}{dY^{2n+1}}\frac{(-2ay)^{2n+1}}{(2n+1)!} .$$

The first series above is a function of only even powers of y, while the second series has both even and odd powers. Thus we set this second series equal to zero and after dividing by *2ay* we get

$$0 = \sum_{n=0}^{\infty} \frac{d^{2n+1}T(y^2)}{dY^{2n+1}}\frac{4^n a^{2n} y^{2n}}{(2n+1)!} .$$

Since we are using this differential equation to solve for T , we can replace y^2 by the dummy variable Y and we have derived Euler's relation (1).

Does the differential equation of infinite order have solutions?

In this segment we will try to find a solution of (1) in the form of a power series in *Y*. We obtain the general form of series but are unable to verify that it converges. However, numerical results suggest that it does converge.

Let us try to find a solution of (1) in the form

$$T(Y) = \sum_{k=0}^{\infty} b_k Y^k .$$

Now

$$\frac{d^{2n+1}T(Y)}{dY^{2n+1}} = \sum_{k=0}^{\infty} b_k \frac{d^{2n+1}Y^k}{dY^{2n+1}} = \sum_{k=0}^{\infty} b_k \frac{k!Y^{k-2n-1}}{(k-2n-1)!},$$

and (1) becomes

$$\sum_{n=0}^{\infty} \frac{(4aY)^n}{(2n+1)!} \sum_{k=0}^{\infty} b_k \frac{k!Y^{k-2n-1}}{(k-2n-1)!} = 0 .$$

We write this as

$$\sum_{n=0}^{\infty} \frac{(4a)^n}{(2n+1)!} \sum_{k=2n+1}^{\infty} b_k \frac{k!Y^{k-n-1}}{(k-2n-1)!} = 0 .$$

Now let $j = k-2n-1$ and get

$$\sum_{n=0}^{\infty} \frac{(4a)^n}{(2n+1)!} \sum_{j=0}^{\infty} b_{j+2n+1} \frac{(j+2n+1)!Y^{j+n}}{j!} = 0 .$$

We can write this double sum as

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{(j+2n+1)!}{(2n+1)!} \frac{(4a)^n b_{j+2n+1}}{j!} Y^{j+n} = 0 ,$$

which is

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \binom{j+2n+1}{j} (4a)^n b_{j+2n+1} Y^{j+n} = 0 .$$

Let $p = j+n$ and get

$$\sum_{p=0}^{\infty} \sum_{n=0}^{p} \binom{p+n+1}{p-n} (4a)^n b_{p+n+1} Y^p = 0 .$$

Thus the coefficients of Y must vanish and we have

$$\sum_{n=0}^{p}\binom{p+n+1}{p-n}(4a)^n b_{p+n+1}=0,$$

for $p=0,\ 1,\ 2,\ \cdots$. We get

For p = 0, $\binom{1}{0}b_1=0$.

p = 1, $\binom{2}{1}b_2+\binom{3}{0}(4a)b_3=0$,

p = 2, $\binom{3}{2}b_3+\binom{4}{1}(4a)b_4+\binom{5}{0}(4a)^2b_5=0$,

p = 3, $\binom{4}{3}b_4+\binom{5}{2}(4a)b_5+\binom{6}{1}(4a)^2b_6+\binom{7}{0}(4a)^3b_7=0$,

p = 4, $\binom{5}{4}b_5+\binom{6}{3}(4a)b_6+\binom{7}{2}(4a)^2b_7+\binom{8}{1}(4a)^3b_8+\binom{9}{0}(4a)^4b_9=0$,

p = 5, $\binom{6}{5}b_6+\binom{7}{4}(4a)b_7+\binom{8}{3}(4a)^2b_8+\binom{9}{2}(4a)^3b_9+\binom{10}{1}(4a)^4b_{10}+\binom{11}{0}(4a)^5b_{11}=0$.

We see at once from the first of these equations that $b_1=0$. Let b_2 be given, then from the second equation we have

`
$$b_3=-\frac{\binom{2}{1}b_2}{\binom{3}{0}(4a)}=-\frac{b_2}{2a}.$$

From now we will take $b_{2n}=0$ for n = 2, 3, 4, … .From the third equation we get

$$b_5=-\frac{\binom{3}{2}b_3}{\binom{5}{0}(4a)^2}=-\frac{3b_3}{(4a)^2}=\frac{\binom{3}{2}}{(4a)^3}2b_2=\frac{3}{32a^3}b_2=0.09375\frac{b_2}{a^3}.$$

From the fourth equation we get

$$b_7=-\frac{\binom{5}{2}(4a)b_5}{\binom{7}{0}(4a)^3}=-\frac{10b_5}{(4a)^2}=-\frac{\binom{3}{2}\binom{5}{2}}{(4a)^5}2b_2=-\frac{15}{256a^5}b_2=-0.05659375\frac{b_2}{a^5}.$$

From the fifth equation we get

$$b_9 = -\frac{\binom{5}{4} b_5 + \binom{7}{2}(4a)^2 b_7}{\binom{9}{0}(4a)^4} = -\frac{\binom{3}{2}\binom{5}{4} - \binom{3}{2}\binom{5}{2}\binom{7}{2}}{(4a)^7} 2b_2 = \frac{615}{8192a^7} b_2 = 0.0750732 \frac{b_2}{a^7} .$$

From the sixth equation we have

$$b_{11} = -\frac{\binom{7}{4}(4a)b_7 + \binom{9}{2}(4a)^3 b_9}{\binom{11}{0}(4a)^5} = \frac{\binom{3}{2}\binom{5}{2}\binom{7}{4} + \binom{9}{2}\left(\binom{3}{2}\binom{5}{4} - \binom{3}{2}\binom{5}{2}\binom{7}{2}\right)}{(4a)^9} 2b_2 = \frac{-21090}{131072a^9} b_2 =$$

$$-0.16090393 \frac{b_2}{a^9}$$

.

Thus we have

$$T(Y) = b_2\left(Y^2 - \frac{0.5}{a}Y^3 + \frac{0.09375}{a^3}Y^5 - \frac{0.05659375}{a^5}Y^7 + \frac{0.0750732}{a^7}Y^9 \; - \frac{0.16090393}{a^9}Y^{11} + \cdots\right)$$

.

