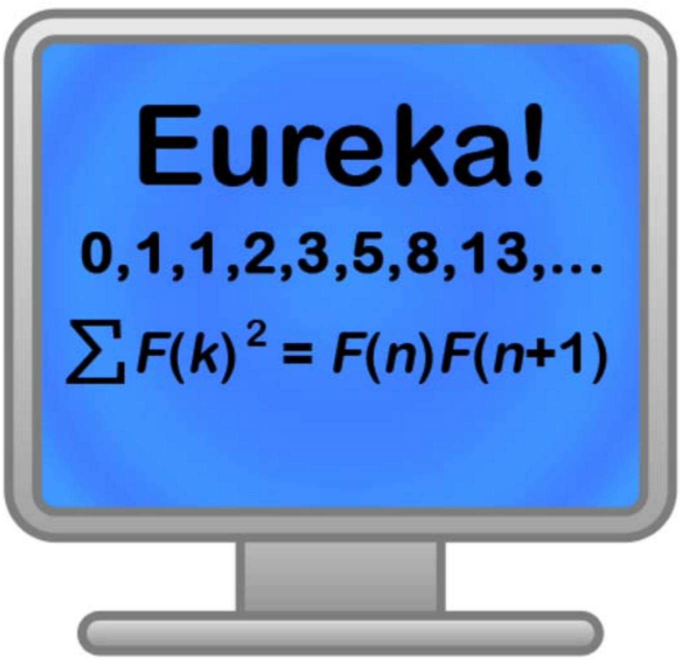
Mathematics by Experiment:

Exploring Patterns of Integer Sequences

Based on *Mathematica* 8.0



Hieu D. Nguyen

Table of Contents

Chapter 3. Classical Numerical Experiments

Chapter 4. Modern Numerical Experiments

Chapter 5. Projects to Explore on Your Own

Preface

Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.

W. S. Anglin, "Mathematics and History", Mathematical Intelligencer, v. 4, no. 4.

Welcome to the jungle! This book is an attempt to excite the reader about the beauty of mathematics through exploration and experimentation with number patterns arising from integer sequences. It seeks to demonstrate how these fascinating patterns can be discovered with the help technology, in particular computer algebra systems such as Stephen Wolfram's *Mathematica* and Internet resources such as Neil Sloan's Online Encyclopedia of Integer Sequences (OEIS). Those who find joy in playing with integer sequences and hunting for formulas experimentally by computer will hopefully find this book worthwhile to read.

More concretely, this book seeks to expose the reader to a wide range of number patterns that can be extracted from experimental data and to intuitively convince (or deceive at times) the reader of their validity before subjecting them to rigorous proof. This requires very good bookkeeping and presentation of numerical data. Thus, the author has put considerable effort into grooming numerical data so that patterns become clear to reader and formulas can be easily conjectured, or if no pattern is evident, then use tools at our disposable to systematically determine if such a formula exists.

Mathematics by Experiment

Mathematics is not a deductive science -- that's a cliche. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork.

Paul R. Halmos, I Want to be a Mathematician, Washington: MAA Spectrum, 1985.

According to the author it seems that little emphasis is placed on exploration and discovery in mathematics education today. No systematic approach is formally taught to our students, who are typically asked to solve problems that have been so well paved or narrowly constructed that there is little room for creativity and experimentation. If students are merely taught to prove theorems, then who will be the explorers of new conjectures?

Mathematical experimentation has largely been overshadowed by mathematical proof. Certainly, developing students' ability to understand and formulate mathematical arguments is a worthy goal in any mathematics course; unfortunately, many students easily get frustrated when they are first exposed to mathematical proofs and asked to construct them independently on their own. The road to mathematical "nirvana" can be long arduous for these students. However, by instilling in them a deep appreciation and ownership of the material through exploration and experimentation, the author believes that students will be much more motivated to tackle mathematical proofs. This issue not only affects college students majoring in mathematics, but more broadly and more importantly, it also affects liberal arts students and high school students alike.

Experimental Mathematics

Fortunately, there is a new movement to regain the flavor of mathematical exploration and experimentation by computer as championed by Jonathan Borwein and David Bailey in their series of books on experimental mathematics [References]. Of course, the intuitive or investigative approach to learning and discovering mathematics is not new. Some middle-school text-books already emphasize such an approach, for example the Connected Mathematics curriculum, and are divided into units consisting of “Investigations”. However, the availability of powerful computers, advanced software such as computer algebra systems, and online mathematical databases to perform numerical experiments and search for deep patterns has given birth to the digital world of experimental mathematics.

It should be emphasized that experimental mathematics is not the same as statistical data analysis or data mining although they do share some similarities and overlap in certain topics. To this end, Borwein and Bailey give an accurate working definition of experimental mathematics in [BB]:

> Experimental mathematics is the methodology of doing mathematics that includes the use of computations for:
>
> 1. Gaining insight and intuition.
> 2. Discovering new patterns and relationships.
> 3. Using graphical displays to suggest underlying mathematical principles.
> 4. Testing and especially falsifying conjectures.
> 5. Exploring a possible result to see if it is worth formal proof.
> 6. Suggesting approaches for formal proof.
> 7. Replacing lengthy hand derivations with computer-based derivations.
> 8. Confirming analytically derived results.

Thus, experimental mathematics refers to an approach of studying mathematics where a field can be effectively studied using advanced computing technology such as computer algebra systems. This approach developed and grown so quickly as the present. *Number crunching* is what computers do best; therefore, any field that requires significant computation or symbolic manipulation has or will be touched by experimental mathematics. However, it has not always been this way, as observed by Borwein and Bailey in [BB]:

> *One of the greatest ironies of the information technology revolution is that while the computer was conceived and born in the field of pure mathematics, through the genius of the giants such as John von Neumann and Alan Turing, until recently this marvelous technology had only a minor impact within the field that gave it birth.*

Aims of This Book

This book provides an introduction to elementary techniques for analyzing integer sequences and extracting patterns from them. The well-trained student of mathematics will be able to recognize many of the number patterns appearing in this book. Yet the author hopes that the wide variety of number patterns will bring a certain amount of joy to all readers regardless of their mathematical backgrounds. However, to truly appreciate the full range of techniques discussed in this book, the reader should have a basic understanding of algorithms and some experience with computer programming and computer algebra systems.

The best way to teach an experimental approach to mathematics is by way of examples; thus, the reader will find this book to be full of them. In fact, examples, exercises, and projects form the core material for this book, which was written to be a resource for high school teachers and college professors who teach discrete math courses, general education math courses, or any math course where integer sequences make an appearance.

The paper edition of this book is intended for those who prefer to cozy up under a blanket on a cold winter night and read it at their own leisure. However, the author hopes that this edition will then pique your interest in browsing through the interactive *Mathematica* edition, which contains code for all of the *Mathematica* subroutines used to generate the data sets and integer sequences discussed in this book, so that you may experiment and conjure up your own number patterns. Regardless of the edition, what’s important is to follow Paul Halmo’s famous battle cry in his ‘automathography’ (a mathematical autobiography), *I Want to be a Mathematician* (see [Ha]):

> *Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?*

Now that we’ve described what this book is about, let us say a few words about what it is not about (of course this can be deduced mathematically by taking the complement of what we’ve already mentioned, but to avoid claims of false advertisement, it's best to be redundant). First of all, this book is not intended for the expert mathematician looking to learn the most advanced techniques for analyzing integer sequences. On the contrary, this book discusses only elementary techniques and does not address those such as the partial sums least squares (PSLQ) algorithm (a generalization of Euclid's algorithm useful for confirming whether a given numerical value expressed with high precision is a rational linear combination of known mathematical constants) or the Wilf-Zeilberger algorithm (a method for finding closed formulas of hypergeometric sums). Moreover, statistical and data mining techniques will not be found in this book, although in the last chapter we discuss a data mining approach to discovering new mathematical identities by perfoming a computer-automated analysis of integer sequences.

DISCLAIMER: this book is certainly not intended to be an exhaustive collection of integer sequences comparable to the Encyclopedia of Integer Sequences ([]). The online version, OEIS, is an extremely useful database for identifying whether a given sequence has already been studied and to provide formulas for generating it and references to known results. However, the author believes that the next step to improving OEIS is to categorize all known number patterns involving an integer sequence, or more systematically, to develop tools and techniques for generating them automatically, without sending the user on a literature hunt for them. For example, given the Fibonacci sequence $\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, ...\}$, it would be useful to not only identify it as such and provide formulas, either as a recurrence $F_{n+1} = F_n + F_{n-1}$ or explicitly as $F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n \sqrt{5}}$ (Binet's formula), but also to *automatically* generate other related identities, such as $\sum_{k=0}^{n} F_k = F_{n+2} - 1$, $\sum_{m=1}^{n} F_k^2 = F_n F_{n+1}$, and $L_n = F_n + 2 F_{n-1}$, where $\{L_n\}$ is the Lucas sequence, a cousin of the Fibonacci sequence that satisfies the same recurrence but begins with the values $L_0 = 2$ and $L_1 = 1$. What is needed then is a study of certain relations common to many integer sequences and program them for detection.

Thus, the last chapter in this book represents an attempt towards this next step, namely to employ a CAS such as *Mathematica* to automatically generate formulas and identities from a given set of integer sequences and to employ databases such as OEIS to recognize them. We believe the perfect technological storm has arrived to achieve through a convergence of powerful computers, large online databases, and sophisticated computer algebra systems. These advances will allow us to leapfrog from merely identifying a single integer sequence to data mining an entire database of integer sequences to reveal all the number patterns and mathematical identities that lie inside it.

Acknowledgements

In writing this book the author was inspired by Robert M. Young's classic text, *Excusions in Calculus: An Interplay of the Continuous and the Discrete* (MAA, 1992), to foster the same spirit of mathematical exploration by which one is guided by George Polya's craft of discovery and the insights of the great masters. Thus, this book seeks by analogy to provide a 21st-century adaptation of Young's work by describing the interplay between the analog (human mind) and the digital (computer) in the exploration of integer sequences and their number patterns.

Jonathan Borwein and David Bailey's series of books on experimental mathematics (see[BBC], [BB], [BBG], [DB]) greatly influenced the author's experimental approach to writing this book.

Neil Sloan's Online Encyclopedia of Integer Sequences (OEIS) has been an invaluable resource. Its overwhelming collection of integer sequences has allowed the author to cherry-pick some of the most fascinating integer sequences to share with the reader.

Many of the examples discussed in this book were taken from a variety of expository math journals, such as those published by the Mathematical Association of America (MAA) and The Fibonaaci Quarterly. Other classic textbooks that have been helpful and should be mentioned include *Concrete Mathematics: A Foundation for Computer Science* (Ronald Graham, Donald Knuth, and Oren Patashnik), *The Art of Computer Programming* (Donald Knuth), and *Unsolved Problems in Number Theory* (Richard Guy).

Lastly, I would like to thank my young daughter, Ai Lan, for her help in typing the title of this book while sitting on her father's lap. There are few simple joys in life more memorable than this.

Mathematica Notes

This book was completely typeset using *Mathematica* 8.0, including all the computations and data that appear in it. Why choose *Mathematica* instead of other computer algebra systems (CAS) that are available, such as Maple, Matlab, and Sage? In the author's opinion all CAS are essentially comparable in terms of features and their library of mathematical functions. Each has its advantages and disadvantages, a debate that the author will not touch on, but refer the reader to Internet discussion boards. The author has chosen *Mathematica* primarily because it is the CAS of choice at his institution and one that he is most comfortable with, having used it for the last fifteen years. Most of the examples presented in this textbook can be easily ported over to other CAS. The point of this book is to demonstrate that modern computer algebra systems such as *Mathematica* are powerful experimental tools for investigating number patterns of integer sequences.

There are two types of *Mathematica* commands that the reader should be aware of:

Built-In Commands: These commands are built into *Mathematica*. Some commands require that the add-on *Mathematica* package *Combinatorica* be loaded. This is automatically done when one loads the *Mathematica* package MathematicsbyExperiment.m created solely for this book, which can also be downloaded from the book's website at:

http://www.rowan.edu/colleges/las/departments/math/facultystaff/nguyen/experimentalmath/index.html

It is recommended that you save this file to the same location as the *Mathematica* notebook containing the online edition of this book. Then choose 'Yes' to evaluate all initialization cells when you first evaluate an input cell in this notebook.

In-House Commands: These commands were programmed by the author. Code for these commands (written as *Mathematica* modules) are contained in the same *Mathematica* package, MathematicsbyExperiment.m. Follow the same instructions as above.

1

The Joy of Number Patterns

Euler's work reminds us of what we should strive for in the teaching of our students and ourselves--to observe, to experiment, to search for patterns, and above all, to rejoice not just in the final rigor, but in that wondrous process of discovery that must always precede it.

Robert M. Young, *Excursions in Calculus*, 1992.

1.1 What's the Pattern?

What person isn't fascinated by patterns in nature? The joy of recognizing patterns begins at an early age when as toddlers we first learn to recognize patterns in music, words, and of course numbers. Of course there many different types of patterns; however, this book celebrates the joy of discovering number patterns of integer sequences. All of us at some point have played the game where we were shown a collection of numbers and asked "What's the pattern?" If not, then this book is your opportunity to play with number patterns and of course learn some fascinating mathematics along the way. Here's a quickie: 0, 1, 1, 2, 3, 5, 8, 13. What's the pattern? What's the next number in the sequence?

A more fundamental question than "What's the pattern?" is "What *is* a pattern?". Niles Eldredge, the world reknown paleontologist and expert on evolutionary theory, posed the following metaphysical question in the his book, *The Pattern of Evolution* ([El]):

> *Do patterns just 'exist' in nature and simply force themselves on our consciousness? Or is it true that we have to learn to see--to become receptive to the pattern before we can even notice it, much less make some sense of it? If we have to learn, what sort of factors enter into the process?*

Regarding *number* patterns, the answer to Eldredge's question is clear. Most scientists have a platonic view of mathematics; thus, they believe that number patterns exist regardless of perception. This issue is not important for us. More important is the fact that we humans have the ability to detect number patterns at all, and history seems to suggest that our ability to so is limitless. For example, there have been mathematicians in the past who have shown the uncanny ability to recognize some deep number patterns. Of course, the computer has become a powerful tool to help us detect patterns. This book aims to share with the reader some of the basic tools and techniques available to analyze data and search for number patterns within integer sequences. It takes a slightly more serious approach to number patterns than brainteaser books intended for the general reader, but yet encourages playfulness and curiosity in hopes that the reader will delve through its pages and explore some of the patterns on his or her own.

1.1.1 Patterns Within Patterns

There is much more joy to be found beyond merely recognizing a number pattern. The story become much more fascinating when we learn about how each pattern arises (especially from multiple contexts), how each pattern may contain other patterns within it, and the clues by which these patterns reveal themselves to us. Number patterns take on different forms depending on the mathematical objects involved. In this book we primarily deal with integer sequences. Thus, we explore number patterns that can be described by algebraic formulas, recurrences, and identities. This book aims to tell the interesting story behind some fascinating patterns of integer sequences, many well known and some not so well known, and to reveal insights that will help us lead to their detection.

1.1.2 Computer Exploration

Computers have become so powerful that they can solve a wide range of problems involving intensive or high-precision computation. Of course the solutions obtained by computers in many cases are not rigorous and demonstrate only existence. Proof of uniqueness may then require human intervention. Also, some will argue that computer solutions yield little insight into the problem, especially if they were obtained by brute force methods. This is a valid point; however, computer solutions should not be viewed as the end-game of a problem, but the beginning. In fact, they can provide clues to help us gain insight, something that the author hopes to forcefully demonstrate through the plethora of examples in this book.

1.2 The Scientific Method (for Mathematicians)

God forbid that Truth should be confined to Mathematical Demonstration!

Blake
Notes on Reynold's Discourses, c. 1808.

Mathematics plays an important role in the sciences, a notion that has been firmly established since the Scientific Revolution. Much has been written about the impact of mathematics on the sciences. However, the reverse impact has received far less attention. One aspect that the author believes where modern science can help mathematics is through the former’s most basic creed, the Scientific Method. Ask a random sample of well educated people to explain the Scientific Method as it was taught to them school and some might remember it correctly as a systematic approach to performing scientific research. Then ask the same group if they recall such a method for mathematics. Most likely there will be little or no response. This is because no such method is taught let alone emphasized throughout their childhood instruction of mathematics. Why is it that scientific experimentation is systemically taught to children in schools at an early age while mathematical experimentation is largely ignored? The instinct for mathematical exploration remains undeveloped and fails to see light in many children's minds even as they reach adulthood.

1.2.1 The Scientific Method

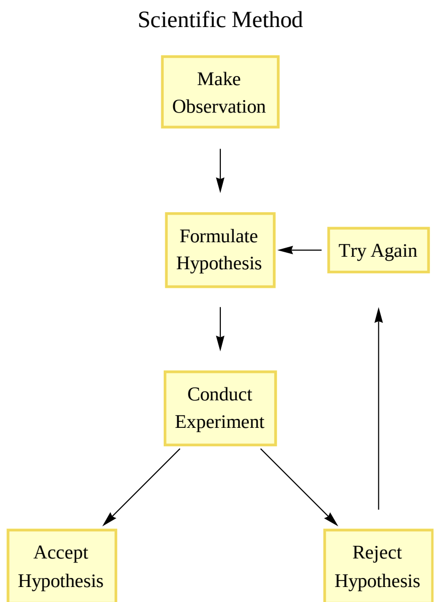
The Scientific Method outlines a systematic set of principles for investigating physical phenomena through experimentation:

1. Observation

2. Hypothesis - Questions are posed of the observations which lead to a hypothesis for explaining the observation.

3. Testing - A controlled experiment is conducted to test the hypothesis.

The flow chart below demonstrates how one typically follows the Scientific Method:

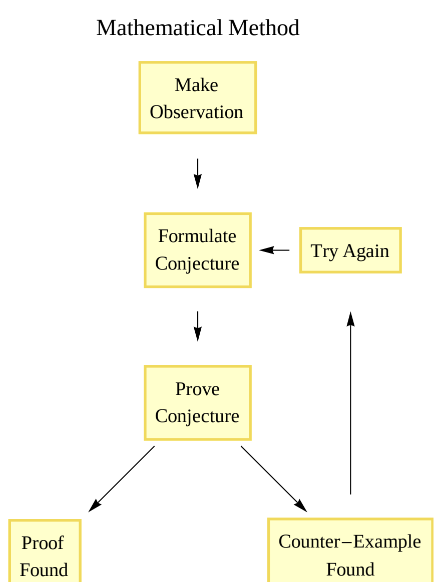


1.2.2 The Mathematical Method

Mathematics should promulgate its own "scientific" method, which by analogy we shall refer to as the Mathematical Method. Unfortunately, it is less well known and rarely emphasized in education in comparison to the Scientific Method. Based on the Lakatos viewpoint [Ref], the Mathematical Method is similar to the Scientific Method except that it replaces testing with proof:

1. Observation
2. Conjecture
3. Proof

Here is the corresponding flow chart:



The emphasis on rigorous proof is what sets mathematics apart from the sciences, the gold standard by which every theorem is measured by. While mathematical proof has received much attention in undergraduate education, mathematical experimentation has been largely ignored in all mathematics courses. Math instructors spend little time on the discovery aspects of the material that they teach; most of their time is spent on proof demonstrations. While the author strongly believes in the importance of proof, he also believes in the power of discovery to motivate students.

1.2.3 The Process of Mathematical Discovery

The stereotype is that only professional mathematicians are in a position to discover new mathematical results. In fact, we argue that in this day and age it is easier for students to make discoveries in mathematics than it is in the sciences. New discoveries in fields such as physics, chemistry and biology today require either advanced scientific knowledge, large collaborations and/or large budgets. On the other hand, an undergraduate student can make discoveries in certain branches of mathematics by merely using a personal computer equipped with a computer algebra system.

This book aims to help shift the discussion more on the process of mathematical discovery and address the question: how should mathematical students go about making observations and asking the right questions to formulate a conjecture? Of course, this book focuses only on those areas of mathematics that are currently amenable to computation and experimentation.

In their paper [MT], C. L. Mallows and J. W. Tukey describe how success in exploratory investigation is crucially dependent on:

i) a willingness to collect and study the data,

ii) use of diagnostic techniques to show the unexpected,

iii) an ability to recognize striking patterns,

iv) enough understanding of the context of the problem to enable these patterns to be recognized as potentially meaningful,

v) avoidance of precipitate committment to models of clearly inadequate complexity, and

vi) energetic follow up of the clues obtained

These aspects demonstrate that the computer alone is not enough to solve problems; human intuition and insight still play a vital role in sucessfully leading the mathematical explorer on the path to discovery.

The real danger is not that computers will begin to think like men, but that men will begin to think like computers.

Harris, Sydney J. (In H. Eves, Return to Mathematical Circles, Boston: Prindle, Weber and Schmidt, 1988)

1.3 The Great Bookkeepers

Like a good bookkeeper, the mathematician should make a serious effort to record his data in painstaking detail and to organize his data in such a manner that allows a pattern to be easily revealed through careful observation. Some of the greatest mathematicians have been excellent bookkeepers such as John Wallis, Isaac Newton and Carl Friedrich Gauss. A fine example of Gauss' outstanding bookkeeping is his calculation of the frequency of primes among the first 50,000 positive integers in intervals of 1000, which lead to his discovery of the Prime Number Theorem. In his article *A History of the Prime Number Theorem*, L. J. Goldstein describes Gauss' achievement (see [Go]):

> *Gauss' calculations are awesome to contemplate, since they were done long before the days of high-speed computers. Gauss' persistence is most impressive... Modern students of mathematics should take note of the great care with which data was compiled by such giants as Gauss. Conjectures in those days were rarely idle guesses. They were usually supported by piles of laboriously gathered evidence.*

However, the greatest bookkeeper of them all must undoubtedly be Leonard Euler. In his textbook *Induction and Analogy in Mathematics*, George Polya describes Euler regarding his nature into making mathematical discoveries (see [Ref]):

> *A master of inductive research in mathematics, he made important discoveries (on infinite series, in the Theory of Numbers, and in other branches of mathematics) by induction, that is, by observation, daring guess, and shrewd verification. In this respect, however, Euler is not unique; other mathematicians,*

> *great and small, used induction extensively in their work.*
>
> *Yet Euler seems to me almost unique in one aspect: he takes pains to present the relevant inductive evidence carefully, in detail, in good order. He presents it convincingly but honestly, as a genuine scientist should do. He presentation is "the candid exposition" of the ideas that led him to those discoveries" and has a distinctive charm. Naturally enough, as any other author, he tries to impress his readers, but, as a really good author, he tries to impress his readers only by such things as have genuinely impressed himself.*

In other words Euler had talent for good bookkeeping. His discovery of the recurrence for the sum of divisors functions and its connection to the Pentagonal Number Theorem is a classic example of first-rate bookkeeping.

In addition to good bookkeeping, the ability (or patience) to compute mathematical values to great accuracy is another trademark of great mathematicians. Their computations were amazingly done by hand with mere pencil and paper. Of course these tools have been replaced by modern calculators and computers, which has allowed even students to compute values quickly and with extremely high precision.

