Synopsis by Section of Leonhard Euler's

METHODVS VNIVERSALIS SERIERVM CONVERGENTIVM SVMMAS QUAM PROXIME INVENIENDI

A general method for finding approximations to the sums of convergent series (E46)

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1. Euler describes the topic of this paper. In modern terms, the paper will show that we can approximate a sum by an integral with a few simple correction terms:

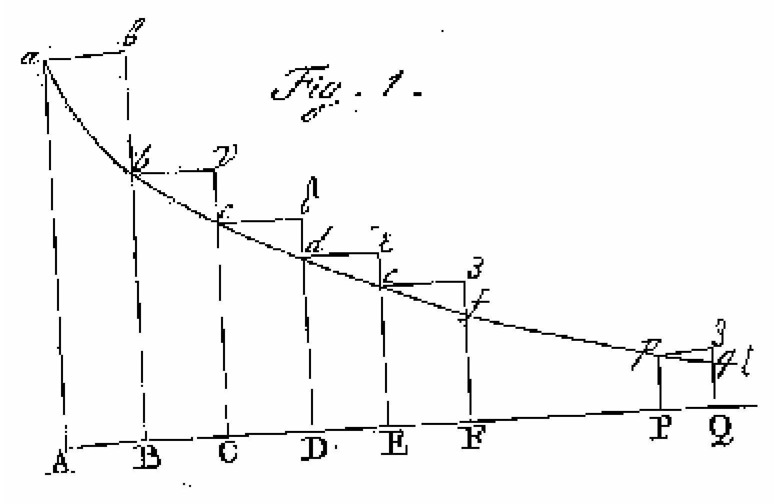
$$\sum_{n=\alpha}^{\beta} f(n) \approx \int_{\alpha}^{\beta} f(x)dx + \frac{f(\beta)+f(\alpha)}{2} + \frac{f(\alpha)-f(\alpha+1)}{12} - \frac{f(\beta)-f(\beta+1)}{12}.$$

Euler does not use this notation. In place of the above he writes

$$a+b+c+.......+x+y = \int ydn + \frac{a+y}{2} + \frac{(a-b)}{12} - \frac{(y-z)}{12}.$$

He does use *n* for the index of summation, but he uses *x* for the term of the sum and not the expression $f(n)$. He uses *s* for the partial sum of the series. The notation for integration is especially confusing, and we will use modern notation in this synopsis.

2. Euler states that he will use the curve shown in Figure 1 to obtain his approximation to the sum, and that he will find upper and lower bounds for this sum.



2 and 3. Euler notes that the series $a+b+c+d+e+etc.$ is the sum of the areas of the rectangles *Aa, Bb, Cc, Dd* etc. shown in Figure 1. We will use the modern notation $f(\alpha)=a$, $f(\alpha+1)=b$, $f(\alpha+2)=c$, …, $f(\beta-1)=p$, $f(\beta)=q$, in place of Euler's *a, b, c, …p, q.* (Euler also uses $f(\beta-1)=x$ and . $f(\beta)=y$. This multiple use of the variables *x* and *y* can be confusing.)

4. Euler describes the curve shown in figure 1. In our notation this is $y=f(x)$, and Euler refers to it as simply *y*.

5. Euler argues the truth of $a+b+c....+x>\int ydn$, which in modern notation is

$$\sum_{n=\alpha}^{\beta-1} f(n) > \int_{\alpha}^{\beta} f(x)dx .$$

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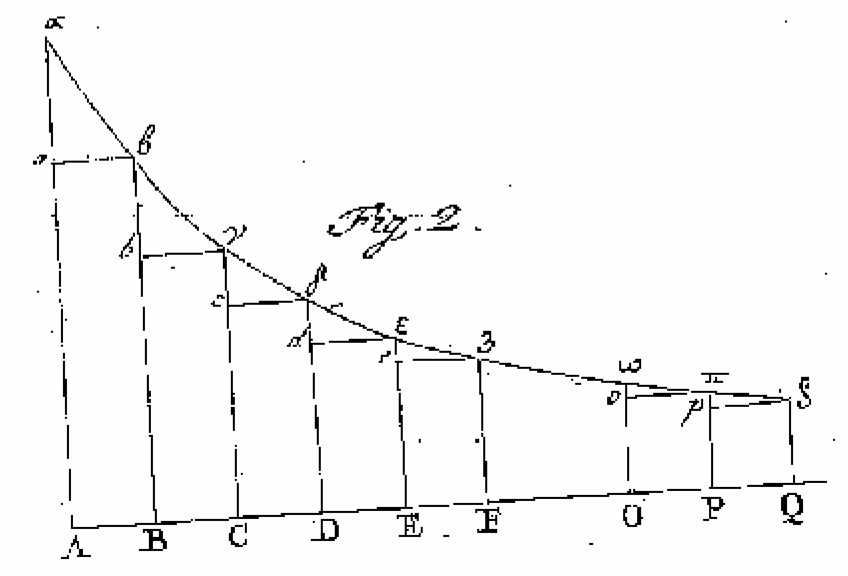


6 and 7. Euler has found a lower limit for his sum, and now he seeks an upper limit. He discusses the curve shown in figure 2, which is the curve shown in figure 1 translated to the right 1 unit. Thus in modern notation this curve is $y = f(x-1)$, and Euler refers to it simply as *x*. Comparing this curve with rectangles we see that