This last result *suggests* that this series converges, possibly for $|Y| < a$ and thus we *conjecture* that nontrivial solutions exist for Euler's differential equation of infinite order. It remains to prove that this last equation really converges.

Notes for Section 11

We repeat Euler's proof that two parallel diameters implies infinitely many, giving more details.

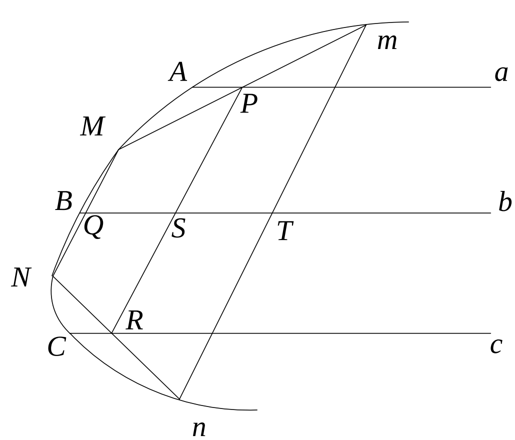


Figure 11.1

Theorem: Given a curve with both *Aa* and *Bb* as two parallel oblique-angled diameters separated by the distance *a*. Then that curve has infinitely many parallel oblique-angled diameters all separated by the same distance *a*.

Proof: Let *mAMBNCn* be the given curve as shown in Figure 11.1. The chord *Mm* is parallel to the tangent at *A* and thus is bisected at point *P*. Both chords *MN* and *mn* are parallel to the tangent at *B* and are thus bisected at *Q* and *T*. Now consider the chord *Nn*, Construct the line *PSR* parallel to *MN*. Clearly point *R* bisects chord *Nn*. Also since *PS* = *RS*, the distance *a* between lines *Aa* and *Bb* equals the distance between lines *Bb* and *Cc*.

We must now show that for any other parallelogram *N'M'm'n'* formed in the same way as parallelogram *NMmn*, the lines *N'n'* and *Nn* are parallel. See Figure 2. We construct *N'M'* parallel to *NM*, *M'm'* parallel to *Mm*, and *m'n'* parallel to *mn*. By the previous reasoning the line *Cc* bisects the chord *m'n'*. Since the line *Bb* bisects the line $\alpha\beta$ and the line *M'N'*, it is clear that $N'\alpha = \beta M'$. For the same reason, $n'\gamma = \delta m'$. Since *M'm'* is parallel to *Mm*, $\beta M' = \delta m'$. Thus $N'\alpha = \gamma n'$ and the chords *Nn* and *N'n'* are parallel. It follows that *Cc* is an oblique-angled diameter, and by repeating this argument again and again, there are infinitely many diameters. The theorem is proved.

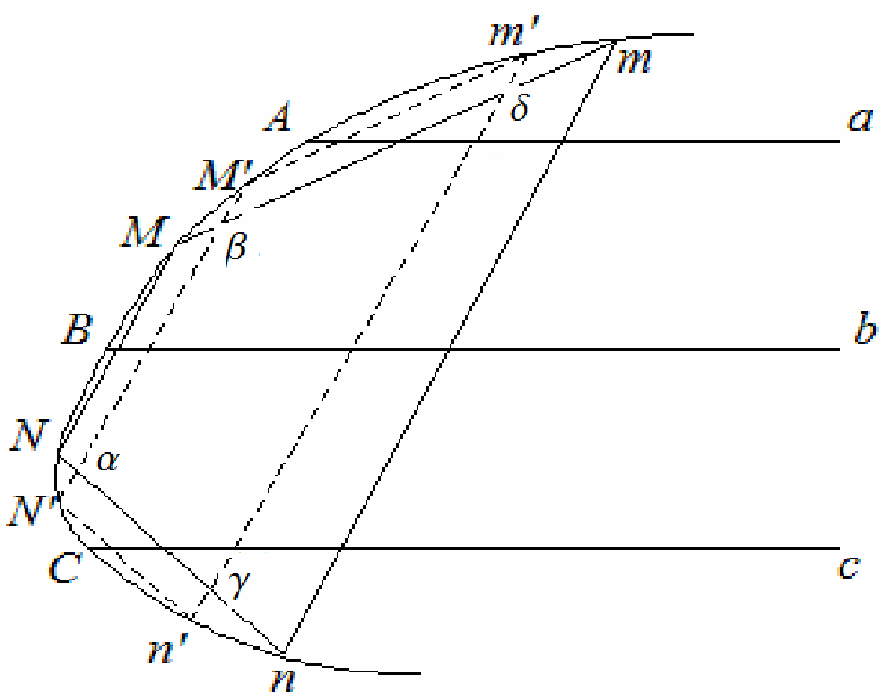


Figure 11.2

Notes for Section 12

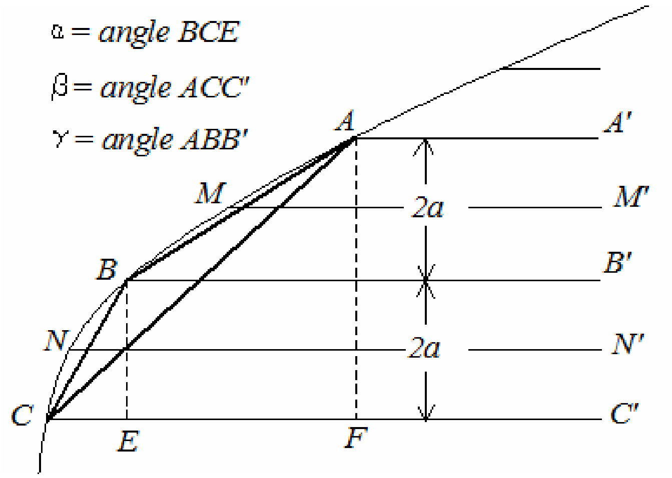


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Figure 12.1

Lemma**:** In the above figure, the curve *AMBNC* has oblique-angled diameters *AA'*, MM', *BB'*, NN', and *CC'*. All these parallel diameters are separated by the distance *a*. Chord *BC* is bisected by and makes angle α with diameter *NN'*, chord *AC* is bisected by and makes angle β with diameter *BB'*, and chord *AB* is bisected by and makes angle γ with diameter *MM'*. Then it follows that

$$\cot\alpha + \cot\gamma = 2\cot\beta .$$

Proof**:** Notice that $CE = 2a\cot\alpha$, $EF = 2a\cot\gamma$, and $CF = 4a\cot\beta$. Since *CF* = *CE* + *EF*, the result follows at once.