Each chapter in this book ends with a mathematical story of a great bookkeeper, including the four mentioned above. The author hopes that the reader will enjoy these stories, revel in their achievements, and be inspired as the author was to explore the mathematical jungle of integer sequences.

Leonard Euler: Sum of Divisors and the Pentagonal Number Theorem

Leonard Euler (1707 – 1783)

http://www-history.mcs.st-and.ac.uk/Mathematicians/Euler.html

Leonard Euler is one of the greatest mathematicians of all time and certainly the most prolific. His works span over 50 years and are accessible online at the Euler Archive (http://www.math.dartmouth.edu/~euler/). Among his many achievements, Euler's application of the Pentagonal number theorem to the divisor function stands out as a mathematical gem, which he recounts in his paper *Discovery of a most extraordinary law of the numbers concerning the sum of their divisors*, written in French and published in 1751 (see [Eu]). We retell the story based on Young's treatment in [Yo], which includes excerpts of Polya's translation of Euler's paper into English (see [Po]).

First studied by Euler, the *sum of divisors* (or *sigma*) function is one of the most interesting mathematical objects in number theory. Define $\sigma(n)$ to be sum of the divisors of n, i.e., $\sigma(n) = \sum_{d|n} d$. For example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Here is a table listing values for $\sigma(n)$ for $n = 1, \ldots, 100$:

Sum of Divisors σ(n)

n	σ(n)	n	σ(n)	n	σ(n)	n	σ(n)	n	σ(n)
1	1	11	12	21	32	31	32	41	42
2	3	12	28	22	36	32	63	42	96
3	4	13	14	23	24	33	48	43	44
4	7	14	24	24	60	34	54	44	84
5	6	15	24	25	31	35	48	45	78
6	12	16	31	26	42	36	91	46	72
7	8	17	18	27	40	37	38	47	48
8	15	18	39	28	56	38	60	48	124
9	13	19	20	29	30	39	56	49	57
10	18	20	42	30	72	40	90	50	93

The tables above show no obvious pattern for the values of $\sigma(n)$, even for n prime, and leads Euler to give the following bleak diagnosis:

> *If we examine a little the sequence of these numbers, we are almost driven to despair. We cannot hope to discover the least order. The irregularity of the primes is so deeply involved in it that we must think it impossible to disentangle any law governing this sequence, unless we know the sequence of the primes itself. It could appear even that the sequence before us is still more mysterious than the sequence of the primes.*

However, after such dire words, Euler, like a mathemagician, reveals that he has cracked the code to the sigma function:

> *Nevertheless, I observed that this sequence is subject to a completely definite law and could even be regarded as a recurring sequence. This mathematical expression means that each term can be computed from the foregoing terms, according to an invariable rule.*

To imitate Euler's discovery, we begin by exploring recursive patterns between consecutive elements $\sigma(n)$, $\sigma(n-1)$, and $\sigma(n-2)$:

n	σ(n)	σ(n–1)	σ(n–2)
1	1	--	--
2	3	1	--
3	4	3	1
4	7	4	3
5	6	7	4
6	12	6	7
7	8	12	6
8	15	8	12
9	13	15	8
10	18	13	15

Immediately, we see the following sum relations:

$\sigma(3) = 4 = 3 + 1 = \sigma(2) + \sigma(1)$

$\sigma(4) = 7 = 4 + 3 = \sigma(3) + \sigma(2)$

Unfortunately, this pattern fails to hold for $\sigma(5)$ through $\sigma(10)$. When a pattern fails, it's useful to determine the way in which it failed in hopes of adapting the pattern as Euler did. In this case we find that

$\sigma(5) = 6 = 7 + 4 - 5 = \sigma(4) + \sigma(3) - 5$

Does this new pattern hold for $\sigma(6)$ by subtracting $\sigma(5) + \sigma(4)$ by 6? Unfortunately, no. However, what we find is that

$\sigma(6) = 12 = 6 + 7 - 1 = \sigma(5) + \sigma(4) - \sigma(1)$

where we have interpreted the subtraction of 1 as subtraction by $\sigma(1)$.

What about $\sigma(7)$? In this case we observe that

$\sigma(7) = 8 = 12 + 6 - 10 = \sigma(6) + \sigma(5) - 10$

How can we interpret the subtraction by 10? It doesn't seem to follow either of the previous patterns of subtraction by 7 or subtraction by $\sigma(2) = 3$? Aha! What about subtraction by both terms:

$\sigma(7) = 8 = 12 + 6 - 3 - 7 = \sigma(6) + \sigma(5) - \sigma(2) - 7$

Examining further values, we discover that

$\sigma(8) = 15 = 8 + 12 - 4 - 1 = \sigma(7) + \sigma(6) - \sigma(3) - \sigma(1)$

$\sigma(9) = 13 = 15 + 8 - 7 - 3 = \sigma(8) + \sigma(7) - \sigma(4) - \sigma(2)$

$\sigma(10) = 18 = 13 + 15 - 6 - 4 = \sigma(9) + \sigma(8) - \sigma(5) - \sigma(3)$

We now seem to be on to something, but in order to confirm this emerging pattern we should experiment on a larger data set that includes more columns of previous values of σ:

n	σ(n)	σ(n–1)	σ(n–2)	σ(n–5)	σ(n–7)
10	18	13	15	6	4
11	12	18	13	12	7
12	28	12	18	8	6
13	14	28	12	15	12
14	24	14	28	13	8
15	24	24	14	18	15

Further experimentation shows that the sigma function obeys a remarkable recurrence, which can be inferred from the table below:

$\sigma(1) = 1$

$\sigma(2) = \sigma(1) + 2$

$\sigma(3) = \sigma(2) + \sigma(1) = 3 + 1 = 4$

$\sigma(4) = \sigma(3) + \sigma(2) = 4 + 3 = 7$

$\sigma(5) = \sigma(4) + \sigma(3) - 5 = 7 + 4 - 5 = 6$

$\sigma(6) = \sigma(5) + \sigma(4) - \sigma(1) = 6 + 7 - 1 = 12$

$\sigma(7) = \sigma(6) + \sigma(5) - \sigma(2) - 7 = 12 + 6 - 3 - 7 = 8$

$\sigma(8) = \sigma(7) + \sigma(6) - \sigma(3) - \sigma(1) = 8 + 12 - 4 - 1 = 15$

$\sigma(9) = \sigma(8) + \sigma(7) - \sigma(4) - \sigma(2) = 15 + 8 - 7 - 3 = 13$

$\sigma(10) = \sigma(9) + \sigma(8) - \sigma(5) - \sigma(3) = 13 + 15 - 6 - 4 = 18$

$\sigma(11) = \sigma(10) + \sigma(9) - \sigma(6) - \sigma(4) = 18 + 13 - 12 - 7 = 12$

$\sigma(12) = \sigma(11) + \sigma(10) - \sigma(7) - \sigma(5) + 12 = 12 + 18 - 8 - 6 + 12 = 28$

$\sigma(13) = \sigma(12) + \sigma(11) - \sigma(8) - \sigma(6) + \sigma(1) = 28 + 12 - 15 - 12 + 1 = 14$

$\sigma(14) = \sigma(13) + \sigma(12) - \sigma(9) - \sigma(7) + \sigma(2) = 14 + 28 - 13 - 8 + 3 = 24$

$\sigma(15) = \sigma(14) + \sigma(13) - \sigma(10) - \sigma(8) + \sigma(3) + 15 = 24 + 14 - 18 - 15 + 4 + 15$

Euler actually writes out additional examples, in fact up to $\sigma(20)$, and comments “I think these examples are sufficient to discourage anyone from imagining that it is by mere chance that my rule is in agreement with the truth”. So what is the rule that Euler is referring to here? Clearly it involves the sequence of values $\{1,2,5,7,12,15,...\}$. In fact, this sequence complete determines the recurrence pattern for computing $\sigma(n)$ as follows:

1. $\sigma(n)$ equals the sum of the previous terms $\sigma(n-1)$, $\sigma(n-2)$, $\sigma(n-5)$, $\sigma(n-7)$, etc., whose signs alternate every second term.

2. If n equals one of the values $\{1,2,5,7,12,15,...\}$, then this value is also added to (or subtracted from) $\sigma(n)$ according to the same sign convention.

Is there a pattern to the sequence $\{1,2,5,7,12,15,...\}$? As Euler observes, this sequence is "a mixture of two sequences with a regular law", namely, the odd terms 1, 5, 12, ..., are in fact pentagonal numbers (discussed in Chapter 3) and follow the formula $n(3n-1)/2$, whereas the even terms 2, 7, 15, ..., follow the formula $n(3n+1)/2$ (Euler had written down such formulas in an earlier work). Together, both the odd and even terms form the generalized pentagonal numbers:

$$p(n) = \begin{cases} n(3n-1)/2, & \text{if } n \text{ is odd} \\ n(3n+1)/2, & \text{if } n \text{ is even} \end{cases} \tag{1.1}$$

The following is Euler’s remarkable explanation on how he recognized the divisor sequence:

> *I confess that I did not hit on this discovery by mere chance, but another proposition opened the path to this beautiful property...In considering the partition of numbers, I examined, a long time ago, the expression*
>
> $$(1-x)\left(1-x^2\right)\left(1-x^3\right)\left(1-x^4\right)\left(1-x^5\right)\left(1-x^6\right)\left(1-x^7\right)\left(1-x^8\right)\cdots$$
>
> *in which the product is assumed to be infinite. In order to see what kind of series will result, I multiplied actually a great number of factors and found*
>
> $$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40}\cdots$$
>
> *The exponents of x are the same which enter into the above formulas; also the signs + and - arise twice in succession.*

This result known as the Pentagonal Number Theorem. Thus, not only did Euler discover a wonderful pattern involving the sum of divisors function, but made a profound connection with a generating function for the generalized pentagonal numbers defined $p(n)$:

Theorem (Sum of Divisors): Let n be a positive integer and p(K) be the largest generalized pentagonal number less than n. Then $\sigma(n)$ satisfies the recurrence

$$\sigma(n) = \sum_{k=1}^{K} (-1)^{\lfloor (k-1)/2 \rfloor} \sigma(n - p(k)) + d(n) \tag{1.2}$$

where $d(n) = \begin{cases} (-1)^{\lfloor (N-1)/2 \rfloor} n, & \text{if } n = p(N) \\ 0, & \text{otherwise} \end{cases}$.

What a remarkable discovery!

2

Exploring Patterns of Integer Sequences

If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.

Kurt Godel, *Some Basic Theorems on the Foundations*, 1951

The Fibonacci sequence $\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$ is one of the most recognized integer sequences in the world (recorded as entry A000045 in the Online Encyclopedia of Integer Sequences (OEIS)). It is named after Fibonacci (also known as Leonardo of Pisa), who first wrote on such numbers in his most famous work, the Liber Abaci (Book of Calculations), published in 1202 AD. However, there is evidence that indicates Fibonacci numbers were studied as early as 700 AD by Indian mathematicians (see [Fi]). It is the only sequence to sport its own publication, the 48-year old journal *The Fibonacci Quarterly* (http://www.fq.math.ca/), which publishes mathematical results having connections to the Fibonacci sequence.

The Fibonacci sequence appears in the Liber Abaci [Fi] as a solution to a counting problem involving a population of rabbits: How Many Pairs of Rabbits Are Created by One Pair in One Year.

> *A certain man had one pair of rabbits in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also. Because the abovewritten pair in the first month bore, you will double it; there will be two pairs in one month. One of these, namely the first, bears in the second month, and thus there are in the second month 3 pairs; of these in one month two are pregnant, and in the third month 2 pairs of rabbits are born, and thus there are 5 pairs in the month; in this month 3 pairs are pregnant, and in the fourth month there are 8 pairs, of which 5 pairs bear another 5 pairs; these are added to the 8 pairs making 13 pairs in the fifth month; those 5 pairs that are born in this month do not mate in this month, but another 8 pairs are pregnant, and thus there are in the sixth month 21 pairs; to these are added 13 pairs that are born in the seventh month; there will be 34 pairs in this month;...there will be 377 pairs, and this many pairs are produced from the abovewritten pair in the mentioned place at the end of one year.*
>
> *You can see in the margin* [see below] *how we operated, namely that we added the first number to the second, namely the 1 to the 2, and the second to the third, and the third to the fourth, and the fourth to the fifth, and thus one after another until we added the tenth to the eleventh, namely the 144 to the 233, and we had the abovewritten sum of rabbits, namely 377, and thus you can in order to find it for an unending number of months.*

beginning	*1*
first	*2*
second	*3*

third	*5*
fourth	*8*
fifth	*13*
sixth	*21*
seventh	*34*
eighth	*55*
ninth	*89*
tenth	*144*
eleventh	*233*
end	*377*

Thus, Fibonacci recognized the simple recurrence pattern, $F_{n+1} = F_n + F_{n-1}$, that governs these numbers. And yet from simple recurrence, thousands of number patterns, formulas and identities involving the Fibonacci sequence have been discovered, including the well-known identity discovered by Cassini,

$$F_{n-1}\, F_{n+1} - F_n^2 = (-1)^n \tag{2.1}$$

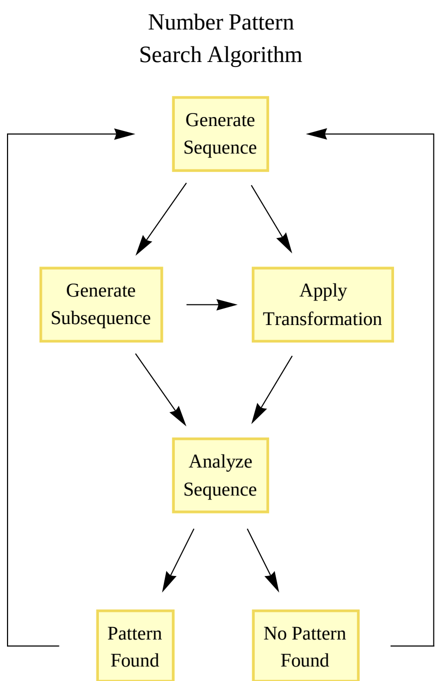
and the remarkable Millin series,

$$\sum_{n=0}^{\infty} \frac{1}{F_{2\,n}} = \frac{7-\sqrt{5}}{2} \tag{2.2}$$

Even to this day, the Fibonacci sequence continues to fascinate mathematicians and be a fruitful subject of investigation.

So how doe one join the hunt to experimentally discover patterns of integer sequences such as the Fibonacci sequence? As a start, given a sequence $\{a_n\}$, one should generate enough data, say be calculating a sufficient number of terms of the sequence, to allow a number pattern to emerge. If necessary, massage the data and systematically present it in such a way that either an explicit formula, recurrence, or algorithm becomes evident. Additional patterns can then be obtained by considering subsequences and transformations of $\{a_n\}$. It is the myriad of such transformations and the creative techniques for exttracting patterns that will make our exploration interesting.

Thus, this book discusses methods and tools used to transform integer sequences and analyze them in order to obtain interesting formulas, identities and connections to other sequences.The following algorithm describes the approach that we shall take to explore integer sequences:



We demonstrate this with the Fibonacci sequence $\{F_n\}$:

1. Generate Sequence: We generate by computer the first, say, 10 terms of the Fibonacci sequence using its recurrence:

```
ColumnDisplay[n, Fibonacci[n], 0, 9, 10, "n", "Fn", ""]
```

n	F_n
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34

2. Generate Subsequence and/or Apply Transformation: We apply the partial sums transformation to $\{F_n\}$ to obtain a new sequence $\{S_n\}$ defined by

$$S_n = \sum_{k=0}^{n} F_k \tag{2.3}$$

Thus, $S_0 = 0$, $S_1 = 0 + 1 = 1$, $S_2 = 0 + 1 + 1 = 2$, $S_3 = 0 + 1 + 1 + 2 = 4$, etc. (A000071)

3. Analyze Transformed Sequence: We analyze $\{S_n\}$ for patterns. Since $\{S_n\}$ can be considered an offspring of the parent sequence $\{F_n\}$, it is natural to make a comparison between the two:

```
nMax = 9;
dataFibonaccinumbersandpartialsums =
  Table[{n, Fibonacci[n], Sum[Fibonacci[k], {k, 0, n}]}, {n, 0, nMax}];
ColumnDataDisplay[dataFibonaccinumbersandpartialsums,
 10, {"n", "Fn", "Sn=∑_{k=0}^{n} Fn"}, ""]
```

n	F_n	$S_n=\sum_{k=0}^{n} F_n$
0	0	0
1	1	1
2	1	2
3	2	4
4	3	7
5	5	12
6	8	20
7	13	33
8	21	54
9	34	88

Unfortunately, the two columns $\{F_n\}$ and $\{S_n\}$ do not appear to be correlated if we make a direct comparison between the values in each row. However, if we realign these two rows by shifting $\{F_n\}$ up by two rows, i.e. replacing $\{F_n\}$ with $\{F_{n+2}\}$, then a pattern emerges. Can you describe it?

```
nMax = 9;
dataFibonaccinumbersandpartialsums =
  Table[{n, Fibonacci[n + 2], Sum[Fibonacci[k], {k, 0, n}]}, {n, 0, nMax}];
ColumnDataDisplay[dataFibonaccinumbersandpartialsums,
 10, {"n", "Fn+2", "Sn=∑_{k=0}^{n} Fn"}, ""]
```

n	F_{n+2}	$S_n=\sum_{k=0}^{n} F_n$
0	1	0
1	2	1
2	3	2
3	5	4
4	8	7
5	13	12
6	21	20
7	34	33
8	55	54
9	89	88

4. Pattern Found or No Pattern Found: In this case, a pattern was found: $\{F_{n+2}\}$ and $\{S_n\}$ differ by 1. Thus, we've discovered the classic identity

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1 \tag{2.4}$$

Of course, we've only experimentally verified that this identity is true and only for a finite number of rows. It remains to prove (2.4) deductively, say by mathematical induction. Those acquainted with techniques of proof should have no difficulty in proving this identity. For those new or still transitioning to mathematical proofs (see Appendix A for an introduction to techniques of proof), below is a proof by mathematical induction.

Proof of (2.4): The base case $n = 0$ is clearly true:

$$\sum_{k=0}^{0} F_k = F_0 = 0$$

$$F_{0+2} - 1 = 1 - 1 = 0$$

As for the inductive step, assume that the n-th case holds. Then merely add F_{n+1} to both sides of (2.4) and apply the Fibonacci recurrence $F_{n+3} = F_{n+2} + F_{n+1}$ to demonstrate that the $(n + 1)$-th case is true:

$$\sum_{k=0}^{n} F_k + F_{n+1} = F_{n+2} - 1 + F_{n+1} \Rightarrow \sum_{k=0}^{n+1} F_k = F_{n+3} - 1$$

This completes the proof and guarantees that (2.4) holds for all n.

A word of caution is in order. The example presented above involved a pattern that was relatively easy to detect. However, not all patterns will be this easy. Some will force us to jump through many hoops and hurdles before revealing themselves. Of course, there will be integer sequences where no patterns exist at all (for now). Fortunately, there are tools at our disposable to increase our chances of success.

2.1 Tools of the Trade

Look for the obvious. This describes the best approach to discovering patterns. Unfortunately, the difficulty lies in making a pattern obvious to the *observer*. Thus, as in any skilled trade, having the right tools for the job is vital.

The toolkit for the experimental mathematician has grown over the last couple of decades and to such an extent that it has become quite effective at solving many mathematical problems. New technological tools have significantly altered how we perform research in mathematics, tools that we believe will lead to a renaissance in experimental mathematics in the 21st century.

2.1.1 Computer Algebra Systems

Computer algebra systems (CAS) are software tools that perform symbolic and high-precision computation. They are generally more advanced than scientific calculators in terms of both computing power and graphics generation, although this boundary continues to shift as personals computers and hand-held electronic devices converge in size. Some well known CAS include *Mathematica*, Maple, Matlab and Sage. In this book, we limit our discussion to *Mathematica*, the CAS created by Stephen Wolfram. However, this book can of course be used with most other CAS since they all possess an equivalent library of mathematical functions.

2.1.1.1 Mathematica

Mathematica is a powerful computer algebra system capable of symbolic and high-precision computation. Its large library of built-in functions, including many that are useful for performing numerical experiments, makes it an indispensable tool for the experimental mathematician.

Below we give a brief description of some *Mathematica* commands, based on version 8.0, that will be useful for generating sequence data and detecting their number patterns. Information about a *Mathematica* command can be displayed by inserting a question mark (?) before the command and evaluating it as input. Complete documentation on all *Mathematica* commands is available through the program's help menu or online at:

http://reference.wolfram.com/mathematica/guide/Mathematica.html

Those with access to *Mathematica* will benefit from the *Mathematica* version of this book, which allows the user to evaluate all of the *Mathematica* programs that appear in it. Please download and run the *Mathematica* package MathematicsbyExperiment.m to load those commands not built into *Mathematica*. This package is available at http://www.rowan.edu/math/facultystaff/nguyen/experimentalmath/index.html.

FindInstance

```
? FindInstance
```

FindInstance[*expr*, *vars*] finds an instance of *vars* that makes the statement *expr* be True.
FindInstance[*expr*, *vars*, *dom*] finds an instance over the domain *dom*. Common choices of *dom* are Complexes, Reals, Integers and Booleans.
FindInstance[*expr*, *vars*, *dom*, *n*] finds *n* instances. ≫

NOTE: **FindInstance** is not the same command as **Solve**; the former attempts to find particular solutions to a set of equations over a specified domain while the latter attempts to find the most general solution.

Example 2.1 - Pell's Equation

Any Diophantine equation of the form $x^2 - n \cdot y^2 = 1$, where n is not a perfect square, is called Pell's equation. These equations were originally studied by ancient Greek and Indian mathematicians who recognized their usefulness in approximating square roots. In this example we shall investigate positive integer solutions to the most elementary case, $x^2 - 2\,y^2 = 1$.

NOTE: If $n = m^2$ is a perfect square, then $x^2 - n \cdot y^2 = 1$ reduces to finding Pythagorean triples of the form $(1, my, x)$.

a) Using *Mathematica*'s **FindInstance** command, we find that there are five solutions over the domain $0 \le x \le 1000$ and $0 \le y \le 1000$.

```
Pellsolutions = Sort[FindInstance[
   x^2 - 2 y^2 == 1 && 0 < x < 1000 && 0 < y < 1000, {x, y}, Integers, 10]]
```

```
{{x → 3, y → 2}, {x → 17, y → 12}, {x → 99, y → 70}, {x → 577, y → 408}}
```

Let's display these solutions in column form to help facilitate our recognition of patterns:

Solutions to Pell's Equation $x^2 - 2y^2 = 1$

x	y
3	2
17	12
99	70
577	408

Do you recognize a pattern for the solutions (x, y)? The shrewd pattern hunter might recognize a recurrence formula from just these four solutions, from which an explicit formula can be obtain by applying the theory of recursive sequences. For those new to this game, there is fortunately a *Mathematica* command that can easily find this explicit formula.