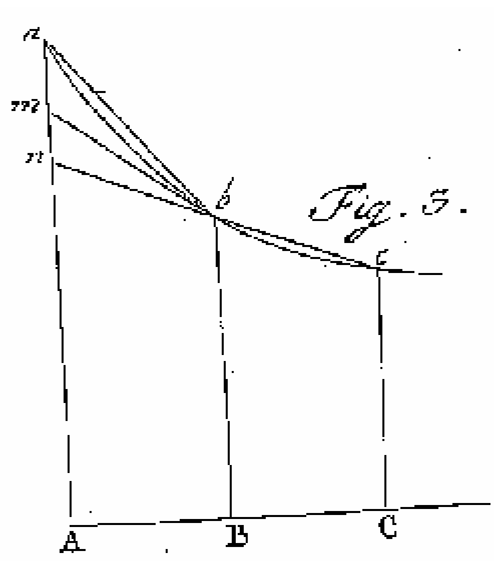
$$\sum_{n=\alpha}^{\beta-1} f(n) < \int_{\alpha}^{\beta} f(x-1)dx,$$

which Euler writes as $a+b+c....+x < \int xdn$. Thus he has found lower and upper limits for his sum.

8. We have obtained close approximations to the sums of these series by neglecting the small curvilinear triangles in each figure. From figure 1, we must add the area of these curvilinear triangles to $\int_{\alpha}^{\beta} f(x)dx$. Each curvilinear triangle is the sum of a rectilinear triangle and a curvilinear segment. The sum of all the rectilinear triangles is $\dfrac{f(\alpha)-f(\beta)}{2}$ which Euler writes as $\dfrac{a-y}{2}$. (Note that he has used *y* rather than *q* for his last ordinate. Why?) He now has the improved lower bound

$$(1) \qquad \int_{\alpha}^{\beta} f(x)dx + \frac{f(\alpha)-f(\beta)}{2} < \sum_{n=\alpha}^{\beta-1} f(n),$$

which he writes as $a+b+c....+x>\int ydn+\frac{a-y}{2}$.



9. Now Euler looks at figure 2 and subtracts the same rectilinear triangles to obtain an upper bound for his sum

$$(2)\qquad \sum_{n=\alpha}^{\beta-1} f(n)<\int_{\alpha}^{\beta} f(x-1)dx-\frac{f(\alpha)-f(\beta)}{2},$$

which he writes as $a+b+c....+x<\int xdn-\frac{a-y}{2}$.

10. Euler states that (1) is a closer approximation than (2). To improve the approximation we must add the areas of the curvilinear segments, the first two of which are shown in his figure 3. He argues, without proof, that the area of the curvilinear segment is approximately one sixth the area of the triangle *abn.* We prove the following lemma in the notes that follow the translation:

Lemma**:** Let the arc *abc* be a segment of a parabola, and let *nbc* be a chord as shown in figure 3. Then the area of the curvilinear segment *aba* equals one sixth the area of the triangle *abn*.

Thus Euler's estimate is based on approximating the curve $y = f(x)$ by a parabola over each rectangle.

11. Using the above lemma Euler estimates the sum of all curvilinear segments to be

$$\frac{f(\alpha)-f(\alpha+1)}{12}-\frac{f(\beta)-f(\beta+1)}{12},$$

which he writes as $\frac{a-b}{12}-\frac{y-z}{12}$. Thus he obtains

(3) $$\sum_{n=\alpha}^{\beta} f(n) \approx \int_{\alpha}^{\beta} f(x)dx + \frac{f(\beta)+f(\alpha)}{2} + \frac{f(\alpha)-f(\alpha+1)}{12} - \frac{f(\beta)-f(\beta+1)}{12}.$$

12. Euler states his strategy for the successful use of (3) To sum a series $\sum_{n=n_o}^{\beta} f(n)$, he will first sum *exactly* from $n = n_0$ to $n = \alpha - 1$, then use (3) to *approximate* the sum of the remainder of the series. He also notes that for an infinte series (3) simplifies to

(4) $$\sum_{n=\alpha}^{\infty} f(n) \approx \int_{\alpha}^{\infty} f(x)dx + \frac{7f(\alpha)}{12} - \frac{f(\alpha+1)}{12}.$$

13. Euler gives his first example. He sums the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{1000000}$.

To this end he first sums exactly $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{10} = 2.928968$. He then uses (3) to estimate the remainder $\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\cdots+\frac{1}{1000000} = 11.463701643751678813$. (He does not display this last sum.) His final result is 14.392669.

14. In this final section Euler estimates $\sum_{n=0}^{\infty}\frac{1}{n^2}$. Again he sums the first ten terms of the series exactly, then uses (4) to sum the remainder of the series. His final result is 1.644920. Since Euler does not end by comparing this approximation with his discovery of the exact result

$$\sum_{n=0}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} = 1.6449340668482264365,$$

it is easy to suspect that this paper was written before he found this closed form. However, we have no definitive proof that this is true. Ed Sandifer tells us that ” Euler wrote both E-46 (finding approximate sums) and E-41 (Basel problem) in 1735. It's a little hard to know which came first, and there's a good chance he was working on both of them at the same time.”