To understand the relation between the angles obtained by Euler, we use the following notation for the angles made by chords with successive diameters. Referring to the figure we will use $\theta_1 = \alpha$, the angle made by the chord *BC* with the first diameter *NN'*, $\theta_2 = \beta$, the angle made by the chord *AC* with the second diameter *BB'*, etc. Then we have shown that

$$\cot\theta_3 = 2\cot\theta_2 - \cot\theta_1 ,$$

and it follows that $\cot\theta_4 = 2\cot\theta_3 - \cot\theta_2 = 2\left(2\cot\theta_2 - \cot\theta_1\right) - \cot\theta_2$.

$= 3\cot\theta_2 - 2\cot\theta_1$ We now have

$$\cot\theta_4 = 3\cot\theta_2 - 2\cot\theta_1 .$$

Now we have $\cot\theta_5 = 2\cot\theta_4 - \cot\theta_3 = 2\left(3\cot\theta_2 - 2\cot\theta_1\right) - \left(2\cot\theta_2 - \cot\theta_1\right)$

$= 4\cot\theta_2 - 3\cot\theta_1$. Thus we have

$$\cot\theta_5 = 4\cot\theta_2 - 3\cot\theta_1,$$

And in general

$$\cot\theta_n = (n-1)\cot\theta_2 - (n-2)\cot\theta_1.$$

Notes for Section 13

Euler's conclusion, that he can take the limit as $a \to 0$ in his differential equation of infinite order does not seem valid by modern standards. A more careful examination is called for.

Euler does not give any examples of specific transcendental curves with parallel diameters. One easy example is the curve

$$x = \cos y$$

which has orthogonal diameters given by $y = \pi n$ for all integer *n*.

We found a simple theorem which helps us find additional curves of this type.

Theorem: Let the curves $x = f_1(y)$ and $x = f_2(y)$ both have an oblique angled diameter at $y = d$. Then the curve $x = f_3(y) = a\, f_1(y) + b\, f_2(y)$, where *a* and *b* are any constants, also has the line $y = d$ as an oblique angled diameter.

Proof: The value of $\dfrac{dx}{dy}$ at the point where the diameter $y = d$ intersects the curve $x = f_1(y)$ is $f_1'(d)$. Therefore values of *y* at the points where the line $x = f_1'(d)y + c$ intersects the curve $x = f_1(y)$ are the roots of the equation

$$f_1(y) - f_1'(d)y - c = 0.$$

Because the resulting chord is bisected, these roots are of the form $y = d \pm \Delta_1(c)$. Here $\Delta_1(c)$ is the vertical distance from the diameter to the ends of the chord, and is obviously a function of *c*. Another way of looking at this is say that if $f_1(y) - f_1'(d)y$ takes the value *c* for $y = d + \Delta$, then it takes on the same value *c* for $y = d - \Delta$. Similar results hold for the curve $x = f_2(y)$.

Now consider the curve $x = f_3(y) = a\, f_1(y) + b\, f_2(y)$. The value of $\dfrac{dx}{dy}$ at the point where the diameter $y = d$ intersects is curve is $af_1'(d) + bf_2'(d)$. Thus we must examine the intersections of the line $x = (af_1'(d) + bf_2'(d))y + C$ with this curve to determine if the chord is bisected. The values of *y* where this intersection occurs are the roots of the equation

$a\, f_1(y) + b\, f_2(y) = (af_1'(d) + bf_2'(d))y + C$. This equation can be written as

$a(f_1(y) - f_1'(d)y) + b(f_2(y) - f_2'(d)y) = C$. Suppose $y = d + \Delta$ is a root of this equation. Is then $y = d - \Delta$ also a root? The answer is yes because both of the terms in brackets have the same value for $y = d \pm \Delta$ as was just shown. Thus the theorem is proved.

As an example of this theorem, take $f_1(y) = \cos y$ and $f_2(y) = y^2$. Both curves have the lines $y = \pi n$ for integer *n*, as diameters. Thus the curves $x = a\cos y + by^2$ have these same lines as diameters.

Notes for Section 23 and 24

Euler has shown that an arbitrary function $W(Z,V) = 0$ of the variables

$$Z = y^2 + \frac{1}{3}\theta^2 x^2 \quad \text{and} \quad V = \frac{1}{3}\theta x y^2 - \frac{1}{27}\theta^3 x^3,$$

will be symmetric about the *y* axis and possess an oblique angled diameter making angle $\tan^{-1}\theta$ with the negative *x* axis. The chords that this diameter bisects make angle $\tan^{-1}(\theta/3)$. He calls the curves given by the special case

$$W(Z,V) = bZ + V - a^3$$

redundant hyperbolas of Newton. These reduce to

$$a^3 = by^2 + \frac{1}{3}\theta^2 bx^2 + \frac{1}{3}\theta x y^2 - \frac{1}{27}\theta^3 x^3.$$

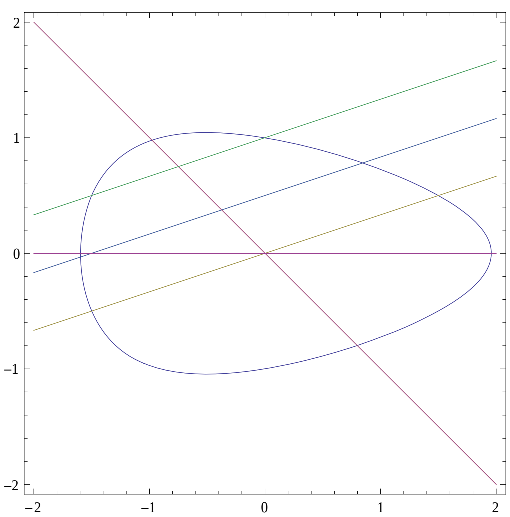

Figure 23.1

Example 23.1: Let *a* = *b* = $\theta = 1$ and get

$$y^2 + x^2/3 + (x\, y^2)/3 - x^3/27 = 1,$$

shown in Figure 23.1. The diameter is the line $y = -x$ and three corresponding bisected chords with slope 1/3 are shown. In Figure 23/2 we see the same curve and diameter over a wider range of x and y.