FindSequenceFunction

```
? FindSequenceFunction
```

FindSequenceFunction[{a_1, a_2, a_3, …}] attempts to find a simple function that yields the sequence a_n when given successive integer arguments.
FindSequenceFunction[{{n_1, a_1}, {n_2, a_2}, …}] attempts to find a simple function that yields a_i when given argument n_i.
FindSequenceFunction[*list*, *n*] gives the function applied to *n*. ≫

NOTE: To increase *Mathematica*'s chances of finding a formula for a given sequence using **FindSequenceFunction**, it may be necessary to input up to ten terms depending on the complexity of the formula (if it exists). Beware however that it may give incorrect results; some examples of this will be discussed later in this section. Thus, it is good practice to always confirms formulas generated from this command.

Example 2.2

a) Consider the finite sequence {1, 4, 9, 16, 25, 36, 49, 64, 81}. Applying the **FindSequenceFunction** command to it yields the formula for the perfect squares, n^2 (A000290), as expected.

```
FindSequenceFunction[{1, 4, 9, 16, 25, 36, 49, 64, 81}, n]
```

n^2

b) However, *Mathematica* does not always produce a formula that one expects from a given pattern. Consider the sequence consisting of all natural numbers NOT divisible by 3: $\{a_n\} = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, \ldots\}$ (A001651). Applying the **FindSequenceFunction** command to the first ten terms of this sequence yields the formula:

```
FindSequenceFunction[{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16}, n]
```

$$\frac{1}{4}\left(-3-(-1)^n+6\,n\right)$$

This formula reveals little of the fact that it excludes multiples of 3, but is in fact correct. Can you prove that it generates a_n?

NOTE: Another formula for a_n is given by

$$a_n = n + \lfloor (n+1)/2 \rfloor - 1$$

Here, $\lfloor x \rfloor$ refers to the floor function (also called the *greatest integer* function), defined to be the greatest integer less than or equal to x. Can you prove that this formula and the one given by *Mathematica* are equivalent? HINT: Equate the two formulas and simplify to obtain

$$\text{Simplify}\left[\frac{1}{4}\left(-3-(-1)^n+6\,n\right) == n + \text{Floor}[(n+1)/2] - 1\right]$$

$$(-1)^n + 4\,\text{Floor}\left[\frac{1+n}{2}\right] == 1 + 2\,n$$

Now consider separately the two cases where n is odd and n is even.

■ Example 2.1 - Pell's Equation (continued)

Let's go back to the Pell equation $x^2 - 2\,y^2 = 1$ and apply the **FindSequenceFunction** command to the four solutions obtained earlier to see if formulas can be gotten for x and y.

```
FindSequenceFunction[{3, 17, 99, 577}, n]
FindSequenceFunction[{2, 12, 70, 408}, n]
```

```
FindSequenceFunction[{3, 17, 99, 577}, n]

FindSequenceFunction[{2, 12, 70, 408}, n]
```

Unfortunately, it appears that *Mathematica* did not evaluate our **FindSequenceFunction** commands; however, this only indicates that it was not able to find formulas for x and y, most likely because we did not provide *Mathematica* with enough terms. Thus, we'll need to enlarge our solution set, say double the number of solutions to eight. As the output below shows, this requires enlarging the domain that needs to be search by several orders of magnitude. Fortunately, this poses no difficulty for *Mathematica* given the computer powerful of today's desktop computers.

```
Pellsolutionsdouble = Sort[FindInstance[
   x^2 - 2 y^2 == 1 && 0 < x < 1000000 && 0 < y < 1000000, {x, y}, Integers, 10]]
```

```
{{x → 3, y → 2}, {x → 17, y → 12}, {x → 99, y → 70},
 {x → 577, y → 408}, {x → 3363, y → 2378}, {x → 19601, y → 13860},
 {x → 114243, y → 80782}, {x → 665857, y → 470832}}
```

Feeding these eight solutions into the **FindSequenceFunction** now yields the following formulas:

```
dataPellsolutionsx =
 Table[Pellsolutionsdouble[[k, 1, 2]], {k, 1, Length[Pellsolutionsdouble]}]
dataPellsolutionsy = Table[Pellsolutionsdouble[[k, 2, 2]],
  {k, 1, Length[Pellsolutionsdouble]}]
```

```
{3, 17, 99, 577, 3363, 19601, 114243, 665857}
```

```
{2, 12, 70, 408, 2378, 13860, 80782, 470832}
```

```
Clear[x0, y0];
x0 = FindSequenceFunction[dataPellsolutionsx, n]
y0 = FindSequenceFunction[dataPellsolutionsy, n]
```

$$\frac{1}{2}\left(\left(3-2\sqrt{2}\right)^n+\left(3+2\sqrt{2}\right)^n\right)$$

$$-\frac{\left(4+3\sqrt{2}\right)\left(\left(3-2\sqrt{2}\right)^n-\left(3+2\sqrt{2}\right)^n\right)}{4\left(3+2\sqrt{2}\right)}$$

This teaches us a lesson: don't give up at the first try with **FindSequenceFunction** -- try longer sequences.

It remains to verify that these formulas indeed satisfy the Pell equation $x^2 - 2y^2 = 1$:

```
Simplify[x0^2 - 2 y0^2 == 1]
```

```
True
```

NOTE: We have not demonstrated that these formulas generate ALL positive integer solutions. For a proof of this fact and a complete treatment of Pell equations, see [Ba]. The solutions for x and y appear as entries A001541 and A001542 in OEIS, respectively.

FURTHER EXPLORATION: Find formulas for positive integer solutions to the negative Pell equation $x^2 - 2y^2 = -1$ and confirm that your formulas are valid.

Example 2.3 - Dishonest Men, Coconuts, and a Monkey

Consider the following "Coconut" problem that first appeared in The Saturday Evening Post (October 9, 1926), written by Ben Ames Williams:

> *Five men and a monkey were shipwrecked on a desert island, and they spent the first day gathering coconuts for food. Piled them all up together and then went to sleep for the night.*
>
> *But when they were all asleep one man woke up, and he thought there might be a row about dividing the coconuts in the morning, so he decided to take his share. So he divided the coconuts into five piles. He had once coconut left over, and he gave that to the monkey, and he hid his pile and put the rest all back together.*
>
> *By and by the next man woke up and did the same thing. And he had one left over, and he gave it to the monkey. And all five of the men did the same thing, one after the other; each one taking a fifth of the coconuts in the pile when he woke up, and each one having one left over for the monkey. And in*

the morning they divided what coconuts were left, and they came out in five equal shares. Of course, each one must have known there were coconuts missing; but each one was guilty as the others, so they didn't say anything. How many coconuts were there in the beginning?

An interesting discussion of the solution by Martin Gardner can be found in [Ga], p.3. Here we shall take a more experimental approach to determine the number of coconuts in the original pile.

Let c be the total number of coconuts and s_k be the number of coconuts that the k-th sailor receives from his division of the cocunts during the night, and r the number of coconuts that each sailor receives after the final division in the morning. The problem can then be described by the following system of Diophantine equations:

$$\begin{aligned} c &= 5\,s_1 + 1 \\ 4\,s_1 &= 5\,s_2 + 1 \\ 4\,s_2 &= 5\,s_3 + 1 \\ 4\,s_3 &= 5\,s_4 + 1 \\ 4\,s_4 &= 5\,s_5 + 1 \\ 4\,s_5 &= 5\,r + 1 \end{aligned} \tag{2.5}$$

Eliminating the intermediate variables leads to a single Diophantine equation:

```
Clear[c, s];
Eliminate[{c == 5 s[1] + 1, 4 s[1] == 5 s[2] + 1, 4 s[2] == 5 s[3] + 1, 4 s[3] == 5 s[4] + 1,
  4 s[4] == 5 s[5] + 1, 4 s[5] == 5 r + 1}, {s[1], s[2], s[3], s[4], s[5]}]
```

```
11 529 + 15 625 r == 1024 c
```

It is now easy enough to use Mathematica's **FindInstance** command to find a solution:

```
FindInstance[eq, {c, r}, Integers]
```

```
{{c → 15 621, r → 1023}}
```

Thus, $c = 15\,621$, which of course is an unrealistically large number of coconuts that the sailors would have discovered. To obtain other solutions, we argue as follows: since c is divided six times into 5 piles, then 5^6 can be added to any solution to obtain the next highest solution. It follows that 15621 must be the smallest positive integer solution.

What if we now generalize the problem to n sailors? What is the smallest position integer solution for c? Here is a table listing the values of r and c for $n = 1$ up to $n = 5$:

Solutions to the Coconut Problem

n	r	c
1	3	21
2	15	121
3	63	621
4	255	3121
5	1023	15 621

Thus, the formula for c as a function of n is given by:

```
FindSequenceFunction[{21, 121, 621, 3121, 15 621}, n]
```

$-4 + 5^{1+n}$

The sequence {1, 21, 121, 621, 3121, 15 621, ...} is entry A164785 in OEIS.

FURTHER EXPLORATION: Suppose m coconuts remain and are given to the monkey (instead of one coconut) each time the coconuts are divided into five piles by each of the five sailors. How many coconuts were in the original pile? What if there were n sailors?

Example 2.4 - FindSequenceFunction Flounders

a) WARNING: The **FindSequenceFunction** command does not alway produce the desired answer since it is sensitive to the number of terms used. Consider the finite sequence {2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17} consisting of positive integers that are NOT squares (A000037). Applying the **FindSequenceFunction** to it yields

```
FindSequenceFunction[{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17}, n]
```

$$2+\frac{1}{8}\left(-9-(-1)^{n}+10\,n+\mathrm{Cos}\left[\frac{n\,\pi}{2}\right]-\mathrm{Sin}\left[\frac{n\,\pi}{2}\right]+\sqrt{2}\,\mathrm{Sin}\left[\frac{1}{4}\,(\pi+6\,n\,\pi)\right]\right)$$

Let's now use this formula to regenerate the finite sequence:

$$\mathbf{Table}\left[2+\frac{1}{8}\left(-9-(-1)^{n}+10\,n+\mathrm{Cos}\left[\frac{n\,\pi}{2}\right]-\mathrm{Sin}\left[\frac{n\,\pi}{2}\right]+\sqrt{2}\,\mathrm{Sin}\left[\frac{1}{4}\,(\pi+6\,n\,\pi)\right]\right),\ \{n,1,13\}\right]$$

```
{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 15, 16, 17}
```

In the output above notice that the entry 14, which appears in the original sequence, is missing and that the entry 16 should not be included. Here's a formula, $\left\lfloor n+\sqrt{n+\sqrt{n}}\right\rfloor$, that correctly generates the original finite sequence:

```
Table[Floor[n + Sqrt[Sqrt[n] + n]], {n, 1, 13}]
```

```
{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17}
```

At this point the reader might object by arguing that the **FindSequenceFunction** command would have produced the formula $\left\lfloor n+\sqrt{n+\sqrt{n}}\right\rfloor$ if it had been given more terms, say 20 terms instead of 13? Unfortunately, even then *Mathematica* fails to give this formula. Instead, *Mathematica* gives a formula involving the **DifferenceRoot** function:

```
Table[Floor[n + Sqrt[Sqrt[n] + n]], {n, 1, 20}]
```

```
{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24}
```

```
Clear[formula];
formula[n_] =
 FindSequenceFunction[Table[Floor[n + Sqrt[Sqrt[n] + n]], {n, 1, 20}], n]
```

```
DifferenceRoot[Function[{y, n}, {-1 - y[n] + y[1 + n] == 0, y[1] == 2,
    y[2] == 3, y[3] == 5, y[4] == 6, y[5] == 7, y[6] == 8, y[7] == 10, y[8] == 11,
    y[9] == 12, y[10] == 13, y[11] == 14, y[12] == 15, y[13] == 17}]][n]
```

Let's confirm if this formula is correct. Here's a list of the first 20 terms generated from it:

```
Table[formula[n], {n, 1, 20}]
```

```
{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24}
```

From the output above it does seems that *Mathematica*'s formula gives the same terms as those generated from $\left\lfloor n+\sqrt{n+\sqrt{n}}\right\rfloor$, but alas, if we compare the two formulas out to 30 terms, then we find that they disagree starting with the 21th term (the value at this term should be 26, not 25):

```
Table[{Floor[n + Sqrt[Sqrt[n] + n]], formula[n]}, {n, 1, 30}]
```

```
{{2, 2}, {3, 3}, {5, 5}, {6, 6}, {7, 7}, {8, 8}, {10, 10}, {11, 11}, {12, 12},
 {13, 13}, {14, 14}, {15, 15}, {17, 17}, {18, 18}, {19, 19}, {20, 20},
 {21, 21}, {22, 22}, {23, 23}, {24, 24}, {26, 25}, {27, 26}, {28, 27},
 {29, 28}, {30, 29}, {31, 30}, {32, 31}, {33, 32}, {34, 33}, {35, 34}}
```

Thus, **FindSequenceFunction** must be used with extreme caution.

b) HELPFUL TIP: The **FindSequenceFunction** may give different, but equivalent, formulas for the same sequence, depending on whether the sequence has been shifted. *Mathematica* assumes that the first element entered into this command corresponds to $n = 1$. For example, recall from an earlier example the finite sequence of perfect squares, {1, 4, 9, 16, 25, 36, 49, 64, 81}, which *Mathematica* recognized as being generated from the formula n^2. If we now prepend the element 0 to this sequence, then *Mathematica* easily shifts the formula accordingly:

```
FindSequenceFunction[{0, 1, 4, 9, 16, 25, 36, 49}, n]
```

$(-1 + n)^2$

On the other hand, *Mathematica* fails to do this for the Fibonacci sequence. For example, inputting the finite sequence {1, 1, 2, 3, 5, 8, 13, 21}, which begins with the term $F_1 = 1$, yields the Fibonacci sequence as expected:

```
FindSequenceFunction[{1, 1, 2, 3, 5, 8, 13, 21}, n]
```

```
Fibonacci[n]
```

However, prepending the element $F_0 = 0$ doesn't produce the desired shift in the formula, F_{n-1}, that we would expect:

```
FindSequenceFunction[{0, 1, 1, 2, 3, 5, 8, 13, 21}, n]
```

$\frac{1}{2}$ (-Fibonacci[n] + LucasL[n])

This is not necessary a bad thing. In this case, equating the two formulas leads to an identity between the Fibonacci sequence and its cousin, the *Lucas* sequence L_n (A000032), defined by the same recurrence, $L_{n+1} = L_n + L_{n-1}$, but with different initial values, namely $L_0 = 2$ and $L_1 = 1$:

$$F_{n-1} = (L_n - F_n)/2 \tag{2.6}$$

or equivalently,

$$L_n = F_n + 2\,F_{n-1} \tag{2.7}$$

The next *Mathematica* command will be helpful in simplifying complicated expressions.

Simplify

```
? Simplify
```

Simplify[*expr*] performs a sequence of algebraic and other transformations on *expr*, and returns the simplest form it finds.

Simplify[*expr*, *assum*] does simplification using assumptions. ≫

Example 2.5

a) Here's an example where the **Simplify** command comes to the rescue to simplify a trigonometric identity that *Mathematica* does not recognize:

```
Sum[Cos[n * Pi / 100], {n, 0, 100}]
```

$$\frac{1}{4}\left(-1-\sqrt{5}\right)+\frac{1}{4}\left(1-\sqrt{5}\right)+\frac{1}{4}\left(-1+\sqrt{5}\right)+\frac{1}{4}\left(1+\sqrt{5}\right)$$

```
Simplify[%]
```

0

b) However, *Mathematica* isn't always saavy when it comes to reducing formulas, even with the **Simplify** command. For example, the following double sum reduces to a constant value, namely 1, which we easily discover by computing some test values. However, *Mathematica* does not seem to recognize this or at least cannot prove it to be true based on its methods.

```
TimeConstrained[Simplify[Sum[(-1)^j / (i! j!), {i, 0, n}, {j, 0, n - i}]], 30]
```

```
$Aborted
```

```
Table[Sum[(-1)^j / (i! j!), {i, 0, n}, {j, 0, n - i}], {n, 1, 10}]
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

2.1.1.2 *Mathematica* Missteps

Mathematica is not a perfect CAS as demonstrated with the **FindSequenceFunction** command In fact, many of its errors have been documented, most due to differences in assumptions between *Mathematica* and the user. Here are some examples to keep in mind:

Example 2.6

The function $f(x) = x^0$ is well defined only for $x > 0$ and in that case equals 1. Thus, be careful with improper evaluation such as $f(0)$:

```
f[x_] = x^0;
f[0]
```

```
1
```

The answer $f(0) = 1$ is WRONG since 0^0 is undefined:

```
0^0
```

Power::indet : Indeterminate expression 0^0 encountered. ≫

```
Indeterminate
```

Example 2.7

Consider Ramanujan's constant $e^{\sqrt{163}\,\pi}$, which is an almost integer (a value that is very close to an integer).

```
R = Exp[Pi Sqrt[163]]
N[R, 32]
```

$e^{\sqrt{163}\,\pi}$

$2.6253741264076874399999999999925 \times 10^{17}$

However, if compute its decimal part, $R - \lfloor R \rfloor$, using *Mathematica*'s approximation command, **N**, but without specifying the *n*-digit precision, then we obtain the wrong answer (recall that the decimal part of every real value is bounded between 0 and 1):

```
N[R - Floor[R]]
```

```
-480.
```

However, we can remedy this by specifying the precision, say to 15 digits:

```
N[R - Floor[R], 15]
```

```
0.999999999999250
```

This serves as a warning that the reader should be careful with approximating extremely large values when *Mathematica*.

Example 2.8

```
Solve[x^2 + 1 / x == x + 1 / x, x]
```

```
{{x → 0}, {x → 1}}
```

Example 2.9

In the evaluation below, *Mathematica* claims that the two sums are equal, where the first defines the harmonic number $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$, i.e., sum of reciprocals of the first n natural numbers (A001008 and A002805), and the second involves a harmonic sum of the number of divisors of the natural numbers up to n.

```
Sum[1 / k, {k, 1, n}]
Sum[1 / (k * Length[Divisors[k]]), {k, 1, n}]
```

```
HarmonicNumber[n]
```

```
HarmonicNumber[n]
```

However, numerical computation shows that this is in fact NOT true:

```
Table[Sum[1 / k, {k, 1, n}], {n, 1, 5}]
Table[Sum[1 / (k * Length[Divisors[k]]), {k, 1, n}], {n, 1, 5}]
```

$$\left\{1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}\right\}$$

$$\left\{1, \frac{5}{4}, \frac{17}{12}, \frac{3}{2}, \frac{8}{5}\right\}$$

2.1.2 Online Databases

2.1.2.1 Online Encyclopedia of Integer Sequences (OEIS) - http://oeis.org

The Online Encyclopedia of Integer Sequences (OEIS) is an online database created by Neil Sloan in 1965 that as of January 2011 contains a collection of almost 200,000 integer sequences and allows the user to search whether a given integer sequence appears in it (see [O]). Each sequence entry in OEIS (or the *Encyclopedia* as we'll sometimes refer to it) contains a host of information about the sequence, including a list of initial terms, formulas, known results, and references. Here's an example using an in-house command called **OEIS**, i.e. a command created by the author and not built into *Mathematica* (see Mathematics-byExperiment.m package file for *Mathematica* code) to query the OEIS website.

OEIS

```
? OEIS
```

OEIS[*sequence*,*n*] queries the Online Encyclopedia of Integer Sequences (OEIS) website (http://oeis.org) to search for *sequence* and displays the first *n* search results. The default value is *n*=1. The maximum value is *n*=10. Requires loading of MathematicsbyExperimentPackage.m package.

Example 2.10 - Searching the Encyclopedia (OEIS)

a) Suppose we searched the Encyclopedia to see if it recognizes the finite sequence {1, 1, 2, 3, 5, 8, 13, 21, 34, 55}.

```
OEIS[{1, 1, 2, 3, 5, 8, 13, 21, 34, 55}]
```

```
OEIS Query: {1, 1, 2, 3, 5, 8, 13, 21, 34, 55}

The On-Line Encyclopedia of Integer
  Sequences, published electronically at http://oeis.org, 2010

Go to OEIS complete search results

Summary display of results 1-1 out of 3186 results found.

    {{A000045,
      Fibonacci numbers: F(n) = F(n- 1 ) + F(n- 2 ), F(0) = 0, F( 1 ) = 1 , F(
        2 ) = 1 , ... (Formerly M0692 N0256) , +180 2136 },
     {0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,
      2584, 4181, 6765, 10 946, 17 711, 28 657, 46 368, 75 025,
      121 393, 196 418, 317 811, 514 229, 832 040, 1 346 269, 2 178 309,
      3 524 578, 5 702 887, 9 227 465, 14 930 352, 24 157 817, 39 088 169}}
```

Not only does OEIS recognize this sequence as being part of the Fibonacci sequence (A000045), but also part of 3185 other sequences. This shows that the Fibonacci numbers, one of the most recognized sequences in the world, arise in many other patterns. To learn more about the Fibonacci sequence, just click on the link for A000045, which will open up your web browser to the OEIS web page describing them. To see the other 3185 sequences, click on the button ‘Go to OEIS complete search results’, which will open up your web browser and show the full search results directly from OEIS.

b) Here's an example of a finite sequence where one would think that because of the extremely large last element, OEIS would find either an exact match or no match at all:

```
OEIS[{1, 1, 3, 21, 987, 2 178 309, 10 610 209 857 723}, 2]
```

```
OEIS Query: {1, 1, 3, 21, 987, 2 178 309, 10 610 209 857 723}

The On-Line Encyclopedia of Integer
  Sequences, published electronically at http://oeis.org, 2010

Go to OEIS complete search results

Summary display of results 1-2 out of 7 results found.

    {{A058635, (2^n)-th Fibonacci number., +120 9 },
     {1, 1, 3, 21, 987, 2 178 309, 10 610 209 857 723, 251 728 825 683 549 488 150 424 261,
      141 693 817 714 056 513 234 709 965 875 411 919 657 707 794 958 199 867,
      44 893 845 313 309 942 978 077 298 160 660 626 646 181 883 623 886 239 791 269 694 466 661 322 \
       268 805 744 081 870 933 775 586 567 858 979 269}}
    {{A050615, Products of distinct terms of A000045 [2^(i+2)]: a(n)
        = Product(F(2^(i+2))^bit(n,i),i=0..[log2(n+ 1 )]).,  +120 1 },
     {1, 3, 21, 63, 987, 2961, 20 727, 62 181, 2 178 309, 6 534 927, 45 744 489,
      137 233 467, 2 149 990 983, 6 449 972 949, 45 149 810 643,
      135 449 431 929, 10 610 209 857 723, 31 830 629 573 169,
      222 814 407 012 183, 668 443 221 036 549, 10 472 277 129 572 601}}
```

Suprisingly, there are seven match results with the first two shown above. This shows that there integer sequences that OEIS cannot match exactly because they are either embedded in other integer sequences or have gone through simplifications during calculation; but even then partial matches can still provide clues to help us find their formulas. This is demonstrated in the next example.

Example 2.11 - Rational sequences and Fibonacci numbers at prime positions

Suppose we wish to find an explicit formula for the recursive rational sequence defined by

```
u[1] = 1;
u[n_] := (3 u[n - 1] + 1) / (5 u[n - 1] + 3)
```

Below is a list of the first twenty values:

```
datau = Table[u[n], {n, 1, 15}]
```

$$\left\{1, \frac{1}{2}, \frac{5}{11}, \frac{13}{29}, \frac{17}{38}, \frac{89}{199}, \frac{233}{521}, \frac{305}{682}, \frac{1597}{3571}, \frac{4181}{9349}, \frac{5473}{12\,238}, \frac{28\,657}{64\,079}, \frac{75\,025}{167\,761}, \frac{98\,209}{219\,602}, \frac{514\,229}{1\,149\,851}\right\}$$

Observe that applying the **FindSequenceFunction** command to this sequence yields a complicated formula:

$$\texttt{FindSequenceFunction}\left[\left\{1, \frac{1}{2}, \frac{5}{11}, \frac{13}{29}, \frac{17}{38}, \frac{89}{199}, \frac{233}{521}, \frac{305}{682}, \frac{1597}{3571}, \frac{4181}{9349}, \frac{5473}{12\,238}, \frac{28\,657}{64\,079}, \frac{75\,025}{167\,761}, \frac{98\,209}{219\,602}, \frac{514\,229}{1\,149\,851}\right\}, n\right]$$

$$-\left(-370\,248\,451\,2^{-1+n} + 165\,580\,141 \times 2^{-1+n}\sqrt{5} - 969\,323\,029\left(7-3\sqrt{5}\right)^{-1+n} + 433\,494\,437\sqrt{5}\left(7-3\sqrt{5}\right)^{-1+n}\right) \Big/ \left(-827\,900\,705\,2^{-1+n} + 370\,248\,451 \times 2^{-1+n}\sqrt{5} + 2\,167\,472\,185\left(7-3\sqrt{5}\right)^{-1+n} - 969\,323\,029\sqrt{5}\left(7-3\sqrt{5}\right)^{-1+n}\right)$$

Let's divide the problem by analyzing the numerators, {1,1,5,13,17,89,...}, and denominators, {1,2,11,29,38,199,...}, separately. Again, we find that the **FindSequenceFunction** command fails to give an elementary pattern for the numerators, expressing them in terms of *Mathematica*'s **DifferenceRoot** function.

```
datanumerator = Numerator[datau]
```

{1, 1, 5, 13, 17, 89, 233, 305, 1597, 4181, 5473, 28 657, 75 025, 98 209, 514 229}

```
FindSequenceFunction[datanumerator, n]
```

```
DifferenceRoot[Function[{y, n}, {y[n] - 18 y[3 + n] + y[6 + n] == 0,
    y[1] == 1, y[2] == 1, y[3] == 5, y[4] == 13, y[5] == 17, y[6] == 89}]][n]
```

On the other hand, feeding the same sequence into OEIS yields three possible matches involving sequences that are related to the Fibonacci sequence:

```
OEIS[datanumerator, 10]
```

```
OEIS Query: {1, 1, 5, 13, 17, 89, 233, 305, 1597, 4181, 5473, 28 657, 75 025, 98 209, 514 229}

The On-Line Encyclopedia of Integer
  Sequences, published electronically at http://oeis.org, 2010

Go to OEIS complete search results

Summary display of results 1-3 out of 3 results found.
```

{{A167808, Numerator of x(n)=x(n- 1)+x(n-2),
x(0)=0, x(1)= 1 /2; denominator= A130196 ., +280 10 },
{0, **1**, **1**, **1**, 3, **5**, 4, **13**, 21, **17**, 55, **89**, 72, **233**, 377, **305**, 987, **1597**,
1292, **4181**, 6765, **5473**, 17 711, **28 657**, 23 184, **75 025**, 121 393,
98 209, 317 811, **514 229**, 416 020, 1 346 269, 2 178 309, 1 762 289,
5 702 887, 9 227 465, 7 465 176, 24 157 817, 39 088 169, 31 622 993}}

{{A079497,
a(1)= 1 , for n>2 a(n) is the smallest integer > a(n- 1) such that sqrt(
5)*a(n) is closer and > to an integer than sqrt(5)*a(n-
1) (i.e. a(n) is the smallest integer > a(n- 1) such that
frac(sqrt(5)*a(n))<frac(sqrt(5)*a(n- 1)))., +280 0 },
{**1**, **5**, 9, **13**, **17**, **89**, 161, **233**, **305**, **1597**, 2889, **4181**, **5473**, **28 657**,
51 841, **75 025**, **98 209**, **514 229**, 930 249,
1 346 269, 1 762 289, 9 227 465, 16 692 641,
24 157 817, 31 622 993, 165 580 141, 299 537 289,
433 494 437, 567 451 585, 2 971 215 073, 5 374 978 561}}

{{A174883, Largest odd divisors of Fibonacci numbers., +280 0 },
{**1**, **1**, **1**, 3, **5**, **1**, **13**, 21, **17**, 55, **89**, 9, **233**, 377, **305**, 987, **1597**, 323, **4181**, 6765,
5473, 17 711, **28 657**, 1449, **75 025**, 121 393, **98 209**, 317 811, **514 229**, 104 005, 1 346 269,
2 178 309, 1 762 289, 5 702 887, 9 227 465, 933 147, 24 157 817, 39 088 169, 31 622 993}}

With this as a clue, the keen bookkeeper will notice that most of the numerators are in fact Fibonacci numbers, namely {1, 1, 5, 13, 89, 233, 1597, 4181, ...}, and in fact those that are *not* Fibonacci numbers, {17, 305, 5473, ...}, occur at every third entry starting with 17.

Using the following in-house command, which identifies the positions of elements within a given integer sequence, we verify that the numerators consist of Fibonnacci numbers of the form F_{2n-1}:

MatchPosition

```
? MatchPosition
```

MatchPosition[*subset*,*sequence*,*n*,*first*,*last*] displays the positions of the elements specified by *subset* within the list generated by the function *sequence*[*n*] from *n*=*first* to *n*=*last*. Requires loading of MathematicsbyExperimentPackage.m package.

```
MatchPosition[datanumerator, Fibonacci, n, 1, 50] // Column
```

```
Fibonacci[{{1}, {2}}]
Fibonacci[{{1}, {2}}]
Fibonacci[{{5}}]
Fibonacci[{{7}}]
Fibonacci[{}]
Fibonacci[{{11}}]
Fibonacci[{{13}}]
Fibonacci[{}]
Fibonacci[{{17}}]
Fibonacci[{{19}}]
Fibonacci[{}]
Fibonacci[{{23}}]
Fibonacci[{{25}}]
Fibonacci[{}]
Fibonacci[{{29}}]
```

As for the corresponding list of denominators, we follow the trail above and eliminate every third entry starting with the denominator 38 from consideration. Then matching these denominators with the Lucas numbers yields the same pattern:

```
datadenominator = Denominator[datau]
```

```
{1, 2, 11, 29, 38, 199, 521, 682, 3571, 9349, 12 238, 64 079, 167 761, 219 602, 1 149 851}
```

```
MatchPosition[datadenominator, LucasL, n, 1, 50] // Column
```

```
LucasL[{{1}}]
LucasL[{}]
LucasL[{{5}}]
LucasL[{{7}}]
LucasL[{}]
LucasL[{{11}}]
LucasL[{{13}}]
LucasL[{}]
LucasL[{{17}}]
LucasL[{{19}}]
LucasL[{}]
LucasL[{{23}}]
LucasL[{{25}}]
LucasL[{}]
LucasL[{{29}}]
```

The formula for u_n in terms of Fibonacci and Lucas numbers is now clear:

$$u_n = \frac{F_{2n-1}}{L_{2n-1}}$$

where F_n and L_n are Fibonacci and Lucas numbers, respectively. The following table confirms this:

```
Table[Fibonacci[2 n - 1] / LucasL[2 n - 1], {n, 1, 10}]
```

$$\left\{1, \frac{1}{2}, \frac{5}{11}, \frac{13}{29}, \frac{17}{38}, \frac{89}{199}, \frac{233}{521}, \frac{305}{682}, \frac{1597}{3571}, \frac{4181}{9349}\right\}$$

2.1.2.2 Inverse Symbolic Calculator (ISC) - http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html

The Inverse Symbolic Calculator (ISC), created by Simon Plouffe in 1995, is useful for searching real numbers that represent the exact value of a given approximation (see [I]). Here is an example using the in-house command, **ISC**, to query the ISC website (see Mathematics by Experiment package file for *Mathematica* code) .

ISC

```
? ISC
```

ISC[*value*,*n*] queries the Inverse Symbolic Calculator (ISC) website (http://oldweb.cecm.sfu.ca/cgi–bin/isc/) to search for a real number with approximation given by *value* and displays the first *n* search results. Requires loading of MathematicsbyExperimentPackage.m package.

For example, feeding the approximation 3.14159 for π into ISC yields many matches (289). The first five matches are shown below:

```
ISC["3.14159", 5]
```

```
Inverse Symbolic Calculator, published electronically
  at http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html
```

Go to ISC complete search results

Summary display of results 1-5 out of 289 results found.

3141592653589793=Pi

3141592920353982=71/226

3141591530269234=1/31831

3141590543309810=1/3183101

3141592653589793=Psi(1/4)-Psi(3/4)

We can, of course, reduce the number of matches by feeding ISC a better approximation:

```
ISC["3.141592653589793", 5]
```

```
Inverse Symbolic Calculator, published electronically
  at http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html
```

Go to ISC complete search results

Summary display of results 1-5 out of 43 results found.

3141592653589793=Pi

3141592653589793=Psi(1/4)-Psi(3/4)

3141592653589793=GAM(1/6)*GAM(5/6)-Pi

3141592653589793=1/2*GAM(1/6)*GAM(5/6)

3141592653589793=GAM(1/4)/sr(2)*GAM(3/4)

Surprisingly, we still found 43 matches, which goes to show that there are many interesting values that approximate π.

Example 2.9 - Pell's Equation (Revisited)

The Pell equation, $x^2 - 2\,y^2 = 1$ (discussed in Example 1.1), and its negative, $x^2 - 2\,y^2 = -1$, share a common pattern. Let's combine their first four solutions:

```
Pellsolutionsplusminus = FindInstance[(x^2 - 2 y^2 == 1 || x^2 - 2 y^2 == -1) &&
   0 < x < 1000 && 0 < y < 1000, {x, y}, Integers, 10]
```

```
{{x → 1, y → 1}, {x → 7, y → 5}, {x → 41, y → 29}, {x → 239, y → 169},
 {x → 3, y → 2}, {x → 17, y → 12}, {x → 99, y → 70}, {x → 577, y → 408}}
```

Let's sort and index these solutions:

Pell Equations $x^2 - y^2 = \pm 1$

n	x_n	y_n
1	1	1
2	3	2
3	7	5
4	17	12
5	41	29
6	99	70
7	239	169
8	577	408

Next, define $r_n = \frac{x_n}{y_n}$. A natural problem to investigate is the limiting value of r_n as $n \to \infty$? Here is a table listing the first ten approximate values of r_n :

Pell Equations $x^2 - y^2 = \pm 1$

n	$r_n = x_n / y_n$
1	1.00000000000000
2	1.50000000000000
3	1.40000000000000
4	1.41666666666667
5	1.41379310344828
6	1.41428571428571
7	1.41420118343195
8	1.41421568627451
9	1.41421319796954
10	1.41421362489487

To obtain an even better approximation of the limiting ratio, we can extend the solution set by using the **FindSequenceFunction** to find formulas for x_n and y_n:

```
x[n_] = FindSequenceFunction[{1, 3, 7, 17, 41, 99, 239, 577}, n]
y[x_] = FindSequenceFunction[{1, 2, 5, 12, 29, 70, 169, 408}, n]
```

$$\frac{1}{2}\left(\left(1-\sqrt{2}\right)^{n}+\left(1+\sqrt{2}\right)^{n}\right)$$

```
Fibonacci[n, 2]
```

The output above refers to the Fibonacci polynomial $F_n(2)$. Here is a list of values for r_n accurate to 12 decimal places:

Pell's Equation $x^2 - y^2 = \pm 1$

n	$r_n = x_n / y_n$	n	$r_n = x_n / y_n$
1	1.00000000000000	11	1.41421355164605
2	1.50000000000000	12	1.41421356421356
3	1.40000000000000	13	1.41421356205732
4	1.41666666666667	14	1.41421356242727
5	1.41379310344828	15	1.41421356236380
6	1.41428571428571	16	1.41421356237469
7	1.41420118343195	17	1.41421356237282
8	1.41421568627451	18	1.41421356237314
9	1.41421319796954	19	1.41421356237309
10	1.41421362489487	20	1.41421356237310

It is now clear that these ratios converge to limit $\sqrt{2}$. Those that did not recognize this value can feed 1.414213562373 into ISC:

```
ISC["1.414213562373", 5]
```

```
Inverse Symbolic Calculator, published electronically
  at http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html
```

```
Go to ISC complete search results
```

```
Summary display of results 1-5 out of 67 results found.
```

1414213562373095=sqrt(2)

1414213562373095=2/sr(2)

1414213562373095=1/16^-1/8

1414213562373095=E^(1/2*ln(2))

1414213562373095=Im((-2+0*I)^(1/2))

This confirms $\sqrt{2}$ as the limit. Of course we can obtain the same answer by also using *Mathematica*'s **Limit** command since exact formulas for x_n and y_n are known:

```
Limit[x[n] / y[n], n → Infinity]
```

$\sqrt{2}$

Example 2.13 - Round Off Error

Suppose we wanted to find the limit of the sequence u_n in Example 1.10 as $n \to \infty$. Here is a table of the first 30 values of u_n:

n	u_n
1	1.000000000
2	0.3333333333
3	0.5000000000
4	0.4285714286
5	0.4545454545
6	0.4444444444
7	0.4482758621
8	0.4468085106
9	0.4473684211
10	0.4471544715

n	u_n
11	0.4472361809
12	0.4472049689
13	0.4472168906
14	0.4472123369
15	0.4472140762
16	0.4472134119
17	0.4472136656
18	0.4472135687
19	0.4472136057
20	0.4472135916

n	u_n
21	0.4472135970
22	0.4472135949
23	0.4472135957
24	0.4472135954
25	0.4472135955
26	0.4472135955
27	0.4472135955
28	0.4472135955
29	0.4472135955
30	0.4472135955

This sequence appears to stabilize out to 10 decimal places to the value 0.4472135955. However, feeding this value into ISC fails to give an answer.

```
ISC["0.4472135955", 5]
```

```
Inverse Symbolic Calculator, published electronically
  at http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html
```

```
Go to ISC complete search results
```

```
Your search resulted in no match.
```

This is because *Mathematica* had rounded off the last significant digit (10-th decimal place). A more precise calculation shows that this digit should be 4 not 5.

```
Table[N[Fibonacci[n] / LucasL[n], 15], {n, 1, 40}]
```

```
{1.00000000000000, 0.333333333333333, 0.500000000000000, 0.428571428571429,
 0.454545454545455, 0.444444444444444, 0.448275862068966, 0.446808510638298,
 0.447368421052632, 0.447154471544715, 0.447236180904523, 0.447204968944099,
 0.447216890595010, 0.447212336892052, 0.447214076246334, 0.447213411871319,
 0.447213665639877, 0.447213568708896, 0.447213605733234, 0.447213591591195,
 0.447213596992973, 0.447213594929677, 0.447213595717786, 0.447213595416755,
 0.447213595531739, 0.447213595487819, 0.447213595504595, 0.447213595498187,
 0.447213595500634, 0.447213595499700, 0.447213595500057, 0.447213595499920,
 0.447213595499972, 0.447213595499952, 0.447213595499960, 0.447213595499957,
 0.447213595499958, 0.447213595499958, 0.447213595499958, 0.447213595499958}
```

Feeding the correct value 0.4472135954 into ISC now yields the exact limit, $\frac{1}{\sqrt{5}}$.

```
ISC["0.4472135954", 5]
```

```
Inverse Symbolic Calculator, published electronically
  at http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html
```

```
Go to ISC complete search results
```

```
Summary display of results 1–5 out of 52 results found.
```

4472135954999579=1/sr(5)

4472135954999579=1/5^1/2

4472135954999579=sqrt(20)

4472135954999579=1/sqrt(5)

4472135954999579=2*exp(1/2)^ln(5)

NOTE: Of course we can obtain the exact answer by also using *Mathematica*'s **Limit** command since exact formulas for F_n and L_n are known:

```
Limit[Fibonacci[n] / LucasL[n], n → Infinity]
```

$$\frac{1}{\sqrt{5}}$$

2.2 Tricks of the Trade

You can use all the quantitative data you can get, but you still have to distrust it and use your own intelligence and judgment.

-- Alvin Toffler (www.quotationspage.com)

In this modern age and with tools such as computer algebra systems, it is quite easy to generate large data sets for analysis. In this section we discuss frequently used techniques (tricks of the trade) for manipulating and transforming integer sequences to obtain number patterns. The examples presented demonstrate that although the computer can be programmed to detect most of these patterns, it still pays to do good bookkeeping and have a keen eye.

2.2.1 Good Bookkeeping

I never guess. It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts.

-- Sir Arthur Conan Doyle (1859 - 1930), The Sign of Four, A Scandal in Bohemia

Good bookkeeping begins with arranging data in an orderly fashion that allows patterns to reveal themselves either through symmetry or periodicity.

2.2.1.1 Partitioning a Sequence

Partitioning a sequence into a collection of subsets or a two-dimensional data set is an effective method to reveal patterns that would be difficult to recognize otherwise.

Example 2.14- Stern's Diatomic Sequence

Define a sequence $\{a_n\}$, called the diatomic sequence (A002487), whose generation proceeds as follows:

1. Begin with the finite sequence {0, 1}.
2. Insert between every two consecutive terms of the sequence a term equal to the sum of those two entries
3. Iterate step 2.

The sequence in the limit is the diatomic sequence. The table below describes the first five iterations.

Generation of Diatomic Sequence

Iteration	Sequence
0	{0, 1}
1	{0, 1, 1}
2	{0, 1, 1, 2, 1}
3	{0, 1, 1, 2, 1, 3, 2, 3, 1}
4	{0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1}
5	{0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4, 5, 1}

NOTE: Observe that this construction has *memory* in the sense that each iteration preserves the sequence from the previous iteration.

The sequence $\{a_n\}$ can be programmed in *Mathematica* as follows:

```
Clear[a, stern];
nMax = 11;
a[0] = 0;
a[1] = 1;
Do[
  stern =
   Riffle[Table[a[k], {k, 0, 2^i - 1}], Table[a[k - 1] + a[k], {k, 1, 2^(i - 1)}]];
  Do[a[2^(i - 1) + k] = stern[[2^(i - 1) + k + 1]], {k, 1, 2^(i - 1)}],
  {i, 1, nMax}];
datastern = Table[{n, a[n]}, {n, 0, 39}];
ColumnDataDisplay[datastern, 10,
 {"n", "a(n)"}, "Generation of Diatomic Sequence", Left]
```

Generation of Diatomic Sequence

n	a(n)
0	0
1	1
2	1
3	2
4	1
5	3
6	2
7	3
8	1
9	4

n	a(n)
10	3
11	5
12	2
13	5
14	3
15	4
16	1
17	5
18	4
19	7

n	a(n)
20	3
21	8
22	5
23	7
24	2
25	7
26	5
27	8
28	3
29	7

n	a(n)
30	4
31	5
32	1
33	6
34	5
35	9
36	4
37	11
38	7
39	10

a) Observe that the value 1 seems to repeat itself and occur at positions equal to a power of 2. In other words, we have $a_{2^n} = 1$. Let's use these positions as markers then to create an array (ignore the first element a_0) whose rows begin with 1 and whose lengths increase by powers of two:

1																									
1	2																								
1	3	2	3																						
1	4	3	5	2	5	3	4																		
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5										
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11	3	1

A patter now emerges if we sum each row:

```
Table[Sum[a[k], {k, 2^n, 2^(n+1) - 1}], {n, 0, 5}]
```

```
{1, 3, 9, 27, 81, 243}
```

Thus, the row sums are equal to powers of 3:

$$\sum_{k=2^n}^{2^{n+1}-1} a_k = 3^n. \tag{2.8}$$

b) Next, notice that the gaps along each column is constant, i.e. the difference between any two terms in each column is the same.

Gaps Along Each Column

Column	Gap
1	0
2	1
3	1
4	2
5	1
6	3
7	2
8	3
9	1
10	4

Do you recognize this sequence of gaps? You should since it is just the diatomic sequence itself. Thus, we've found the identity

$$a_{2^{n+1}+n} - a_{2^n+n} = a_n \tag{2.9}$$

c) Also, observe that values in certain columns above repeat themselves. For example, column 2 is repeated as columns 3, 5, 9, 17, etc. (positions that are one more than a power of 2) as shown in red below:

```
Grid[Table[If[n == 2^p + 1 || n == 2^p + 2 || n == 2^p + 4 || n == 2^p + 8 || n == 2^p + 16,
    Style[a[n], Red], a[n]], {p, 0, 5}, {n, 2^p, 2^(p + 1) - 1}],
  Frame → All, Background → {None, {{None, LightGray}}}]
```

1																									
1	2																								
1	3	2	3																						
1	4	3	5	2	5	3	4																		
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5										
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11	3	1

This leads to the identity

$$a_{2^n+1} = a_{2^n+2} = a_{2^n+4} = .\,.. = a_{2^n+2^m} = .\,.. \tag{2.10}$$

NOTE: It seems that each entry is the sum of two entries above similar to Pascal's triangle; however, this pattern fails for those entries that are repeated from the previous column. Therefore, if we consider instead a diatomic array where we pad 1's at the end of each row to make it palindromic and arrange the entries into columns as shown below for the first five rows:

```
diatomicarray[nMax_] := Table[ReplacePart[
    PadRight[{}, 2^(nMax - 1) + 1, ""], (1 + 2^(nMax - n) * # → a[2^(n - 1) + #]) & /@
     (Range[0, 2^(n - 1)])], {n, 1, nMax}] // Grid
```

```
diatomicarray[5]
```

```
1                               1
1               2               1
1       3       2       3       1
1   4   3   5   2   5   3   4   1
1 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5 1
```

Thus we see that each entry is either the sum of the two entries above it or if there is an entry directly above it, then it takes on the same value as that entry. Recursively, the next row in the array can be constructed by copying the previous row and inserting the sum of every two consecutive entries in the previous row as a new entry.

2.2.2 Sequences and Their Subsequences

We've already seen examples of various kinds of sequences and subsequences in the examples above. We treat in further detail recursive sequences and special subsequences are arise quite frequently.

2.2.2.1 Recursive sequences

A sequence $\{a_n\}$ is *recursive* if it satisfies a recurrence, a relationship where each element a_n is expressed in terms of any of the previous elements $a_{n-1}, a_{n-2}, \ldots, a_1, a_0$. A linear homogeneous recurrence with constant coefficients is one of the form

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \ldots + c_{n-1} \cdot a_1 + c_n \cdot a_0$$

where $\{c_1, c_2, \ldots, c_n\}$ are constant real values. This recurrence will be called a $\{c_1, c_2, \ldots, c_n\}$-recurrence. For example, recall that the Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$ has a $\{1, 1\}$-recurrence: $F_{n+1} = F_n + F_{n-1}$.

Generating Recursive Sequences

In many cases generating integer sequences by recursion is more efficient than using an explicit formula. There are many methods to generate recursive sequences in *Mathematica*. We demonstrate four different methods to illustrate their strengths and weaknesses.

Example: Let's generate the sequence a_n defined by a $\{5, -1\}$-recurrence:

$$a_n = 5\, a_{n-1} - a_{n-2}$$

METHOD 1 - We define the sequence a_n as a delayed function **a[n]** using the **SetDelayed** assignment specified by the symbol ":=" (colon-equal sign):