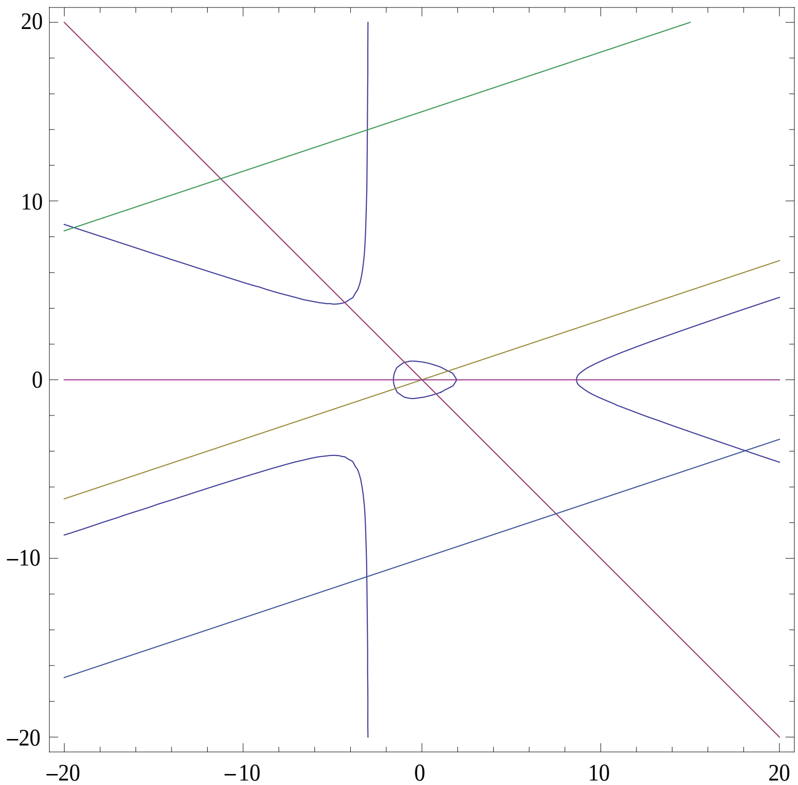

Figure 23.2

Example 23.2 Now consider $W(Z,V) = Z^2V^2 - 1 = 0$ which reduces to

$$(y^2 + x^2/3)^2 \; ((x\,y^2)/3 - x^3/27)^2 - 1 = 0$$

with $\theta = 1$. In Figure 23.3 we see the curve with the diameter $y = -x$ and three chords with the corresponding slope 1/3 that are bisected.

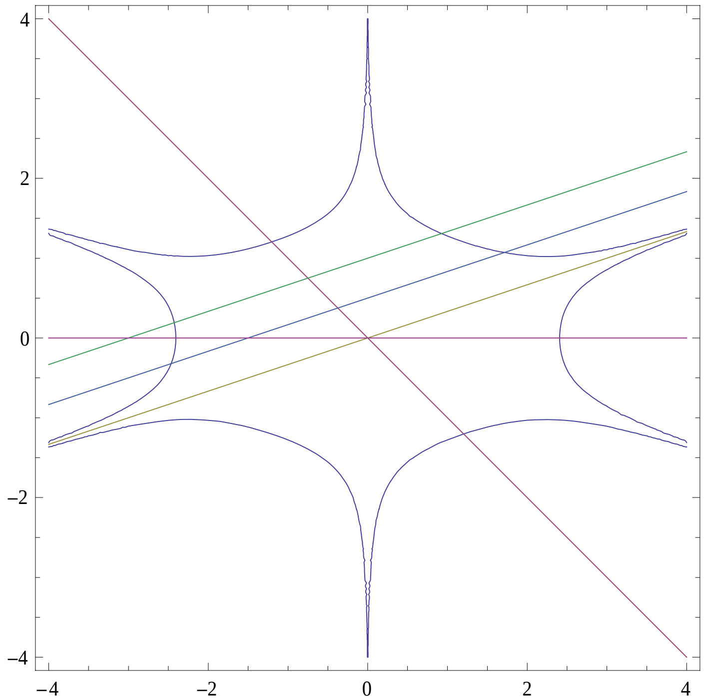

Figure 23.3

Example 23.3 Finally we examine the transcendental function

$$W(Z,V) = Z + 3V + \sin(ZV) - 1 = 0,$$

which simplifies to

$(y^2 + x^2/3) + 3\ ((x\ y^2) / 3 - x^3 / 27) + \sin[(y^2 + x^2/3)\ ((x\ y^2) / 3 - x^3/27) - 1 = 0.$

Again we take $\theta = 1$. In Figure 23.4 we see the curve with the diameter $y = -x$ and three chords with the corresponding slope 1/3 that are bisected.