```
Clear[a];
a[0] = 2;
a[1] = 5;
a[n_] := 5 a[n - 1] - a[n - 2]
Timing[Table[a[n], {n, 0, 25}]]
```

```
{3.291, {2, 5, 23, 110, 527, 2525, 12 098, 57 965, 277 727, 1 330 670, 6 375 623, 30 547 445,
  146 361 602, 701 260 565, 3 359 941 223, 16 098 445 550, 77 132 286 527, 369 562 987 085,
  1 770 682 648 898, 8 483 850 257 405, 40 648 568 638 127, 194 758 992 933 230,
  933 146 396 028 023, 4 470 972 987 206 885, 21 421 718 540 006 402, 102 637 619 712 825 125}}
```

The **Timing** command shows that it took *Mathematica* about 3 seconds to generate the first 25 terms, which is quite slow. This is because previous terms, instead of being stored to memory, are always recalculated (due to the **SetDelayed** assignment) in generating the next term of the sequence.

METHOD 2 - To force *Mathematica* to store previous terms, we adapt our definition of **a[n]** as follows:

```
ClearAll[a];
a[0] = 2;
a[1] = 5;
a[n_] := a[n] = 5 a[n - 1] - a[n - 2]
Timing[Table[a[n], {n, 0, 25}]]
```

```
{0., {2, 5, 23, 110, 527, 2525, 12 098, 57 965, 277 727, 1 330 670, 6 375 623, 30 547 445,
  146 361 602, 701 260 565, 3 359 941 223, 16 098 445 550, 77 132 286 527, 369 562 987 085,
  1 770 682 648 898, 8 483 850 257 405, 40 648 568 638 127, 194 758 992 933 230,
  933 146 396 028 023, 4 470 972 987 206 885, 21 421 718 540 006 402, 102 637 619 712 825 125}}
```

Observe that this methods drastically reduces the run-time to almost zero. In fact, calculating the first 1000 terms requires less than a second (output is suppressed):

```
ClearAll[a];
a[0] = 2;
a[1] = 5;
a[n_] := a[n] = 5 a[n - 1] - a[n - 2]
Timing[Table[a[n], {n, 0, 1000}];]
```

```
{0.032, Null}
```

METHOD 3 - An equivalent approach to Method 2 is to program a loop to generate a_n inside a *Mathematica* module (subroutine):

```
ClearAll[a];
sequence[nMax_] := Module[{a, n},
  a[0] = 2;
  a[1] = 5;
  Do[
   a[n] = 5 a[n - 1] - a[n - 2],
   {n, 2, nMax}
  ];
  Table[a[n], {n, 0, nMax}]
 ]
Timing[sequence[1000];]
```

```
{0.032, Null}
```

METHOD 4

One can also use *Mathematica*'s **LinearRecurrence** command, which is useful for generating *linear* recursive sequences.

LinearRecurrence

```
? LinearRecurrence
```

LinearRecurrence[*ker*, *init*, *n*] gives the sequence of length *n*
obtained by iterating the linear recurrence with kernel *ker* starting with initial values *init*.
LinearRecurrence[*ker*, *init*, {n_{min}, n_{max}}] yields terms n_{min} through n_{max} in the linear recurrence sequence. ≫

```
Timing[LinearRecurrence[{5, -1}, {2, 5}, 1000];]
```

```
{0.016, Null}
```

Besides being simple to use, timing shows that **LinearRecurrence** is twice as fast as Methods 2 and 3 in generating the first 1000 terms.

NOTE: We can of course use the **FindSequenceFunction** command to find an explicit formula for a_n based on say the first ten terms:

```
Simplify[FindSequenceFunction[LinearRecurrence[{5, -1}, {2, 5}, 10], n]]
```

$$\frac{1}{2}\left(-\left(-5+\sqrt{21}\right)\left(\frac{1}{2}\left(5+\sqrt{21}\right)\right)^n+\left(\frac{1}{2}\left(5-\sqrt{21}\right)\right)^n\left(5+\sqrt{21}\right)\right)$$

However, we find that this formula is much slower in generating the first 1000 terms:

```
ClearAll[a]
```

$$\text{a[n\_]} = \frac{1}{2}\left(-\left(-5+\sqrt{21}\right)\left(\frac{1}{2}\left(5+\sqrt{21}\right)\right)^n+\left(\frac{1}{2}\left(5-\sqrt{21}\right)\right)^n\left(5+\sqrt{21}\right)\right);$$

```
Timing[Table[Simplify[a[n]], {n, 1, 1000}];]
```

```
{5.163, Null}
```

This is because considerable computation is required to simplify the binomial expressions for a_n to an integer value.

Finding Linear Recurrences

If a recurrence is not known for a finite sequence, then the following command attempts to find such a recurrence.

FindLinearRecurrence

```
? FindLinearRecurrence
```

FindLinearRecurrence[*list*] finds if possible the minimal linear recurrence that generates *list*.

FindLinearRecurrence[*list*, *d*] finds if possible the linear recurrence of maximum order *d* that generates *list*. ≫

Example 2.15 - Coconut Problem Revisited

Suppose we wish to find a linear recurrrence for the set of solutions $r(n)$ to the Coconut Problem discussed in Example :

```
FindLinearRecurrence[{21, 121, 621, 3121, 15 621}]
```

```
{6, -5}
```

Thus, $r(n)$ satisfies the recurrence $r(n) = 6\,r(n-1) - 5\,r(n-2)$, which we now use to generate the next five solutions:

```
LinearRecurrence[{6, 5}, {21, 121}, 10]
```

```
{21, 121, 831, 5591, 37 701, 254 161, 1 713 471, 11 551 631, 77 877 141, 525 021 001}
```

Example 2.16 - Rearranging the Integers

The integers, commonly expressed as a sequence which runs in both directions, $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, satisfy a simple recurrence: $n = (n-1)+1$. In this example we will show that rearranging them can produce some rather interesting recurrences (see [MP]).

a) Let's rearranging the integers into a sequence that begins with 0 and then alternatees between n and $-n$: $\{0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$. What recurrence does this sequence satisfy?

```
FindLinearRecurrence[{0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5}]
```

```
{-1, 1, 1}
```

b) How about the recurring sequence in which every integer appears exactly twice? Let's try $\{0, 0, 1, 1, -1, -1, 2, 2, -2, -2, 3, 3, -3, -3, 4, 4, -4, -4, \ldots\}$:

```
FindLinearRecurrence[
 {0, 0, 1, 1, -1, -1, 2, 2, -2, -2, 3, 3, -3, -3, 4, 4, -4, -4, 5, 5, -5, -5}]

{1, -2, 2, -1, 1}
```

c) Find a formula for the recurrence of a corresponding sequence in which every integer occurs exactly *n*-times?

NOTE: As its name implies, **FindLinearRecurrence** is only useful in finding constant-coefficient linear recurrences (of homogeneous type). Thus, it is not able to find more general linear recurrences satisfied by sequences such the factorials $a(n) = n!$, which has the recurrence

$$a(1) = 1;\ a(n) = n \cdot a(n-1)$$

or nonlinear recurrences such as $a(n) = a(n-1)\,a(n-2)$ with initial values $a(0) = 1$ and $a(1) = 2$.. Try it for yourself to see how **FindLinearRecurrence** evaluates these examples.

Example 2.17 - Summing House Numbers

A classic problem asks for a house number in Louvain, Belgium where the sum of the house numbers before it equals the sum of the house numbers after it (see [Pa]). It is assumed that house numbers take on integers from 1 to k. For example, if $k = 8$, then the desired house number is 6 since

$$1 + 2 + 3 + 4 + 5 = 7 + 8$$

However, not all integer values of k will yield an integer solution for h (the reader can easily check this for $k = 5$). In fact, as we shall see later, such solutions are sparse.

More generally, given a positive integer k, we seek to find another positive integer h such that

$$1 + 2 + \ldots + (h-1) = (h+1) + (h+2) + \ldots + k \qquad (2.11)$$

Simplifying the equation above in *Mathematica* yields

```
Simplify[Sum[i, {i, 1, h - 1}] == Sum[i, {i, h + 1, k}]]

2 h^2 == k + k^2
```

Therefore, it suffices to solve for h and check which values of k will yield integers solutions for h. Here is a table listing the first six solutions where k is less than 10,000:

Solutions to the House Number Problem

h	k
1	1
6	8
35	49
204	288
1189	1681
6930	9800

NOTE: Observe that *Mathematica* used the summation formulas $\sum_{i=1}^{h-1} i = h(h-1)/2$ and $\sum_{j=h+1}^{k} j = (h+k+1)(k-h)/2$ to simplify equation (2.11) above. Without these formulas, the run time required to generate the table of solutions above would be much longer. However, even with these formulas, finding the next five solutions, which grow relatively quickly, will require an exceedingly long time.

A more efficient approach to finding additional solutions is to find a recursion satisfied by the first six values for h:

```
FindLinearRecurrence[{1, 6, 35, 204, 1189, 6930}]
```

```
{6, -1}
```

This recursion allows us to now quickly generate the first ten solutions:

```
LinearRecurrence[{6, -1}, {1, 6}, 10]
```

```
{1, 6, 35, 204, 1189, 6930, 40 391, 235 416, 1 372 105, 7 997 214}
```

NOTES:

1. The equation $2h^2 = k^2 + k$ can be transformed into a Pell equation by completing the square in k and making an appropriate change of variables:

$$2h^2 = k^2 + k = (k + 1/2)^2 - 1/4 \Rightarrow 8h^2 = (2k + 1)^2 - 1$$

Setting $x = 2k + 1$ and $y = 2h$ leads to the Pell equation

$$x^2 - 2y^2 = 1$$

which we discussed in Example (2.1). The solutions for (x, y) correspond precisely to solutions for (h, k) as seen in the following table:

```
ClearAll[x, y]; Pellsolutionsplus = FindInstance[
  (x^2 - 2 y^2 == 1) && 0 < x < 1000 && 0 < y < 1000, {x, y}, Integers, 10]
dataPellsolutionsplus = Table[{Sort[Pellsolutionsplus][[k, 1, 2]],
   Sort[Pellsolutionsplus][[k, 2, 2]]}, {k, 1, Length[Pellsolutionsplus]}]
ColumnDataDisplay[Table[{n, dataPellsolutionsplus[[n, 1]],
   dataPellsolutionsplus[[n, 2]], dataPellsolutionsplus[[n, 2]] / 2,
   (dataPellsolutionsplus[[n, 1]] - 1) / 2}, {n, 1, Length[dataPellsolutionsplus]}],
 10, {"n", "xn", "yn", "hn=yn/2", "kn=(xn-1)/2"}, "Pell Equations x2-2y2=1"]
```

```
{{x → 3, y → 2}, {x → 17, y → 12}, {x → 99, y → 70}, {x → 577, y → 408}}
```

```
{{3, 2}, {17, 12}, {99, 70}, {577, 408}}
```

Pell Equations $x^2-2y^2=1$

n	x_n	y_n	$h_n = y_n/2$	$k_n = (x_n - 1)/2$
1	3	2	1	1
2	17	12	6	8
3	99	70	35	49
4	577	408	204	288

2. The values for h also correspond to the products $x_n\, y_n$, where $(x_n\, y_n)$ is a solution to the Pell equations $x^2 - 2y^2 = \pm 1$:

```
{{x → 1, y → 1}, {x → 7, y → 5}, {x → 41, y → 29}, {x → 239, y → 169},
 {x → 3, y → 2}, {x → 17, y → 12}, {x → 99, y → 70}, {x → 577, y → 408}}
```

Pell Equations $x^2-2y^2=\pm 1$

n	x_n	y_n	$h_n = x_n \cdot y_n$
1	1	1	1
2	3	2	6
3	7	5	35
4	17	12	204
5	41	29	1189
6	99	70	6930
7	239	169	40 391
8	577	408	235 416

Can you explain why? In fact, the sequence {1, 6, 35, 204, 1189, 6930, ...} are square-triangular numbers (A001109), which we discuss further in the next chapter. Note also that the solutions for x_n and y_n share the same recurrence:

```
FindLinearRecurrence[Table[dataPellsolutionsplusminus[[n, 1]],
  {n, 1, Length[dataPellsolutionsplusminus]}]]
FindLinearRecurrence[Table[dataPellsolutionsplusminus[[n, 2]],
  {n, 1, Length[dataPellsolutionsplusminus]}]]
```

```
{2, 1}

{2, 1}
```

FURTHER EXPLORATION: Find a formula involving x_n, y_n, and y_{2n}. Then prove your formula.

2.2.2.2 Special Subsequences

Sequences sometimes reveal themselves through patterns involving their subsequences. Given a sequence $\{a_n\}$, a subsequence $\{b_n\}$ is a sequence whose elements come from $\{a_n\}$ and are listed in the same order. More formally, $\{b_n\}$ is called a subsequence of $\{a_n\}$ if $b_n = a_{f(n)}$ where $f(n)$ is an increasing integer function, i.e., $f(n) < f(n+1)$. Here are some common subsequences that we shall consider in this book:

S1. Even Elements: $\{a_{2n}\} = \{a_0, a_2, a_4, a_6, \ldots\}$

S2. Odd Elements: $\{a_{2n+1}\} = \{a_1, a_3, a_5, a_7, \ldots\}$

S3. Elements at Square Positions: $\{a_{n^2}\} = \{a_1, a_4, a_9, a_{16}, \ldots\}$

S4. Elements at Powers of 2: $\{a_{2^n}\} = \{a_1, a_2, a_4, a_8, \ldots\}$

S5. Elements at Prime Positions: $\{a_{\text{Prime}(n)}\} = \{a_2, a_3, a_5, a_7, \ldots\}$, where Prime($n$) refers to the n-th prime, starting with Prime(1) = 2.

Example 2.18 Stern's Diatomic Sequence (Continued)

Let's continue our search for patterns with the diatomic sequence $\{a_n\}$ discussed in Example 2.13 and compare the sequence itself with some of its subsequences, for example, those at even positions and at odd positions.

Diatomic sequence

n	a_n
0	0
1	1
2	1
3	2
4	1
5	3
6	2
7	3
8	1
9	4

Even Elements

n	a_{2n}
0	0
1	1
2	1
3	2
4	1
5	3
6	2
7	3
8	1
9	4

Odd Elements

n	a_{2n+1}
0	1
1	2
2	3
3	3
4	4
5	5
6	5
7	4
8	5
9	7

Patterns for these subsequences now become clear: $a_{2n} = a_n$ and $a_{2n+1} = a_n + a_{n+1}$.

Thus, the *diatomic sequence* can be alternatively defined through the recurrence

$$\begin{cases} a_{2n} = a_n \\ a_{2n+1} = a_n + a_{n+1} \end{cases} \tag{2.12}$$

with $a_0 = 0$, $a_1 = 1$. We generate the first 10 terms using this recurrence to confirm that it agrees with the original definition.

```
a[0] = 0;
a[1] = 1;
a[n_] := If[Mod[n, 2] == 0, a[n / 2], a[(n - 1) / 2] + a[(n + 1) / 2]]
Table[a[n], {n, 0, 9}]
```

```
{0, 1, 1, 2, 1, 3, 2, 3, 1, 4}
```

2.2.3 Sequence Transformations

Transformations of sequences allow us to generate new sequences (or subsequences). Patterns from these new sequences can be either intereseting in their own right or help us to understand the original sequence. In this subsection, we shall consider some well known transformations that have become part of the standard toolkit for studying integer sequences experimentally. A *sequence transformation* T is a function which defines one sequence, $\{b_n\}$, in terms of another sequence, $\{a_n\}$.

Special Transformations

Here is a list of some common transformations of a given sequence $\{a_n\}$.

T1. Partial Sums: $b_n = a_1 + a_2 + \ldots + a_n$

T2. Linear Weighted Partial Sums: $b_n = a_1 + 2\,a_2 + \ldots + n \cdot a_n$

T3. Squares: $b_n = a_n^2$

T4. Partial Sums of Squares: $b_n = a_1^2 + a_2^2 + \ldots + a_n^2$

T5. Product of Two Consecutive Elements: $b_n = a_n \cdot a_{n+1}$

T6. Determinant: $b_n = \begin{vmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{vmatrix} = a_n \cdot a_{n+2} - a_{n+1}^2$

Example 2.19 - Sums of Fibonacci Numbers

a) At the beginning of this chapter we had found a pattern for the partial sums of the Fibonacci sequence $\{F_n\}$ based on the following table of values:

Fibonacci Numbers Versus Its Partial Sums

n	F_n	$\sum_{k=0}^{n} F_k$
0	0	0 = 0
1	1	1 = 0+1
2	1	2 = 0+1+1
3	2	4 = 0+1+1+2
4	3	7 = 0+1+1+2+3
5	5	12 = 0+1+1+2+3+5
6	8	20 = 0+1+1+2+3+5+8
7	13	33 = 0+1+1+2+3+5+8+13
8	21	54 = 0+1+1+2+3+5+8+13+21
9	34	88 = 0+1+1+2+3+5+8+13+21+34

Shifting the column for $\{F_n\}$ up two rows revealed the following identity:

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1 \tag{2.13}$$

For this simple example, we can avoid the above analysis altogether by asking *Mathematica* to evaluate the partial sums transformation symbolically, which yields the same identity:

```
Sum[Fibonacci[k], {k, 0, n}]
```

```
-1 + Fibonacci[2 + n]
```

b) Next, we consider the sum of the odd Fibonacci numbers F_{2n+1}:

Partial Sums of Odd Fibonacci Numbers

n	F_n	$\sum_{k=0}^{n} F_{2k+1}$
0	0	1 = 1
1	1	3 = 1+2
2	1	8 = 1+2+5
3	2	21 = 1+2+5+13
4	3	55 = 1+2+5+13+34
5	5	144 = 1+2+5+13+34+89
6	8	377 = 1+2+5+13+34+89+233
7	13	987 = 1+2+5+13+34+89+233+610
8	21	2584 = 1+2+5+13+34+89+233+610+1597
9	34	6765 = 1+2+5+13+34+89+233+610+1597+4181

It is clear that these sums are given by the Fibonacci numbers at even positions (A001906):

$$\sum_{k=0}^{n} F_{2k+1} = F_{2(n+1)} \tag{2.14}$$

Again, this identity is confirmed by *Mathematica*;

```
Sum[Fibonacci[2 k + 1], {k, 0, n}]
```

```
Fibonacci[2 (1 + n)]
```

NOTE: If one uses the index $2k-1$ instead of $2k+1$ to represent the odd Fibonacci numbers, then *Mathematica* doesn't recognize their partial sums as an identity, but instead gives an explicit formula.

```
Sum[Fibonacci[2 k - 1], {k, 1, n}]
```

$$\frac{4^n\left(1+\sqrt{5}\right)^{-2n}\left(-1+\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^{4n}\right)}{\sqrt{5}}$$

This follows from *Mathematica*'s knowledge of Binet's formula for the Fibonacci numbers:

$$F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}} \tag{2.15}$$

To obtain identity (2.14), it suffices to verify that *Mathematica*'s explicit formula is equal to F_{2n}, which follows from replacing n by $2n$ in Binet's formula.

c) What do you expect the formula to be for the sum of the even Fibonacci numbers (make a guess before performing your experiment)? Let's verify your conjecture:

Partial Sums of Odd Fibonacci Numbers

n	F_n	$\sum_{k=0}^{n} F_{2k}$
0	0	0 = 0
1	1	1 = 0+1
2	1	4 = 0+1+3
3	2	12 = 0+1+3+8
4	3	33 = 0+1+3+8+21
5	5	88 = 0+1+3+8+21+55
6	8	232 = 0+1+3+8+21+55+144
7	13	609 = 0+1+3+8+21+55+144+377
8	21	1596 = 0+1+3+8+21+55+144+377+987
9	34	4180 = 0+1+3+8+21+55+144+377+987+2584

This time a slight twist emerges in the pattern: the sum of the first n even Fibonacci numbers is one less than the Fibonacci number F_{2n+1}:

$$\sum_{m=1}^{n} F_{2k} = F_{2n+1} - 1 \tag{2.16}$$

Mathematica again gives the same identity:

```
Sum[Fibonacci[2 k], {k, 0, n}]
```

```
-1 + Fibonacci[1 + 2 n]
```

d) How about the partial sum of the squares of the Fibonacci numbers?

Partial Sums of Squares of Fibonacci Numbers

n	F_n	$\sum_{k=0}^{n} F_k^2$
0	0	0 = 0
1	1	1 = 0+1
2	1	2 = 0+1+1
3	2	6 = 0+1+1+4
4	3	15 = 0+1+1+4+9
5	5	40 = 0+1+1+4+9+25
6	8	104 = 0+1+1+4+9+25+64
7	13	273 = 0+1+1+4+9+25+64+169
8	21	714 = 0+1+1+4+9+25+64+169+441
9	34	1870 = 0+1+1+4+9+25+64+169+441+1156

Is there a pattern? Let's consider their divisors:

```
dataFibonacciPartialSumsSquares =
 Table[{n, Fibonacci[n], Sum[Fibonacci[k]^2, {k, 0, n}],
   Divisors[Sum[Fibonacci[k]^2, {k, 0, n}]]}, {n, 1, 9}];
ColumnDataDisplay[dataFibonacciPartialSumsSquares, 10,
 {"n", "Fn", "∑_{k=0}^{n} F_k^2", "Divisors of ∑_{k=0}^{n} F_k^2"},
 "Partial Sums of Squares of Fibonacci Numbers", Left]
```

Partial Sums of Squares of Fibonacci Numbers

n	F_n	$\sum_{k=0}^{n} F_k^2$	Divisors of $\sum_{k=0}^{n} F_k^2$
1	1	1	{1}
2	1	2	{1, 2}
3	2	6	{1, 2, 3, 6}
4	3	15	{1, 3, 5, 15}
5	5	40	{1, 2, 4, 5, 8, 10, 20, 40}
6	8	104	{1, 2, 4, 8, 13, 26, 52, 104}
7	13	273	{1, 3, 7, 13, 21, 39, 91, 273}
8	21	714	{1, 2, 3, 6, 7, 14, 17, 21, 34, 42, 51, 102, 119, 238, 357, 714}
9	34	1870	{1, 2, 5, 10, 11, 17, 22, 34, 55, 85, 110, 170, 187, 374, 935, 1870}

Yes, in this case each sum is equal to the product of two consecutive Fibonacci numbers, also known as the golden rectangle numbers (A001654):