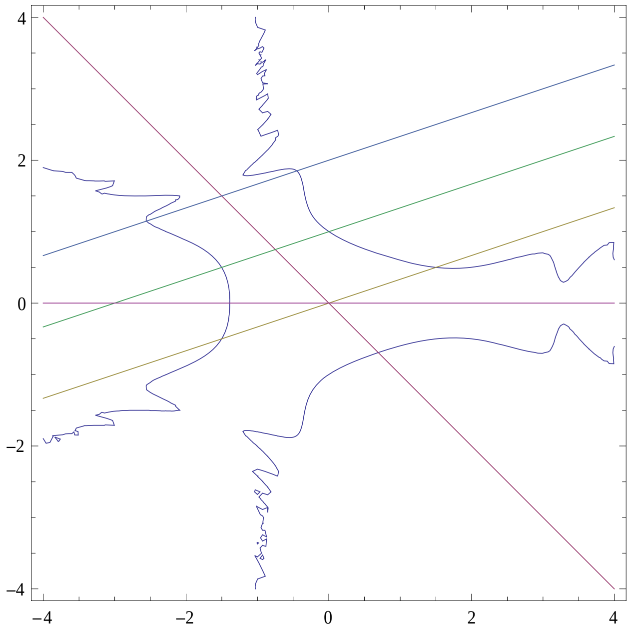

Figure 23.4

Notes for Section 26

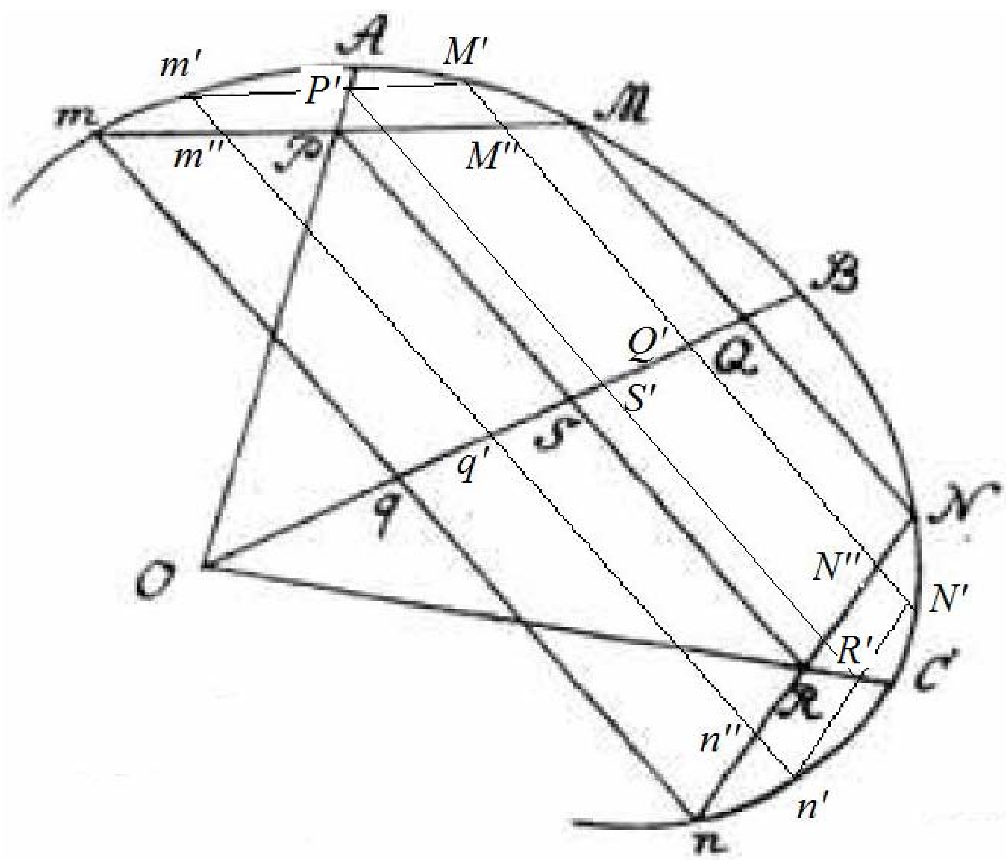

Figure 26.1

We have modified Euler's figure by adding several points and line to help in the proof of the following theorem:

Theorem 26.1: *If a curve has two oblique angled diameters OA and OB that meet at the point O, then it has an additional diameter OC.*

Proof: Let *mM* be a chord that is bisected by the diameter *OA*. From the points *m* and *M* extend parallel chords *mn* and *MN* that are bisected by the diameter *OB*. Now draw the line *nN*. From the point *P* at which *OA* bisects *mM*, extend a line parallel to *mn* and *MN* that intersects *nN* at *R*. It is clear that the chord *nN* is also bisected at *R*. Now draw the line *ORC*.

We must now show that *ORC* is a diameter. We have already shown that it bisects the chord *nN*. Now starting at the point *m'*, we construct the quadrilateral *m'M'N'n'* in the same way that we constructed the quadrilateral *mMNn*. We must show that [1] the chord *n'N'* is parallel to *nN*, and that [2] this chord is bisected by *OC*.

[1] From the figure and our constructions it is clear that *MQ* = *QN*, *M'Q'* = *Q'N'*, and *M''Q'* = *Q'N''*. It now follows that *M'M''* = *N''N'*, and in the same way we see that *m'm''*=*n''n'*. But *m'm''* = *M'M''* because *mM* and *m'M'* are parallel. It is now clear that the chord *n'N'* is parallel to *nN*.

[2] Notice that *PS* = *SR* and *P'S'* = *S'R'*. It follows then that the point *R'* must be on the line *OC* since any line parallel to *mn* joining *OA* and *OC* is bisected by *OB*. This completes the proof.

Theorem 26.2: *Let OA, OB and OC be consecutive oblique angled diameters, where the points A, B and C are on the curve. Then the chord AC is parallel to mn and is bisected by the diameter OB.*

Proof: Using the notation defined in the above theorem, imagine the points *m'* and *M"* converging to the point *A*. Then the quadrilateral *m'M'N'n'* degenerates into the line *AC*, and the theorem is proved.

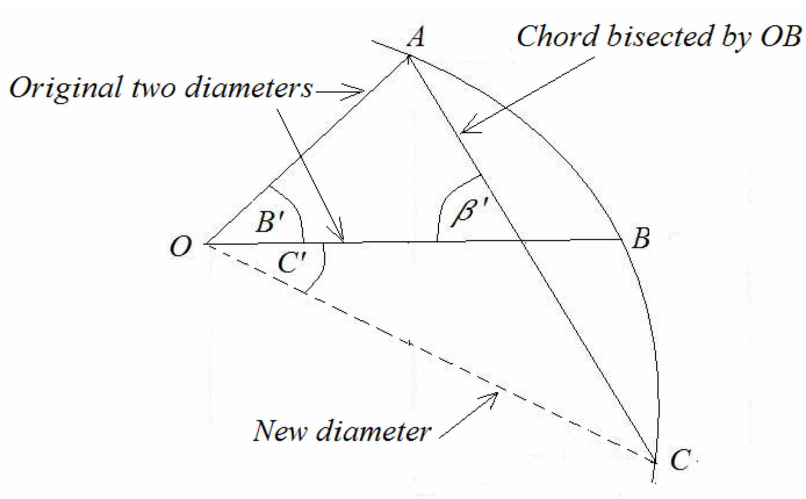

Figure 26.2

Using this last theorem we have now a neat way of constructing new diameters, once two are known. In Figure 26,2 we see a curve *ABC* with two given oblique angled diameters *OA* and *OB*. From point *A* draw the chord *AC* whose slope equals the tangent to the curve at *B*. By Theorem 26.2, *OC* is a new oblique angled diameter.

Notes for Section 27

We give derivations of Euler's equations which he only states in this section.

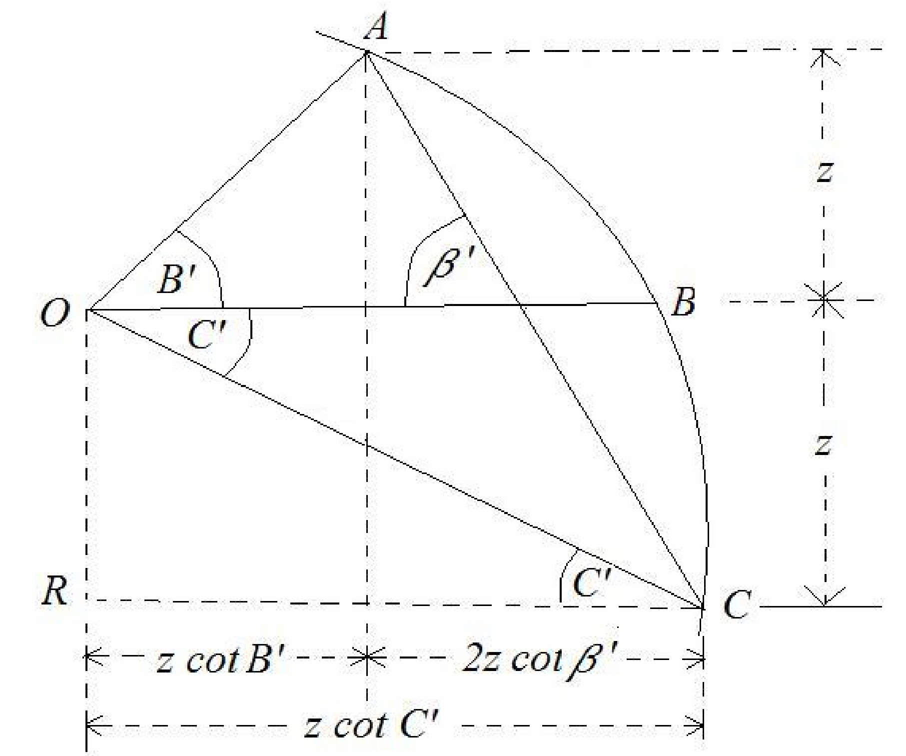

Figure 27.1

In the above Figure 26.2, *OA*, *OB* and *OC* are consecutive oblique angled diameters. By Theorem 26.2, the chord *AC* is bisected by the diameter *OB*. (The diameter *OB* is shown as a horizontal line in this figure, but this is only for convenience. It is not the axis of symmetry.) All the angles shown in the figure are indicated with a "prime". We use the convention that $\tan x' = x$. This now integrates nicely with Euler's notation in this section.

We are now ready to prove Euler's relation that the "cotangent of the angle *BOC* $= \frac{1}{B} + \frac{2}{\beta}$".

From the figure we see that $z\cot C' = z\cot B' + 2z\cot\beta'$. Thus it follows immediately that