```
Sum[Fibonacci[k]^2, {k, 1, n}]
```

```
Fibonacci[n] Fibonacci[1 + n]
```

Thus we've discovered the identity:

$$\sum_{m=1}^{n} F_k^2 = F_n\, F_{n+1} \tag{2.17}$$

NOTE: Observe in the table above that the two Fibonacci divisors F_n and F_{n+1} seem to always be consecutive in the list of divisors for $\sum_{m=1}^{n} F_k^2$. In other words, there are no divisors of $\sum_{m=1}^{n} F_k^2$ that lie between F_n and F_{n+1}. Can you explain why?

e) Let's apply the determinant transformation to the Fibonacci sequence:

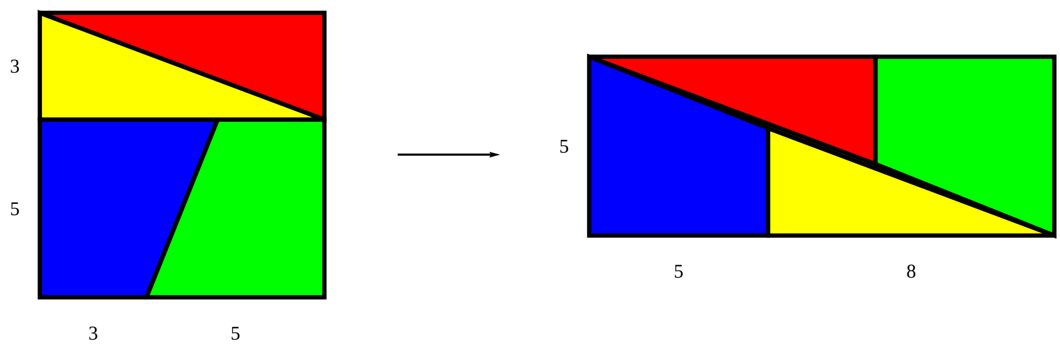
Determinant Transformation of Fibonacci Sequence

n	$F_n F_{n+2} - F_{n+1}^2$
0	−1
1	1
2	−1
3	1
4	−1
5	1
6	−1
7	1
8	−1
9	1

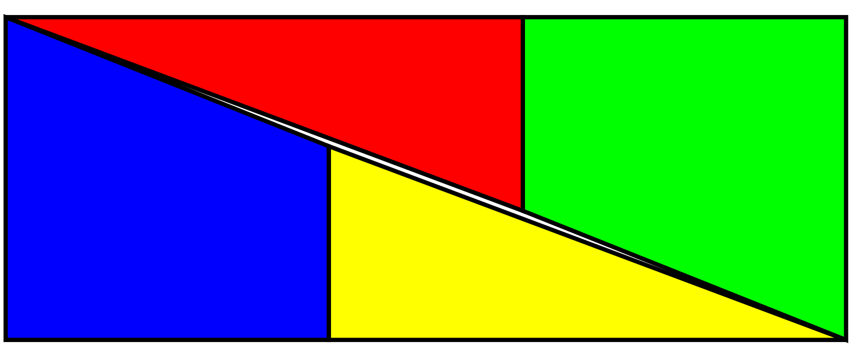
The pattern in this case is quite simple and yields Cassini's identity:

$$F_n\, F_{n+2} - F_{n+1}^2 = (-1)^{n+1} \tag{2.18}$$

NOTE: Cassini's identity is the basis for the following visual deception known as the Missing Square Puzzle: Take a square whose sides are 8 units in length. Divide the square into four pieces according to the illustration below and rearrange them to form a 5 x 13 rectangle:



However, the area the square is $8^2 = 64$ square units, whereas the area of the rectangle is $5 \times 13 = 65$ square units. What happened to the missing square? The answer lies in the fact that the four pieces do not form an exact rectangle; their inclines have different slopes. There is a gap in the shape of a parallelogram in the middle of the rectangle. This becomes more visible if we zoom in:



FURTHER EXPLORATION: Explore on your own to find similar patterns for the determinant transformation of even Fibonacci numbers. Repeat for the odd Fibonacci numbers.

Example 2.20 - Power Sums

John Wallis, in his investigation of areas under polynomial functions, discovered some remarkable simple patterns regarding ratios of *power sums*, i.e., sums of consecutive integers of the form $1^p + 2^p + ... + n^p$. Define

$$R(n,\ p) = \frac{1^p + 2^p + ... + n^p}{n^p + n^p + ... + n^p} = \frac{\sum_{k=1}^n k^p}{n^{p+1}} \tag{2.19}$$

where the denominator is the sum of n copies of n^p. Wallis was able to detect patterns for $R(n,\ p)$, which we demonstrate for $R(n,\ 2)$. Towards this end, let's examine the values of $R(n,\ 2)$ for $1 \le n \le 10$ as shown in the table below.

n	$R(n,2)$
1	$\frac{0^2+1^2}{1^2+1^2} == \frac{1}{2}$
2	$\frac{0^2+1^2+2^2}{2^2+2^2+2^2} == \frac{5}{12}$
3	$\frac{0^2+1^2+2^2+3^2}{3^2+3^2+3^2+3^2} == \frac{7}{18}$
4	$\frac{0^2+1^2+2^2+3^2+4^2}{4^2+4^2+4^2+4^2+4^2} == \frac{3}{8}$
5	$\frac{0^2+1^2+2^2+3^2+4^2+5^2}{5^2+5^2+5^2+5^2+5^2+5^2} == \frac{11}{30}$

n	$R(n,2)$
6	$\frac{0^2+1^2+2^2+3^2+4^2+5^2+6^2}{6^2+6^2+6^2+6^2+6^2+6^2+6^2} == \frac{13}{36}$
7	$\frac{0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2}{7^2+7^2+7^2+7^2+7^2+7^2+7^2+7^2} == \frac{5}{14}$
8	$\frac{0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2}{8^2+8^2+8^2+8^2+8^2+8^2+8^2+8^2+8^2} == \frac{17}{48}$
9	$\frac{0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2}{9^2+9^2+9^2+9^2+9^2+9^2+9^2+9^2+9^2+9^2} == \frac{19}{54}$
10	$\frac{0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2}{10^2+10^2+10^2+10^2+10^2+10^2+10^2+10^2+10^2+10^2+10^2} == \frac{7}{20}$

Visual inspection suggests that the numerators in the fractions above consist of the odd integers and the denominators consists of multiples of 6, although the pattern is not consistent due to cancellation of factors. *Mathematica* confirms that this is indeed the pattern:

```
FindSequenceFunction[{1 / 2, 5 / 12, 7 / 18, 3 / 8, 11 / 30, 13 / 36, 5 / 14}, n]
```

$$\frac{1 + 2\,n}{6\,n}$$

Thus,

$$R(n,\ 2) = \frac{1 + 2\,n}{6\,n} \tag{2.20}$$

NOTE : For those with a background in calculus, observe that the limiting value of $R(n,\ 2)$ as $n \to \infty$ equals 1/3, which gives the exact area under the parabola $f(x) = x^2$ along the interval [0, 1]. This is because $R(n,\ 2)$ represents an approximation of this area by n rectangles, which can be demonstrated as follows: partition the interval [0, 1] into n subintervals having a uniform width of $1/n$. Each subinterval $[(k-1)/n,\ k/n]$ then defines the base of a rectangle R_k whose height is specified by the value of $f(x)$ at the right endpoint of the subinterval. The area of each rectangle is given by

$$\text{area}(R_k) = \frac{1}{n} \cdot f\left(\frac{k}{n}\right) = \frac{k^2}{n^3} \tag{2.21}$$

and the total of these areas, called a *Riemann sum*, is precisely $R(n,\ 2)$:

$$\text{area}(R_1) + \text{area}(R_2) + ... + \text{area}(R_n) = \frac{1^2}{n^3} + \frac{2^2}{n^3} + ... + \frac{n^2}{n^3} = \frac{1^2 + 2^2 + ... + n^2}{n^3} = \frac{1^2 + 2^2 + ... + n^2}{n^2 + n^2 + ... + n^2} = R(n,\ 2) \tag{2.22}$$

FURTHER EXPLORATION:

1. Find a formula for $R(n,\ 3)$ and $R(n,\ 4)$.

2. Find a general formula for $R(n,\ p)$.

2.2.3.2 Finite Differences

The method of finite differences is very powerful tool for determining formulas of sequences generated by a polynomial. In a collection of his 50 most interesting columns (as judged by reader response), *The Colossal Book of Mathematics*, Martin Gardner writes (p. 15):

> *Recreational problems involving permutations and combinations often contain low-order formulas that can be correctly guessed by the method of finite differences and later (one hopes) proved.*

Given a sequence $\{a_1, a_2, a_3, \ldots\}$, we define its *first differences* (or *gap*) to be

$$\Delta a_n = a_{n+1} - a_n \tag{2.23}$$

and more generally its d-th differences (or differences to *level d*) to be

$$\Delta^d a_n = \Delta^{d-1} a_{n+1} - \Delta^{d-1} a_n \tag{2.24}$$

for $p \geq 1$ where $\Delta^0 a_n = a_n$.

If a formula is known for the m-th differences, then this yields a recurrence for the original sequence by backwards computation:

$$\begin{aligned} \Delta^d a_n &= \Delta^{d-1} a_{n+1} - \Delta^{d-1} a_n \\ &= \Delta^{d-2} a_{n+2} - 2\,\Delta^{d-2} a_{n+1} + \Delta^{d-2} a_n \\ &= \Delta^{d-3} a_{n+3} - 3\,\Delta^{d-3} a_{n+2} + 3\,\Delta^{d-3} a_{n+1} - \Delta^{d-3} a_n \\ &\ldots \\ &= \sum_{i=0}^{d} (-1)^i \binom{d}{i} a_{n+d-i} \end{aligned} \tag{2.25}$$

On the other hand, if all d-differences $\Delta^d a_k$ are known for some fixed k, then an explicit formula for the original sequence is given by

$$a_{n+k} = \sum_{m=0}^{n} \binom{n}{m} \Delta^m a_k \tag{2.26}$$

See [GKP], pp. 187-192 for a derivation of this formula.

The following *Mathematica* command allows one to compute the difference table of a sequence to any level:

Differences

```
? Differences
```

Differences[*list*] gives the successive differences of elements in *list*.

Differences[*list*, *n*] gives the n^{th} differences of *list*.

Differences[*list*, $\{n_1, n_2, \ldots\}$] gives the successive $n_k{}^{\text{th}}$ differences at level k in a nested list. ≫

Example 2.21

Consider the sequence $\{a_n\}$ consisting of sums of squares as defined by $a_n = 1^2 + 2^2 + \ldots + n^2$:

Sums of Squares

n	a_n
1	$1 == 1^2$
2	$5 == 1^2 + 2^2$
3	$14 == 1^2 + 2^2 + 3^2$
4	$30 == 1^2 + 2^2 + 3^2 + 4^2$
5	$55 == 1^2 + 2^2 + 3^2 + 4^2 + 5^2$
6	$91 == 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$
7	$140 == 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$
8	$204 == 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$
9	$285 == 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2$
10	$385 == 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$

The following array lists the successive differences of $\{a_n\}$ up to level 4:

d	$\triangle^d a_n$
1	{4, 9, 16, 25, 36, 49, 64, 81, 100}
2	{5, 7, 9, 11, 13, 15, 17, 19}
3	{2, 2, 2, 2, 2, 2, 2}
4	{0, 0, 0, 0, 0, 0}

Thus, we have the following formulas for its differences up to level 3:

$$\Delta a_n = (n+1)^2$$
$$\Delta^2 a_n = (2n+3)$$
$$\Delta^3 a_n = 2$$

All higher-order differences are equal to zero. As a result, the summation in equation (2.26) may be terminated at $m = 3$, giving an effective formula for $\{a_n\}$:

```
differences[d_, n_] := FindSequenceFunction[Differences[sumsofsquares, d], n];
Sum[Binomial[n, m] * differences[m, 0], {m, 0, 3}]
```

$$n + \frac{3}{2}(-1+n)\,n + \frac{1}{3}(-2+n)(-1+n)\,n$$

Simplifying the expression above now yields the classic formula for sums of squares:

```
Factor[Simplify[%]]
```

$$\frac{1}{6}\,n\,(1+n)\,(1+2n)$$

NOTE: *Mathematica* employs similar techniques to obtain the same answer when we ask it to evaluate a_n:

```
a[n]
```

$$\frac{1}{6}\,n\,(1+n)\,(1+2n)$$

Example 2.22 - Gilbreath's Conjecture

Suppose we revise our definition of d-differences to ignore sign and consider only the absolute value of the differences, which we refer to as $|d|$-differences:

Then computing $|d|$-differences for the sequence of primes reveals an interesting pattern known as Gilbreath's conjecture: each sequence of $|d|$-differences, except for the first ($d = 1$), begins with 1. This is indicated by the red entries in the array below.

```
Clear[s]
s[0, n_] := Prime[n]
s[m_, n_] := Abs[s[m - 1, n + 1] - s[m - 1, n]]
```

```
TableForm[
 Table[If[m > 0 && n == 1, Style[s[m, n], Red], s[m, n]], {m, 0, 11}, {n, 1, 12 - m}]]
```

2	3	5	7	11	13	17	19	23	29	31	37
1	2	2	4	2	4	2	4	6	2	6	
1	0	2	2	2	2	2	2	4	4		
1	2	0	0	0	0	0	2	0			
1	2	0	0	0	0	2	2				
1	2	0	0	0	2	0					
1	2	0	0	2	2						
1	2	0	2	0							
1	2	2	2								
1	0	0									
1	0										
1											

In 1993, this pattern was verified by Odlyzko [Od] for all primes up to 10^{13}; however, no proof is known of Gilbreath's conjecture (see [Gi]).

2.2.3.3 Binomial Transform and Inversion

The binomial transform of a sequence $\{a_n\}$ is defined as

$$b(n) = \sum_{k=0}^{n} \binom{n}{k} a(n) \tag{2.27}$$

Recall that $\binom{n}{k}$ are binomial coefficients defined earlier.

The inversion of a sequence $\{a_n\}$ (also referred to as the binomial transform) is defined as

$$c(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a(n) \tag{2.28}$$

Inversion refers to the fact that this transformation is equal to its inverse transform so that one can recover $\{a_n\}$ using the same formula:

$$a(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} c(n) \tag{2.29}$$

Example 2.23

a) Let's apply the binomial transform to the Fibonacci sequence:

```
Clear[b];
b[n_] = Sum[Binomial[n, k] * Fibonacci[k], {k, 0, n}]
```

$$\frac{-\left(\frac{1}{2}\left(3-\sqrt{5}\right)\right)^n+\left(\frac{1}{2}\left(3+\sqrt{5}\right)\right)^n}{\sqrt{5}}$$

Although *Mathematica* does not recognize it, a comparison of the first several values b_n reveals the pattern to be nothing more than the even Fibonacci numbers, F_{2n}.

n	F(n)	$\sum_{k=0}^{n}\binom{n}{k}$F(n)
1	1	1
2	1	3
3	2	8
4	3	21
5	5	55
6	8	144
7	13	377
8	21	987
9	34	2584
10	55	6765

Thus,

$$\sum_{k=0}^{n}\binom{n}{k}F_k=F_{2n} \tag{2.30}$$

Do you think this pattern is special to the Fibonacci sequence? What about other sequences having the same {1, 1} recurrence, but with different initial values, such as the Lucas sequence $\{L_n\}$?

$$L_{n+1}=L_n+L_{n-1};\, L_0=2,\ L_1=1 \tag{2.31}$$

Here are the first ten Lucas numbers, generated using the *Mathematica* function **LucasL**.

```
Table[LucasL[n], {n, 0, 9}]
```

```
{2, 1, 3, 4, 7, 11, 18, 29, 47, 76}
```

Applying the binomial transform shows that it yields a similar formula as the Fibonacci numbers:

```
Clear[c];
c[n_] = Sum[Binomial[n, k] * LucasL[k], {k, 0, n}]
```

$$\left(\frac{1}{2}\left(3-\sqrt{5}\right)\right)^n+\left(\frac{1}{2}\left(3+\sqrt{5}\right)\right)^n$$

Again, comparing the first several values of c_n shows that indeed they are the Lucas numbers at even positions (A005248):

n	L(n)	$\sum_{k=0}^{n} \binom{n}{k}$ L(n)
1	1	3
2	3	7
3	4	18
4	7	47
5	11	123
6	18	322
7	29	843
8	47	2207
9	76	5778
10	123	15 127

This leads us to the same identity:

$$\sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n} \tag{2.32}$$

FURTHER EXPLORATION: Explore whether this identity holds for all {1, 1}-recurrences.

b) Now consider the inversion of the Fibonacci sequence:

```
Clear[b];
b[n_] = Sum[(-1)^k * Binomial[n, k] * Fibonacci[k], {k, 0, n}]
```

$$\frac{\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^n - \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n}{\sqrt{5}}$$

Do you recognize this formula? You should since it is nothing more than (the negative of) Binet's formula for the Fibonacci numbers as seen from its initial values given below.

```
Table[Simplify[b[n]], {n, 0, 10}]
```

{0, -1, -1, -2, -3, -5, -8, -13, -21, -34, -55}

Thus,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_k = -F_n \tag{2.33}$$

NOTE: Observe that *Mathematica* fails to recognize b_n as the negative Fibonacci numbers, even if we apply the **FindSequence-Function** command to its first ten values:

```
Table[Simplify[b[n]], {n, 0, 10}]
FindSequenceFunction[Table[Simplify[b[n]], {n, 0, 10}], n]
```

{0, -1, -1, -2, -3, -5, -8, -13, -21, -34, -55}

$\frac{1}{2}$ (Fibonacci[n] - LucasL[n])

Let's proceed to the inversion of the Lucas numbers:

```
Clear[c];
c[n_] = Sum[(-1)^k * Binomial[n, k] * LucasL[k], {k, 0, n}]
```

$$\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^n+\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n$$

Here, we find that the Lucas sequence is preserved under inversion:

```
Table[Simplify[c[n]], {n, 0, 10}]
```

{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123}

Thus,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_k = L_n \tag{2.34}$$

NOTE: For those with a background in linear algebra, inversion is an involution, i.e., a linear transformation I whose square is the identity transformation, $I^2 = \text{Id}$. It follows that 1 and -1 are its only eigenvalues. The example above demonstrates then that the Lucas and Fibonacci sequences are eigenvectors corresponding to these eigenvalues, respectively. Can you find other sequences that are eigenvectors of the inversion transform?

2.2.3.4 Convolution and Generating Functions

Convolution

Given two sequences $\{a_n\}$ and $\{b_n\}$, we define their convolution to be the sequence

$$c_n = \sum_{k=0}^{n} a_k\, b_{n-k} \tag{2.35}$$

Convolution arises natural in many applications such as signal processing and statistics. Later in this chapter, we'll describe how convolution formulas can be easily derived for many different sequences using generating functions.

Example 2.24 - Convolution of Recurring Sequences

Let's investigate how sequences like the Fibonacci sequence behave under self-convolution. *Mathematica* gives the following unappealing convolution formula:

```
Clear[c];
c[n_] = Sum[Fibonacci[k] * Fibonacci[n - k], {k, 0, n}]
```

$$\frac{1}{5\left(5+\sqrt{5}\right)}\left(-2\left(1+\sqrt{5}\right)\right)^{-n}\left(\left(1+\sqrt{5}\right)\left(4^n-\left(-2\left(3+\sqrt{5}\right)\right)^n\right)+\left(5+\sqrt{5}\right)\left(4^n+\left(-2\left(3+\sqrt{5}\right)\right)^n\right)n\right)$$

Instead, we'll call on *Mathematica* to find a recurrence for $\{c_n\}$ (A001629):

```
Table[Simplify[c[n]], {n, 1, 10}]
```

{0, 1, 2, 5, 10, 20, 38, 71, 130, 235}

```
FindLinearRecurrence[{0, 1, 2, 5, 10, 20, 38, 71, 130, 235}]
```

{2, 1, -2, -1}

Let's compare this recurrence with that of the convolved Lucas numbers.

```
Clear[c];
c[n_] = Sum[LucasL[k] * LucasL[n - k], {k, 0, n}]
```

$$\frac{1}{5+\sqrt{5}}\left(-2\left(1+\sqrt{5}\right)\right)^{-n}\left(4^{n}\left(9+\sqrt{5}\right)+\left(-2\left(3+\sqrt{5}\right)\right)^{n}\left(11+3\sqrt{5}\right)+\left(5+\sqrt{5}\right)\left(4^{n}+\left(-2\left(3+\sqrt{5}\right)\right)^{n}\right)n\right)$$

```
Table[Simplify[c[n]], {n, 1, 10}]
```

{4, 13, 22, 45, 82, 152, 274, 491, 870, 1531}

```
FindLinearRecurrence[{4, 13, 22, 45, 82, 152, 274, 491, 870, 1531}]
```

{2, 1, -2, -1}

Thus, we see that both recurrences are the same. As a last experiment, let's convolve the Fibonacci sequence with the Lucas sequence.

```
Clear[c];
c[n_] = Sum[Fibonacci[k] * LucasL[n - k], {k, 0, n}]
```

$$\frac{1}{\sqrt{5}}\left(\frac{1}{2}\left(-1-\sqrt{5}\right)\right)^{-n}\left(-1+\left(\frac{1}{2}\left(-3-\sqrt{5}\right)\right)^{n}\right)(1+n)$$

```
dataConvolveFibonacciLucas = Table[Simplify[c[n]], {n, 1, 10}]
```

{2, 3, 8, 15, 30, 56, 104, 189, 340, 605}

```
FindLinearRecurrence[dataConvolveFibonacciLucas]
```

{2, 1, -2, -1}

Again, we obtain the same recurrence for $\{c_n\} = \{0, 2, 3, 8, 18, 47, \ldots\}$ (A099920). We also observe the following: the first four initial values of $\{c_n\}$ are Fibonacci numbers. This leads us to conjecture that perhaps Fibonacci numbers appear as factors. Indeed, by generating a table of values for the ratio c_n / F_n (except for $n = 0$), we discover the following pattern:

n	c_n / F_n
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9
9	10
10	11

This results in the identity

$$\sum_{k=0}^{n} F_k L_{n-k} = (n+1) F_n \tag{2.36}$$

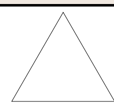
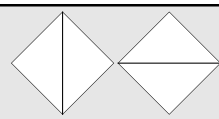
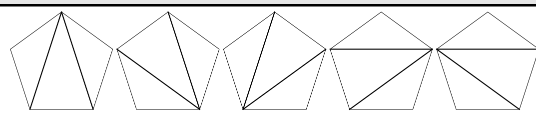
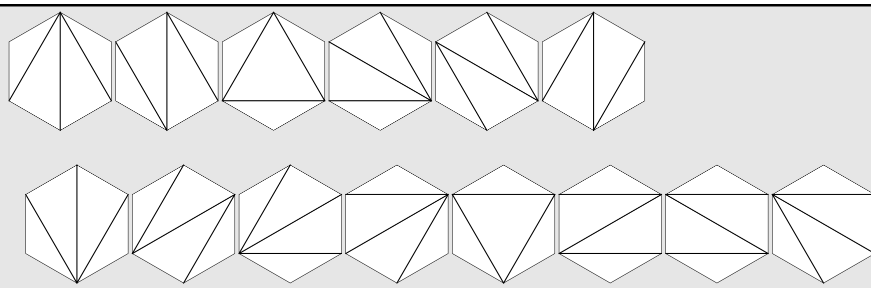
FURTHER EXPLORATION:

a) Investigate whether convolutions of any two {1, 1}-sequences must satisfy a {2, 1, −2, −1}-recurrence. Can you prove this?

b) Investigate recurrences for the convolution of $\{m, 1\}$-sequences by experimenting with various positive integer values for m. Can you find a general formula for the recurrence?

Example 2.25 - Catalan Numbers and Euler's Polygon Division Problem

In 1751, Leonard Euler proposed to Christian Goldbach the problem of finding the number of ways that a n-sided regular polygon can be divided into triangles using its diagonals, which we denote by E_n. Here are plots of the solutions for the first few values of n:

n	Polygon Divisions
3	
4	
5	
6	

Further calculations give us the following values for E_n up to $n = 8$:

n	Number of solutions
3	1
4	2
5	5
6	14
7	42
8	132

Feeding these values into the **FindSequenceFunction** yields

```
FindSequenceFunction[datapolygontriangles, n]
CatalanNumber[-2 + n]
```

Thus, E_n is given by the Catalan number C_{n-2} (A000108), which is known to have formula

$$C_n = \frac{(2n)!}{n!\,(n+1)!} \tag{2.37}$$

Thus,

$$E_n = \frac{(2(n-2))!}{(n-2)!\,(n-1)!} \tag{2.38}$$

The Catalan numbers arise in many other applications in combinatorics (see [St] for 66 different ways of defining the Catalan numbers) and have many interesting properties. For example, suppose we convolve the sequence of Catalan numbers with itself by defining

$$c_n = \sum_{k=0}^{n} C_k\, C_{n-k} \tag{2.39}$$

Here's a table listing the first ten values of c_n:

n	C(n)	c(n)
0	1	1
1	1	2
2	2	5
3	5	14
4	14	42
5	42	132
6	132	429
7	429	1430
8	1430	4862
9	4862	16 796

This shows that convolution shifts the Catalan numbers (as well as the values E_n) and yields the following recurrence formula:

$$C_{n+1} = \sum_{k=0}^{n} C_k\, C_{n-k} \tag{2.40}$$

Generating Functions

An extremely useful analytic approach to studying convolution of sequences is via generating functions. A function $G(x)$ is called a generating function for a sequence $\{a_n\}$ if its elements describe the power series coefficients of $G(x)$, i.e.

$$G(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{2.41}$$

The connection with convolution arises when the generating functions for two sequences $\{a_n\}$ and $\{b_n\}$ are multiplied together; their product yields a function whose power series coefficients give precisely the convolution of $\{a_n\}$ and $\{b_n\}$:

$$\left(\sum_{i=0}^{\infty} a_i\, x^i\right)\left(\sum_{j=0}^{\infty} b_j\, x^j\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i\, b_j\, x^{i+j} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_n\, b_{n-k}\right) x^n = \sum_{n=0}^{\infty} c_n\, x^n \tag{2.42}$$

There are many techniques for finding the generating function for a given sequence. If the sequence has a recurrence, then a formula for the generating function can typically be found by manipulating its power series. We demonstrate this with some examples.

Example 2.26 - Generating function for the Fibonacci sequence

Let F_n denote the Fibonacci sequence as usual and define its generating function by

$$G(x) = \sum_{n=0}^{\infty} F_n\, x^n$$

We then translate the recurrence for the Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$, into an identity involving $G(x)$ as follows:

$$G(x) = \sum_{n=0}^{\infty} F_n\, x^n = 0 + x + \sum_{n=2}^{\infty} F_n\, x^n = x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2})\, x^n = x + x \sum_{n=2}^{\infty} F_{n-1}\, x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2}\, x^{n-2} = x + x\, G(x) + x^2\, G(x)$$

Solving for $G(x)$ gives

$$G(x) = \frac{x}{1 - x - x^2} \tag{2.43}$$

We can obtain the same formulas using *Mathematica*'s **GeneratingFunction** command.

```
GeneratingFunction[Fibonacci[n], n, x]
```

$$-\frac{x}{-1 + x + x^2}$$

Here is a description of the **GeneratingFunction** command:

GeneratingFunction

```
? GeneratingFunction
```

GeneratingFunction[*expr*, *n*, *x*] gives the generating function in x for the sequence whose n^{th} series coefficient is given by the expression *expr*.
GeneratingFunction[*expr*, {n_1, n_2, ...}, {x_1, x_2, ...}] gives the multidimensional generating function in x_1, x_2, ... whose n_1, n_2, ... coefficient is given by *expr*. ≫

A related command, **FindGeneratingFunction**, which combines the two functions **GeneratingFunction** and **FindSequenceFunction**, is useful when a formula for $\{a_n\}$ is not immediately available.

FindGeneratingFunction

```
? FindGeneratingFunction
```

FindGeneratingFunction[{a_1, a_2, …}, *x*] attempts to find a simple generating function in x whose n^{th} series coefficient is a_n.
FindGeneratingFunction[{{n_1, a_1}, {n_2, a_2}, …}, *x*] attempts to find a simple generating function whose ${n_i}^{\text{th}}$ series coefficient is a_i. ≫

Example 2.27 - Generating Functions and Partitions

Generating functions are useful for counting partitions of integers. To demonstrate this, consider the following product of two power series where we multiply term by term and carefully keep track of exponents:

$$\begin{aligned}\sum_{n=0}^{\infty} c_n\, x^n &= \left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{j=0}^{\infty} x^{2\,j}\right) = \left(x^0 + x^1 + x^2 + x^3 + \ldots\right)\left(x^0 + x^{2\cdot 1} + x^{2\cdot 2} + x^{2\cdot 3} + \ldots\right)\\ &= x^{0+0} + x^{1+0} + \left(x^{2+0} + x^{0+2\cdot 1}\right) + \left(x^{3+0} + x^{1+2\cdot 1}\right) + \left(x^{4+0} + x^{2+2\cdot 1} + x^{0+2\cdot 2}\right) + \ldots\\ &= x^0 + x^1 + 2\,x^2 + 2\,x^3 + 3\,x^4 + \ldots\end{aligned} \tag{2.44}$$

Thus, we see that the coefficient c_n counts the number of ways that the integer n can be partitioned as a sum of i copies of 1 and j copies of 2. For example, if $n = 4$, then the expansion above shows that there are three such partitions:

$4 = 1 + 1 + 1 + 1$
$4 = 1 + 1 + 2$
$4 = 2 + 2$

Here's a table listing the number of partitions for n ranging from 1 to 10:

n	Number of Partitions
1	1
2	2
3	2
4	3
5	3
6	4
7	4
8	5
9	5
10	6

The pattern is clear: the values of $\{c_n\}$ consist of positive integers listed in order, but repeated (except for the first value) (A008619). To prove this pattern, we compute the generating functions for the two given power series $\sum_{i=0}^{\infty} x^i$ and $\sum_{j=0}^{\infty} x^{2j}$ by evaluating them directly in *Mathematica*: (the **GeneratingFunction** command is not necessary in this case):

```
G1 = Sum[x^i, {i, 0, Infinity}]
```

$$\frac{1}{1-x}$$

```
G2 = Sum[x^(2 i), {i, 0, Infinity}]
```

$$\frac{1}{1-x^2}$$

Multiplying these two functions together gives us the generating function for $\sum_{n=0}^{\infty} c_n x^n$. To obtain a formula for c_n, it suffices to use Mathematica's **SeriesCoefficient** command

SeriesCoefficient

```
? SeriesCoefficient
```

SeriesCoefficient[*series*, *n*] finds the coefficient of the n^{th}–order term in a power series in the form generated by Series.
SeriesCoefficient[f, {x, x_0, n}] finds the coefficient of $(x - x_0)^n$ in the expansion of f about the point $x = x_0$.
SeriesCoefficient[f, {x, x_0, n_x}, {y, y_0, n_y}, …] finds a coefficient in a multivariate series. ≫

```
SeriesCoefficient[G1 * G2, {x, 0, n}]
```

$$\begin{cases} \frac{1}{4}(3+(-1)^n+2n) & n \geq 0 \\ 0 & \text{True} \end{cases}$$

Thus,

$$c_n = \frac{1}{4}(3 + (-1)^n + 2n) \tag{2.45}$$

This confirms the pattern that we had observed in the table above. Of course, we could have obtained the same answer by computing the coefficient sequence $\{c_n\}$ as the convolution of the coefficient sequences $\{a_n\}$ and $\{b_n\}$ corresponding to the series $\sum_{n=0}^{\infty} a_n x^n = \sum_{i=0}^{\infty} x^i$ and $\sum_{n=0}^{\infty} b_n x^n = \sum_{j=0}^{\infty} x^{2j}$, respectively. Since $a_n = 1$ and $b_n = (1 + (-1)^n)/2$, we find their convolution gives exactly the same formula:

```
Sum[1 * (1 + (-1)^(n - k)) / 2, {k, 0, n}]
```

$$\frac{1}{4}\left(3+(-1)^n+2n\right)$$

An equivalent formula in terms of the floor function is given by $c_n = \left\lfloor \frac{n+2}{2} \right\rfloor$.

NOTE: It makes sense to assume in our definition of a generating function that $G(x) = \sum_{n=0}^{\infty} a_n x^n$ as a power series should converge on some interval of non-zero length; however, this is not always necessary. Even if $\sum_{n=0}^{\infty} a_n x^n$ is a divergent series, we can still algebraically manipulate $\sum_{n=0}^{\infty} a_n x^n$ as a formal power series and hope to get useful results (see [GKP], p. 346 for an example of this involving the sequence of factorials).

2.2.3.5 Continued Fractions

Let a_0 be an integer and $\{a_1, a_2, \ldots, a_n\}$ a finite set of positive integers. The expression

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_n}}}} \tag{2.46}$$

is called a finite continued fraction having $\{a_1, a_2, \ldots, a_n\}$ as quotients. It is also denoted by $[a_0; a_1, \ldots, a_n]$. If the quotients $\{a_1, a_2, \ldots\}$ is infinite, then the corresponding continued fraction $[a_0; a_1, a_2, \ldots]$ is said to be infinite, viewed as a limit of $[a_0; a_1, a_2, \ldots, a_n]$ (called its *convergents*), i.e.,

$$[a_0; a_1, a_2, \ldots] = \lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n] \tag{2.47}$$

Thus, the convergents $p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$ provide rational approximations of $x = [a_0; a_1, a_2, \ldots]$.

Generating Continued Fractions

The continued fraction representation of a real value x can be generated from its integer part $\lfloor x \rfloor$ and fractional part $x - \lfloor x \rfloor$ and repeating this process as follows:

$$a_0 = \lfloor x \rfloor;\ f_1 = x - \lfloor x \rfloor$$

$$a_1 = \left\lfloor \frac{1}{f_1} \right\rfloor;\ f_2 = a_1 - \lfloor a_1 \rfloor$$

$$a_2 = \left\lfloor \frac{1}{f_2} \right\rfloor;\ f_3 = a_2 - \lfloor a_2 \rfloor$$

...

$$a_n = \left\lfloor \frac{1}{f_n} \right\rfloor;\ f_4 = a_3 - \lfloor a_3 \rfloor$$

...

If $f_n = 0$ for some n, then this process is terminated and the resulting continued fraction for $x = [a_0; a_1, a_2, \ldots, a_n]$ is finite.

Example 2.28 - Continued Fraction for π

Let $x = \pi$. Then the first three quotients of the continued fraction for π are given by

$$a_0 = \lfloor \pi \rfloor = 3;\ f_1 = \pi - \lfloor \pi \rfloor = 0.14159 \ldots$$

$$a_1 = \left\lfloor \frac{1}{f_1} \right\rfloor = 7;\ f_2 = a_1 - \lfloor a_1 \rfloor = 0.06251 \ldots$$

$$a_2 = \left\lfloor \frac{1}{f_2} \right\rfloor = 15;\; f_3 = a_2 - \lfloor a_2 \rfloor = 0.99659 \ldots$$

$$a_3 = \left\lfloor \frac{1}{f_2} \right\rfloor = 1;\; f_4 = a_3 - \lfloor a_3 \rfloor = 0.00341 \ldots$$

Mathematica has built-in commands to generate continued fractions and their convergents, for example:

```
ContinuedFraction[Pi, 10]
```

```
{3, 7, 15, 1, 292, 1, 1, 1, 2, 1}
```

```
Convergents[Pi, 10]
```

$$\left\{3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103\,993}{33\,102}, \frac{104\,348}{33\,215}, \frac{208\,341}{66\,317}, \frac{312\,689}{99\,532}, \frac{833\,719}{265\,381}, \frac{1\,146\,408}{364\,913}\right\}$$

Unfortunately, these quotients do not follow a recognizable pattern. However, there are generalized continued fractions of π that do follow regular patterns (see [La]).

NOTE: Observe that the convergents above provide better and better approximations of π, as they should since the sequence of convergents must converge to π:

```
dataconvergentspi =
  Table[{n, Row[{Convergents[Pi, 10][[n]], "=", N[Convergents[Pi, 10][[n]], 9]}]},
   {n, 1, 10}];
ColumnDataDisplay[dataconvergentspi, 10, {"n", "pn/qn"}, "Convergents of π"]
```

Convergents of π

n	p_n/q_n
1	$3=3.00000000$
2	$\frac{22}{7}=3.14285714$
3	$\frac{333}{106}=3.14150943$
4	$\frac{355}{113}=3.14159292$
5	$\frac{103\,993}{33\,102}=3.14159265$
6	$\frac{104\,348}{33\,215}=3.14159265$
7	$\frac{208\,341}{66\,317}=3.14159265$
8	$\frac{312\,689}{99\,532}=3.14159265$
9	$\frac{833\,719}{265\,381}=3.14159265$
10	$\frac{1\,146\,408}{364\,913}=3.14159265$

Example 2.29 - Continued Fractions and Pell Equations

Let $x = \sqrt{2}$. Then the first ten quotients and convergents are

```
ContinuedFraction[Sqrt[2], 10]
```

```
{1, 2, 2, 2, 2, 2, 2, 2, 2, 2}
```

```
Convergents[Sqrt[2], 10]
```

$$\left\{1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985}, \frac{3363}{2378}\right\}$$

Do you recognize the numerators {1, 3, 7, 17, 41, 99, 239, ...} and denominators {1, 2, 5, 12, 29, 70, 169, ...} in the convergents above? They are precisely solutions to the Pell equations $x^2 - 2\,y^2 = \pm 1$ discussed in Example (2.1).

FURTHER EXPLORATION:

1. Explain why the quotients above are all the same, i.e., $a_n = 2$ for all integers $n \geq 1$.

2. Repeat this example by computing the quotients and convergents of $x = \sqrt{3}$. Find patterns for them and determining whether the numerators and denominators of the convergents satisfy Pell equations of any kind.

3. What about $x = \sqrt{n}$ where n is a positive integer?

2.2.4 Methods from Number Theory

2.2.4.1 Divisors and Prime Numbers

An integer d is called a divisor (or factor) of n if d divides n. For example, 4 is a divisor of 12 since 4 divides 12 (i.e. $12/4 = 3$). The *Mathematica* command **Divisors** will generate a list of all the divisors of a given integer. For example, all the divisors of 20 are

```
Divisors[20]
```

```
{1, 2, 4, 5, 10, 20}
```

A positive integer $p > 1$ is called prime if the only divisors are 1 and itself; otherwise, it is said to be composite. For example, 13 is prime since the only divisors are 1 and 13; however, 15 is composite since $15 = 3 \cdot 5$. One can use the *Mathematica* command **PrimeQ** to determine whether an integer is prime.

```
PrimeQ[13]
PrimeQ[15]
```

```
True
```

```
False
```

Example 2.30 - Prime Factorizations of $2^n - 1$

Let's consider the prime factorizations of $2^n - 1$.

n	Prime Factors of 2^n-1
1	$1 = 1^1$
2	$3 = 3^1$
3	$7 = 7^1$
4	$15 = 3^1 \cdot 5^1$
5	$31 = 31^1$
6	$63 = 3^2 \cdot 7^1$
7	$127 = 127^1$
8	$255 = 3^1 \cdot 5^1 \cdot 17^1$
9	$511 = 7^1 \cdot 73^1$
10	$1023 = 3^1 \cdot 11^1 \cdot 31^1$

For $n = 2$, 4, 6, and 10, we observe that $n+1$ (namely 3, 5, 7, and 11) appears as a prime factor of $2^n - 1$. Let's confirm this for the first twenty prime values of $n+1$:

n	Prime Factors of 2^n-1
1	$1 = 1^1$
2	$3 = 3^1$
4	$15 = 3^1 \cdot 5^1$
6	$63 = 3^2 \cdot 7^1$
10	$1023 = 3^1 \cdot 11^1 \cdot 31^1$
12	$4095 = 3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1$
16	$65\,535 = 3^1 \cdot 5^1 \cdot 17^1 \cdot 257^1$
18	$262\,143 = 3^3 \cdot 7^1 \cdot 19^1 \cdot 73^1$
22	$4\,194\,303 = 3^1 \cdot 23^1 \cdot 89^1 \cdot 683^1$
28	$268\,435\,455 = 3^1 \cdot 5^1 \cdot 29^1 \cdot 43^1 \cdot 113^1 \cdot 127^1$

This leads to the following result:

Theorem 2.2: if $n+1$ is an odd prime, then $2^n - 1$ is divisible by $n+1$

Is the converse true? Unfortunately, no. There are composite numbers $n+1$ that divide $2^n - 1$, the first case being $n+1 = 341 = 11\times 31$ which divides $2^{340} - 1$:

```
Do[
 If[Mod[2^n - 1, n + 1] == 0 && ! PrimeQ[n + 1], Print[n + 1]], {n, 1, 1000}]
```

341

561

645

$2^{340}-1 = 3^1 \cdot 5^2 \cdot 11^1 \cdot 31^1 \cdot 41^1 \cdot 137^1 \cdot 953^1 \cdot 1021^1 \cdot 4421^1 \cdot 26\,317^1 \cdot 43\,691^1 \cdot 131\,071^1 \cdot 550\,801^1 \cdot 23\,650\,061^1 \cdot 7\,226\,904\,352\,843\,746\,841^1 \cdot 9\,520\,972\,806\,333\,758\,431^1 \cdot 26\,831\,423\,036\,065\,352\,611^1$

Example 2.31 - Sum of Four Squares

Lagrange's Four-Square Theorem states that any natural number n can be written as a sum of four squares of integers:

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 = n \tag{2.48}$$

But how many ways are there of doing this for each n, i.e. how many solutions are of the form $\{n_1, n_2, n_3, n_4\}$ where we distinguish sign and order? Here are several solutions for $n = 2$:

$1^2 + 1^2 + 0^2 + 0^2 = 2$
$0^2 + 0^2 + 1^2 + 1^2 = 2$
$(-1)^2 + (-1)^2 + 0^2 + 0^2 = 2$

We denote by $r_d(n)$ the number of ways of writing n as a sum of d squares. Then $r_4(2) = 24$. Here's a *Mathematica* command called **SumsOfSquaresRepresentations** for generating all such representations (see http://mathworld.wolfram.com/-SumofSquaresFunction.html):

```
SumOfSquaresRepresentations[4, 2]
```

```
{{-1, -1, 0, 0}, {-1, 0, -1, 0}, {-1, 0, 0, -1}, {-1, 0, 0, 1},
 {-1, 0, 1, 0}, {-1, 1, 0, 0}, {0, -1, -1, 0}, {0, -1, 0, -1}, {0, -1, 0, 1},
 {0, -1, 1, 0}, {0, 0, -1, -1}, {0, 0, -1, 1}, {0, 0, 1, -1}, {0, 0, 1, 1},
 {0, 1, -1, 0}, {0, 1, 0, -1}, {0, 1, 0, 1}, {0, 1, 1, 0}, {1, -1, 0, 0},
 {1, 0, -1, 0}, {1, 0, 0, -1}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 1, 0, 0}}
```

In addition, *Mathematica* has a built-in command for calculating $r_d(n)$: **SquaresR**.

SquaresR

```
? SquaresR
```

SquaresR[d, n] gives the number of ways $r_d(n)$ to represent the integer n as a sum of d squares. ≫

```
SquaresR[4, 2]
```

```
24
```

Here's a table listing the first ten values of $r_4(n)$ (A000118):

n	$r_4(n)$
1	8
2	24
3	32
4	24
5	48
6	96
7	64
8	24
9	104
10	144

Let’s now compare these values with the divisors of n, and in particular with the sum of the divisors of n:

n	$r_4(n)$	Divisors of n	Sum of Divisors of n
1	8	{1}	1
2	24	{1, 2}	3
3	32	{1, 3}	4
4	24	{1, 2, 4}	7
5	48	{1, 5}	6
6	96	{1, 2, 3, 6}	12
7	64	{1, 7}	8
8	24	{1, 2, 4, 8}	15
9	104	{1, 3, 9}	13
10	144	{1, 2, 5, 10}	18

Do you see a connection $r_4(n)$ and the sum of the divisors of n? It seems that the former is 8 times the latter, although this relationship fails for $n = 4$ and $n = 8$. However, if we ignore the divisors 4 and 8, then it is true. Let’s confirm this for higher values of n:

n	$r_4(n)$	Divisors of n	Sum of Divisors of n
11	96	{1, 11}	12
12	96	{1, 2, 3, 4, 6, 12}	28
13	112	{1, 13}	14
14	192	{1, 2, 7, 14}	24
15	192	{1, 3, 5, 15}	24
16	24	{1, 2, 4, 8, 16}	31
17	144	{1, 17}	18
18	312	{1, 2, 3, 6, 9, 18}	39
19	160	{1, 19}	20
20	144	{1, 2, 4, 5, 10, 20}	42

Thus, we see that the $P(n)$ depends only on the sum of those divisors that are NOT divisible by 4.

Theorem: The number of representations of $n \geq 1$ as the sum of four squares is given by

$$r_4(n) = 8 \sum_{d|n, 4 \nmid d} d \tag{2.49}$$

Sum of divisors function:

The sum of the divisors of a positive integer n is traditionally defined by the sigma function, i.e. $\sigma(n) = \sum_{d|n} d$. In *Mathematica*, the sigma function is defined by the command **DivisorSigma**. If the sum of the divisors equals $2n$, i.e. $\sigma(n) = 2n$, then n is said to be a perfect number.

DivisorSigma

```
? DivisorSigma
```

DivisorSigma[k, n] gives the divisor function $\sigma_k(n)$. ≫

For example, the divisors of $n = 6$ are {1, 2, 3, 6}. Thus, $\sigma(6) = 12$ and so 6 is a perfect number.