$$\frac{1}{C} = \frac{1}{\mathrm{B}} + \frac{2}{\beta}.$$

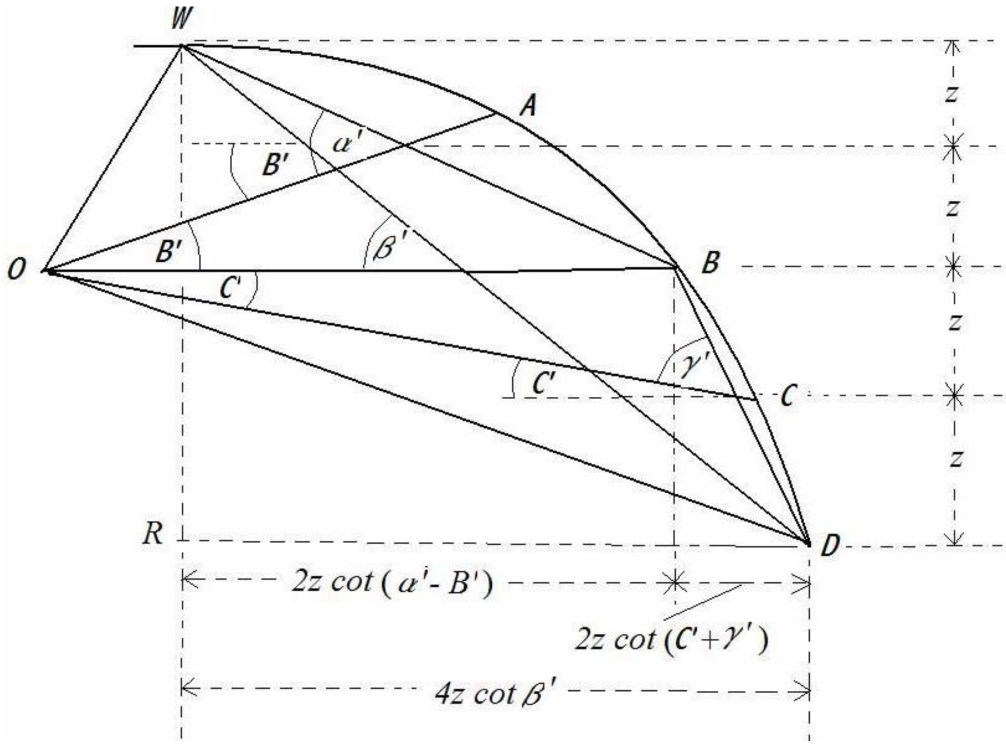

Figure 27.2

Figure 27.2 shows five consecutive oblique angled diameters, *OW*, *OA*, *OB*, *OC* and *OD*. By Theorem 26.2, the chord *WB* is bisected by the diameter *OA*, and likewise the chord *BC* is bisected by the diameter *OC.* In the same way the chord *WD* is bisected by the diameter *OB*. (The diameter *OB* is shown horizontal in the figure, but this is only for convenience. It is not the axis of symmetry.) Five horizontal dashed lines are shown in the figure. The top one passes through the point *W*. The middle one passes through *B* and the bottom one through *D*. The other two pass through the points where the diameters *OA* and *OC* intersect the chords *WB* and *BD* respectively. It now follows that all these lines are equidistant and the distance separating them is shown as *z*.

From Figure 27.2 we see immediately that

$$4z\cot\beta' = 2z\cot(\alpha' - B') + 2z\cot(\gamma' + C')\,.$$

Thus we have

$$\frac{2}{\beta} = \frac{1+\alpha B}{\alpha - B} + \frac{1-\gamma C}{\gamma + C}\,.$$

This simplifies to Euler's relation

$$\gamma = \frac{\alpha\beta^2(1+B^2)}{2\alpha\beta + 4\alpha B - 2\alpha\beta B^2 - \beta^2 - 4\beta B - 4B^2 - \beta^2 B^2}\,.$$

Notes for Section 28

28.1 Reciprocal diameters

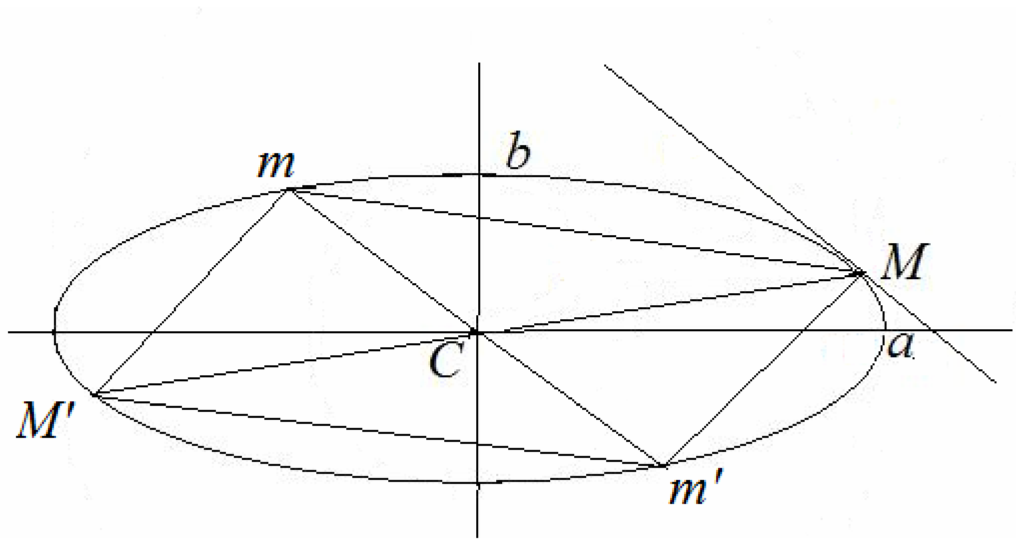

Figure 28.1

The above figure shows an ellipse. Draw any diameter *MM'*. We say that the diameter *mm'* is *reciprocal* to diameter *MM'* if it is parallel to the tangent line to the ellipse at *M*. If we started with diameter *mm*', then *MM'* would be the reciprocal diameter. Euler assumes that we are familiar with reciprocal diameters and that the area of the parallelogram *MmM'm'* is constant, regardless of the choice of the initial diameter, and is equal to 2*ab*. The area of the triangle *CMm* is also constant and equals *ab*/2. In the following sections Euler will look for other curves in which a similar triangular area is constant.

28.2 Parametric equations for the ellipse

These reciprocal diameters have an interesting relation to the parametric form of the equation of the ellipse given by the equations

(28.1) $x = a\sin\theta$ and $y = b\sin\theta$.

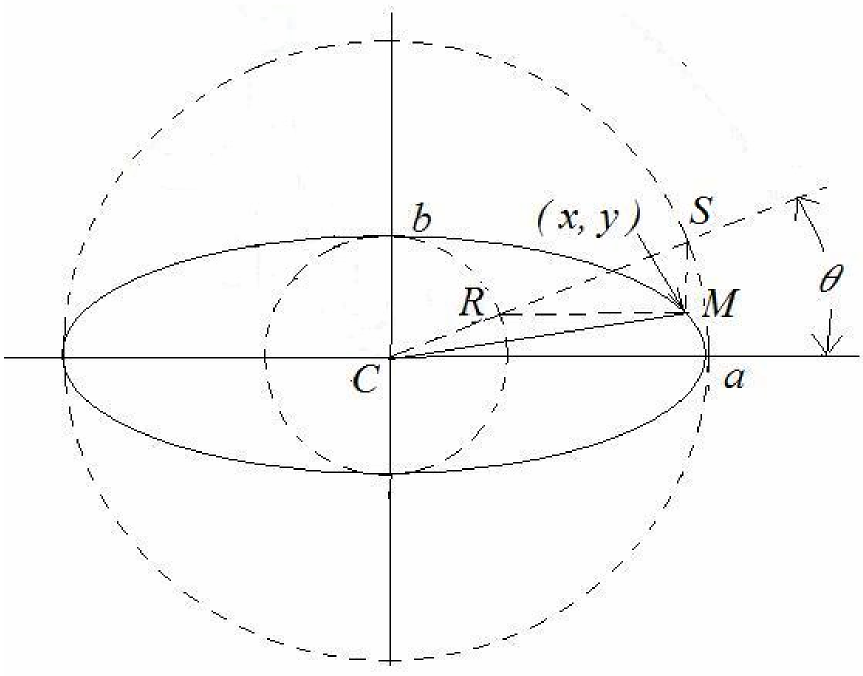

Figure 28.2

In Figure 28.2 we see two circles centered at point *C* with radii *b* and *a*. The ray *CRS* makes angle θ with the *x*-axis and intersects the smaller circle at *R* and the larger circle at *S*. From *R* extend a horizontal line and from *S* drop a vertical line. These two lines intersect at the point *M*. This point *M* is on the ellipse given by the parametric equations given above. As the angle θ varies between 0 and 2π , the point *M* generates the entire ellipse. Notice that the ray *CM* which identifies the point (x, y) differs from the ray *CS* that is made by the parameter θ. The angle θ is known historically as the "eccentric anomaly".

28.3 Reciprocal diameters and the eccentric anomaly

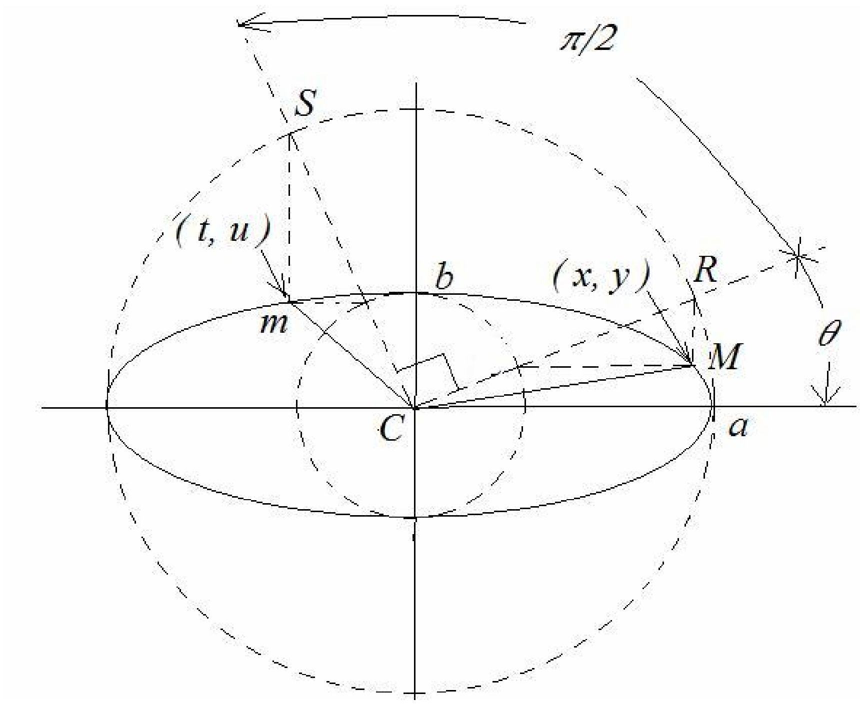

Figure 28.3

To see the relation between the reciprocal diameters and the eccentric anomaly, consider Figure 28.3. Start with the radius *CR* making eccentric anomaly θ, to identify the point on the ellipse *M*. Now increase the eccentric anomaly by $\pi/2$ to identify the radius ray *CS* and corresponding point on the ellipse *m*. This point *m* is reciprocal to *M*. Thus reciprocal points on the ellipse have their related eccentric anomalies separated by the angle $\pi/2$.

To see that this is true, we see that the slope of the tangent at *M* is given by

$$\frac{dy}{dx} = \frac{b\cos\theta\, d\theta}{-a\,\sin\theta\, d\theta} = -\cot\vartheta .$$

Therefore the slope of the ray *CS* is given by $-\cot\theta = \tan(\theta + \pi/2)$ which demonstrates the truth of the relation between *M* and *m* just stated. Thus the coordinates of the point *m* are given by

(28.2) $t = a\cos(\theta + \pi/2) = -a\sin\theta$ and $u = b\sin(\theta + \pi/2) = b\cos\theta$.

28.4 The area of the triangle *CMm* is constant.

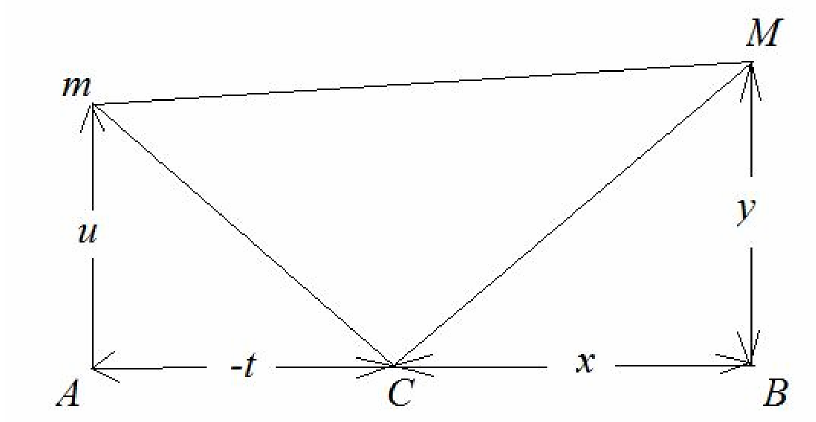

Figure 28.4

The area of the parallelogram *ABMm* shown in Figure 28.4 is $\left(\frac{y+u}{2}\right)(x-t)$. Subtracting the areas of triangles *ACm* and *CBM* we get the area of the triangle *CMm*

Area *CMm* = $\left(\frac{y+u}{2}\right)(x-t)-\frac{(-t)u}{2}-\frac{xy}{2}$.

This simplifies to $\frac{xu-ty}{2}$. Substituting the values of these variables in terms of θ from (28.1) and (28.2) we get

Area *CMm* = $\frac{ab\cos^2\theta+ab\sin^2\theta}{2}=\frac{ab}{2}$.

This proves that the area of the triangle *CMm* is constant.