```
DivisorSigma[1, 6]
```

```
12
```

Some higher perfect numbers are 28, 496, and 8128. It is unknown whether there are infinite many perfect numbers.

Example 2.32 - Sigma function at prime values

Below is a table listing the first thirty values of $\sigma(n)$.

n	σ(n)	n	σ(n)	n	σ(n)
1	1	11	12	21	32
2	3	12	28	22	36
3	4	13	14	23	24
4	7	14	24	24	60
5	6	15	24	25	31
6	12	16	31	26	42
7	8	17	18	27	40
8	15	18	39	28	56
9	13	19	20	29	30
10	18	20	42	30	72

No pattern emerges for $\sigma(n)$ until we restrict our attention to certain subsequences. The following table, which lists $\sigma(n)$ for n prime, clearly shows that $\sigma(n) = n + 1$ in this case:

n	σ(n)
2	3
3	4
5	6
7	8
11	12
13	14
17	18
19	20
23	24
29	30

This of course is due to the fact that any prime integer n has only two divisors, 1 and n (itself).

The next table lists the values of $\sigma(n)$ for those values of n equal to a power of 2:

n	σ(n)
2	3
4	7
8	15
16	31
32	63

It appears that $\sigma(2^n) = 2^{n+1} - 1$ and follows from the fact that the divisors of 2^n are all the non-negative powers of 2 less than or equal to 2^n. Thus, $\sigma(2^n) = 2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1$.

Euler's totient (phi) function:

Two positive integers m and n are said to be relatively prime if they have no commond divisor other than 1. For example, 5 and 11 are relatively prime. However, 6 and 15 are NOT relatively prime, having 3 as a common divisor.

The totient (phi) function $\phi(n)$ is defined to be the number of positive integers less than or equal to n that are relatively prime to n. The corresponding command in *Mathematica* is **EulerPhi**:

EulerPhi

```
?EulerPhi
```

EulerPhi[n] gives the Euler totient function $\phi(n)$. ≫

For example, $\phi(6) = 2$ since there are two positive integers less than or equal to 6 that are relatively prime to 6, namely 1 and 5:

```
EulerPhi[6]
```

```
2
```

Example 2.33 - Euler totient function

a) The following table lists the values of $\phi(n)$ for the first thirty positive integers:

Euler's ϕ function

n	ϕ(n)
1	1
2	1
3	2
4	2
5	4
6	2
7	6
8	4
9	6
10	4

n	ϕ(n)
11	10
12	4
13	12
14	6
15	8
16	8
17	16
18	6
19	18
20	8

n	ϕ(n)
21	12
22	10
23	22
24	8
25	20
26	12
27	18
28	12
29	28
30	8

If we again focus only on primes values of n, then the following pattern emerges:

ϕ(n) for n prime

n	ϕ(n)
2	1
3	2
5	4
7	6
11	10
13	12
17	16
19	18
23	22
29	28

Thus, $\sigma(n) = n - 1$ whenever n is prime. This is because every positive integer less than n is relatively prime to n (there are $n - 1$ such integers).

b) It's clear from the results that we’ve obtained so far that $\phi(n) + \sigma(n) = 2\,n$ whenever n is prime.

```
ColumnDataDisplay[data, 10, {"n", "σ(n)+ϕ(n)"}, ""]
```

n	σ(n)+ϕ(n)
2	4
3	6
5	10
7	14
11	22
13	26
17	34
19	38
23	46
29	58

Let's generalize this problem and consider those values for n where $\phi(n) + \sigma(n) = k\,n$ for some fixed positive integer $k > 2$. For example, here are some solutions for $k = 3$:

```
SigmaPhiSum[k_, number_] := Module[{i = 0, n = 1, list = {}},
  While[i < number,
   If[DivisorSigma[1, n] + EulerPhi[n] == k * n, AppendTo[list, n]; i++]; n++]; list
 ]
```

```
SigmaPhiSum[3, 6]
```

```
{312, 560, 588, 1400, 85632, 147492}
```

Observe that these solutions are of the form $4\,m$, i.e., divisible by 4.

```
Mod[SigmaPhiSum[3, 6], 4]
```

```
{0, 0, 0, 0, 0, 0}
```

Here are some interesting open questions involving solutions to $\phi(n) + \sigma(n) = k\,n$ (see [Gu]):

1. Are there infinitely many solutions for each k?

2. Is there an odd solution?

3. The solutions above are of the form $4\,m$? Are there solutions of the form $4\,m + 2$?

c) What about the product $\sigma(n)\,\phi(n)$? Do you recognize a pattern?

n	σ(n)·ϕ(n)
1	1
2	3
3	8
4	14
5	24
6	24
7	48
8	60
9	78
10	72

Again, if we focus on the primes, then we observe that $\sigma\,(n)\,\phi\,(n)$ is always one less than a square:

n Prime

n	$\sigma(n)\cdot\phi(n)$
2	3
3	8
5	24
7	48
11	120
13	168
17	288
19	360
23	528
29	840

But note that there are composite integers n where $\sigma(n)\,\phi(n)$ is also one less than a square.

```
SigmaPhiProduct[number_] := Module[{i = 0, n = 1, list = {}},
  While[i < number, If[Sqrt[DivisorSigma[1, n] * EulerPhi[n] + 1] ==
      IntegerPart[Sqrt[DivisorSigma[1, n] * EulerPhi[n]] + 1],
    AppendTo[list, n]; i++]; n++]; list
 ]
```

```
SigmaPhiProduct[10]
```

```
{2, 3, 5, 6, 7, 11, 13, 17, 19, 22}
```

Here are some open problems involving $\sigma(n)\,\phi(n)$ (see [Gu]):

1. Are there infinitely values for n where $\sigma(n)\,\phi(n)$ is a perfect square?

2. Are there infinitely many composite values for n where $\sigma(n)\,\phi(n)$ is one less than a square?

2. Characterize those values for n where $\sigma(n)\,\phi(n)$ is divisible by n.

Example 2.34 - Mersenne Primes

Consider the sequence $a(n) = 2^n - 1$.

```
a[n_] := 2^n - 1;
```

a) Let's investigate when $2^n - 1$ is prime. The table below tell us which of the first 20 elements of $a(n)$ are prime:

Table 2.1: Values of $2^n - 1$

n	2^n-1	2^n-1 prime
1	1	False
2	3	True
3	7	True
4	15	False
5	31	True
6	63	False
7	127	True
8	255	False
9	511	False
10	1023	False

n	2^n-1	2^n-1 prime
11	2047	False
12	4095	False
13	8191	True
14	16 383	False
15	32 767	False
16	65 535	False
17	131 071	True
18	262 143	False
19	524 287	True
20	1 048 575	False

If we examine the values for n in which is $2^n - 1$ is prime (called Mersenne primes), we find that they are also prime, namely $n = 2, 3, 5, 7, 13, 17, 19$. Thus, we've discovered the result that

Theorem: If $2^n - 1$ is prime, then n is prime.

NOTE:

1. Observe that the converse is false since $n = 11$ is prime, but $2^{11} - 1 = 2047 = 23 \cdot 89$ is NOT prime.

2. Euclid proved that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is an even perfect number. Euler showed that the converse is also true. However, it is not known whether there exists odd perfect numbers. See Wikipedia entry [Wi-Pe].

2.2.4.2 Congruence and Modular Arithmetic

Every integer n, when divided by a positive integer p, leaves a remainder r. In other words, n can be written in the form

$$n = q \cdot p + r \tag{2.50}$$

with $0 \le r \le n - 1$. Using this fact, we define the congruence of n modulo p to be equal to r (called the residue) and write this in shorthand as $r \equiv n \pmod p$. In the special case where $p = 2$, then $r = 0$ or 1 and determines the parity of n, i.e. n is even if $r = 0$ and n is odd if $r = 1$.

Example 2.35 - Sums of Two Squares

In this example we investigate which positive integers n can be expressed as a sum of two square integers, i.e. $n_1^2 + n_2^2 = n$, where signs and order are distinguished. Then recall that $r_2(n)$ denotes the number of solutions for $\{n_1, n_2\}$. Then $r_2(5) = 8$. Here are the 8 different representations obtained using the **SumOfSquaresRepresentations** command discussed in Example 2.31.

```
SumOfSquaresRepresentations[2, 5]
```

```
{{-2, -1}, {-2, 1}, {-1, -2}, {-1, 2}, {1, -2}, {1, 2}, {2, -1}, {2, 1}}
```

Also recall that we can compute $r_2(5)$ using the **SquaresR** command.

```
SquaresR[2, 5]
```

```
8
```

Here's a table listing the first 20 values of $r_2(n)$ (A004018):

n	$r_2(n)$
1	4
2	4
3	0
4	4
5	8
6	0
7	0
8	4
9	4
10	8

n	$r_2(n)$
11	0
12	0
13	8
14	0
15	0
16	4
17	8
18	4
19	0
20	8

No pattern is evident from the table above. Let's restrict our attention then to prime integers:

n	$r_2(n)$
2	4
3	0
5	8
7	0
11	0
13	8
17	8
19	0
23	0
29	8

n	$r_2(n)$
31	0
37	8
41	8
43	0
47	0
53	8
59	0
61	8
67	0
71	0

We find that starting with $n = 3$, the values of $r_2(n)$ are nonzero for $n = 5$, 13, 17, 29, 37, 41, 53, 61. Is their a pattern to these values? To discover the answer, let's compute the residues of n (modulo 4).

n	n (mod 4)	$r_2(n)$
2	2	4
3	3	0
5	1	8
7	3	0
11	3	0
13	1	8
17	1	8
19	3	0
23	3	0
29	1	8

n	n (mod 4)	$r_2(n)$
31	3	0
37	1	8
41	1	8
43	3	0
47	3	0
53	1	8
59	3	0
61	1	8
67	3	0
71	3	0

Thus, it is now clear that odd prime p is expressible as a sum of two squares if and only if $p \equiv 1 \pmod 4$.

Example 2.36 - Sum of Remainders and Perfect Numbers

a) Let's tabulate the values $n \bmod d$ for $1 \le n \le 10$ and $1 \le d \le 10$:

Table 2.2: Values of $n \bmod d$

	d=1	d=2	d=3	d=4	d=5	d=6	d=7	d=8	d=9	d=10
n=1	0	1	1	1	1	1	1	1	1	1
n=2	0	0	2	2	2	2	2	2	2	2
n=3	0	1	0	3	3	3	3	3	3	3
n=4	0	0	1	0	4	4	4	4	4	4
n=5	0	1	2	1	0	5	5	5	5	5
n=6	0	0	0	2	1	0	6	6	6	6
n=7	0	1	1	3	2	1	0	7	7	7
n=8	0	0	2	0	3	2	1	0	8	8
n=9	0	1	0	1	4	3	2	1	0	9
n=10	0	0	1	2	0	4	3	2	1	0

Let's consider gaps (successive differences) along each column. Define $\Delta r_d(n) = (n+1) \pmod d - n \pmod d$. Then we see that the values of $\Delta r_d(n)$ are either 1 or $1 - d$ (except for the first column):

Table 2.3: Values of $\Delta r_d(n)$

	d=1	d=2	d=3	d=4	d=5	d=6	d=7	d=8	d=9	d=10
n=1	0	−1	1	1	1	1	1	1	1	1
n=2	0	1	−2	1	1	1	1	1	1	1
n=3	0	−1	1	−3	1	1	1	1	1	1
n=4	0	1	1	1	−4	1	1	1	1	1
n=5	0	−1	−2	1	1	−5	1	1	1	1
n=6	0	1	1	1	1	1	−6	1	1	1
n=7	0	−1	1	−3	1	1	1	−7	1	1
n=8	0	1	−2	1	1	1	1	1	−8	1
n=9	0	−1	1	1	−4	1	1	1	1	−9
n=10	0	1	1	1	1	1	1	1	1	1

A little analysis reveals that the choice of value depends on whether or not d divides $(n+1)$. Thus, we have the following formula:

$$\Delta\, r_d(n) = \begin{cases} 1-d, & \text{if } d \mid (n+1) \\ 1, & \text{otherwise} \end{cases} \tag{2.51}$$

b) Define the sum of remainders function $\rho(n) = \sum_{d=1}^{n} (n \bmod d)$ which sums the first n elements along each row in Table 2.2. Here's a listing of the first fifty values of $\rho(n)$ (A004125):

Table 2.4: Sums of Remainders Function $\rho(n)$

```
ρ[n_] := Sum[Mod[n, d], {d, 1, n}]
```

n	ρ(n)
1	0
2	0
3	1
4	1
5	4
6	3
7	8
8	8
9	12
10	13

n	ρ(n)
11	22
12	17
13	28
14	31
15	36
16	36
17	51
18	47
19	64
20	61

n	ρ(n)
21	70
22	77
23	98
24	85
25	103
26	112
27	125
28	124
29	151
30	138

n	ρ(n)
31	167
32	167
33	184
34	197
35	218
36	198
37	233
38	248
39	269
40	258

n	ρ(n)
41	297
42	284
43	325
44	328
45	339
46	358
47	403
48	374
49	414
50	420

Observe that certain values repeat: $\rho(1) = \rho(2) = 0$, $\rho(3) = \rho(4) = 1$, $\rho(7) = \rho(8) = 8$, $\rho(15) = \rho(16) = 36$, $\rho(31) = \rho(32) = 167$.

These positions correspond to powers of 2. Does this pattern hold for all powers of 2? Let's verify for larger values:

Table 2.5: Values of $\rho(2^n)$

n	2^n	$\rho(2^n-1)$	$\rho(2^n)$
1	2	0	0
2	4	1	1
3	8	8	8
4	16	36	36
5	32	167	167
6	64	693	693
7	128	2849	2849
8	256	11 459	11 459
9	512	46 244	46 244
10	1024	185 622	185 622

This confirms the following pattern:

$\rho(2^n) = \rho(2^n - 1)$ for every positive integer n.

Can you prove this?

c) The pattern $\rho(2^n) = \rho(2^n - 1)$ suggests that we should examine successive differences of $\rho(n)$. Define $\Delta\,\rho(n) = \rho(n+1) - \rho(n)$. Here are the first fifty values of $\Delta\,\rho(n)$:

Table 2.6: Values of Δρ(*n*)

Gaps of ρ(n)

n	Δρ(n)	n	Δρ(n)	n	Δρ(n)	n	Δρ(n)	n	Δρ(n)
1	0	11	−5	21	7	31	0	41	−13
2	1	12	11	22	21	32	17	42	41
3	0	13	3	23	−13	33	13	43	3
4	3	14	5	24	18	34	21	44	11
5	−1	15	0	25	9	35	−20	45	19
6	5	16	15	26	13	36	35	46	45
7	0	17	−4	27	−1	37	15	47	−29
8	4	18	17	28	27	38	21	48	40
9	1	19	−3	29	−13	39	−11	49	6
10	9	20	9	30	29	40	39	50	29

Are there any other patterns besides $\Delta\,\rho(2^n - 1) = 0$? Suppose we investigate those values for n where $\Delta\,\rho(n) = 1$. However, an experimental search of the first 10,000 values shows that there are only three cases where $\Delta\,\rho(n) = 1$, namely $n = 2, 9, 135$.

```
Δρ[n_] := ρ[n + 1] - ρ[n];
Do[
 If[ρ[n + 1] - ρ[n] == 1, Print[n]], {n, 1, 10 000}]
```

```
2

9

135
```

Thus, no pattern seems to exist. On the other hand, let’s consider when $\Delta\,\rho(n) = -1$. Here, we find that for the first 10,000 values, there are four cases:

```
Do[
 If[ρ[n + 1] - ρ[n] == -1, Print[n]], {n, 1, 10 000}]
```

```
5
27
495
8127
```

A pattern now emerges: adding 1 to these values for n give the perfect numbers 6, 28, 496, and 8128. This leads us to the following result:

Theorem: n is perfect if and only if $\Delta\,\rho(n-1) = -1$.

See [Sp] for a proof of this theorem.

NOTE: Observe the false pattern in Table 2.6: if $n = 10\,p$, then $\rho(n) = 10\,q + 9$ for some integer q. For example, $\Delta\,\rho(50) = 29$. Alas, this fails for $n = 80$ since $\Delta\,\rho(80) = 40$.

Example 2.37

Let's consider the values of $2^n - 1 \pmod{n}$:

Table 2.7: Values of $2^n - 1 \pmod{n}$

n	2ⁿ−1 mod n
1	0
2	1
3	1
4	3
5	1
6	3
7	1
8	7
9	7
10	3

n	2ⁿ−1 mod n
11	1
12	3
13	1
14	3
15	7
16	15
17	1
18	9
19	1
20	15

It appears that $2^n - 1 \equiv 1 \pmod{n}$ for $n = 2, 3, 5, 7, 11, 13, 17, 19$. These are prime numbers. We check this for the first twenty primes:

n	2ⁿ−1 mod n
2	1
3	1
5	1
7	1
11	1
13	1
17	1
19	1
23	1
29	1

n	2ⁿ−1 mod n
31	1
37	1
41	1
43	1
47	1
53	1
59	1
61	1
67	1
71	1

Thus, we've found strong evidence for the following result:

Theorem 2.6: If n is a prime integer, then $2^n - 1 \equiv 1 \pmod{n}$.

Is the converse true? Unfortunately, no. The first counterexample occurs when $n = 341$. We have that $2^{341} - 1 \equiv 1 \pmod{341}$, but $341 = 11 \cdot 31$ is not prime. Here are the first three counterexamples:

```
Do[
 If[Mod[2^n - 1, n] == 1 && ! PrimeQ[n], Print[n]], {n, 1, 1000}]
```

341

561

645

Observe that these counterexamples correspond to the same counterexamples mentioned at the end of Example 2.30. This is because Theorem 2.6 is equivalent to Theorem 2.2.

Example 2.38 - Power Sums

We shall refer to the sum $S_p(n) = 1^n + 2^n + ... + n^p$ as a power sum. Here is table of power sums for $1 \le n \le 4$ and $1 \le p \le 4$:

Table 2.8: Values of $1^n + 2^n + ... + n^p$

	p=1	p=2	p=3	p=4
n=1	$1^1 == 1$	$1^2 == 1$	$1^3 == 1$	$1^4 == 1$
n=2	$1^1 + 2^1 == 3$	$1^2 + 2^2 == 5$	$1^3 + 2^3 == 9$	$1^4 + 2^4 == 17$
n=3	$1^1 + 2^1 + 3^1 == 6$	$1^2 + 2^2 + 3^2 == 14$	$1^3 + 2^3 + 3^3 == 36$	$1^4 + 2^4 + 3^4 == 98$
n=4	$1^1 + 2^1 + 3^1 + 4^1 == 10$	$1^2 + 2^2 + 3^2 + 4^2 == 30$	$1^3 + 2^3 + 3^3 + 4^3 == 100$	$1^4 + 2^4 + 3^4 + 4^4 == 354$

To find patterns, we consider the remainders of $S_p(n) \pmod n$.

Table 2.9:

	p=1	p=2	p=3	p=4	p=5	p=6	p=7	p=8	p=9	p=10
n=1	0	0	0	0	0	0	0	0	0	0
n=2	1	1	1	1	1	1	1	1	1	1
n=3	0	2	0	2	0	2	0	2	0	2
n=4	2	2	0	2	0	2	0	2	0	2
n=5	0	0	0	4	0	0	0	4	0	0
n=6	3	1	3	1	3	1	3	1	3	1
n=7	0	0	0	0	0	6	0	0	0	0
n=8	4	4	0	4	0	4	0	4	0	4
n=9	0	6	0	6	0	6	0	6	0	6
n=10	5	5	5	3	5	5	5	3	5	5

a) Let's restrict the values of p to prime integers:

Table 2.10:

	p=2	p=3	p=5	p=7	p=11	p=13	p=17	p=19	p=23	p=29
n=1	0	0	0	0	0	0	0	0	0	0
n=2	1	1	1	1	1	1	1	1	1	1
n=3	2	0	0	0	0	0	0	0	0	0
n=4	2	0	0	0	0	0	0	0	0	0
n=5	0	0	0	0	0	0	0	0	0	0
n=6	1	3	3	3	3	3	3	3	3	3
n=7	0	0	0	0	0	0	0	0	0	0
n=8	4	0	0	0	0	0	0	0	0	0
n=9	6	0	0	0	0	0	0	0	0	0
n=10	5	5	5	5	5	5	5	5	5	5

The following pattern emerges from the table above for odd primes p:

Theorem: $S_p(4n+2) = 2n+1$

This is because of Fermat's Little Theorem:

Fermat's Little Theorem: Let p be a prime. Then $p \nmid a$ *implies* $a^{p-1} \equiv 1 \pmod{p}$

$$p \nmid a \Rightarrow a^{p-1} \equiv 1 \pmod{p} \tag{2.52}$$

See if you can prove Theorem using Fermat's Little Theorem.

b) Let's now restrict m to being primes:

Table 2.11:

	p=1	p=2	p=3	p=4	p=5	p=6	p=7	p=8	p=9	p=10	p=11	p=12	p=13	p=14	p=15
n=2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
n=3	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0
n=5	0	0	0	4	0	0	0	4	0	0	0	4	0	0	0
n=7	0	0	0	0	0	6	0	0	0	0	0	6	0	0	0
n=11	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0
n=13	0	0	0	0	0	0	0	0	0	0	0	12	0	0	0
n=17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
n=19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
n=23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
n=29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

This leads us to the following theorem: http://arxiv.org/PS_cache/arxiv/pdf/1011/1011.0076v1.pdf

Theorem: Let n be a prime integer. For p≥1, we have

$$1^p + 2^p + \ldots + n^p = \begin{cases} n-1 \pmod{n} \text{ if } (n-1) \mid p \\ 0 \pmod{n} \text{ if } (n-1) \nmid p \end{cases} \tag{2.53}$$

Example 2.39 - Congruence of Fibonacci Numbers

a) Let's investigate the residues (remainders) of the Fibonacci numbers modulo 2. Below is a table listing the residues of the first twenty Fibonacci numbers:

Table 2.12:

n	F_n mod 2
0	0
1	1
2	1
3	0
4	1
5	1
6	0
7	1
8	1
9	0

n	F_n mod 2
10	1
11	1
12	0
13	1
14	1
15	0
16	1
17	1
18	0
19	1

By examining these residues, it's clear that they repeat every third entry and can be described using the formula

$$F_n \bmod 2 = \begin{cases} 0 \text{ if } n \equiv 0 \bmod 3 \\ 1 \text{ if } n \equiv 1 \bmod 3 \\ 1 \text{ if } n \equiv 2 \bmod 3 \end{cases} \tag{2.54}$$

However, *Mathematica* finds a much cleaner formula:

```
FindSequenceFunction[Table[Mod[Fibonacci[n], 2], {n, 1, 10}], n]
```

$\text{Mod}[n^2, 3]$

To prove this, we consider the three possible remainders for n when divided by 3: 0, 1, 2. This corresponds to $n = 3q$, $n = 3q + 1$, and $n = 3q + 2$, respectively. Then substituting each of these forms for n into n^2 yields

```
Expand[n^2 /. n → {3 q, 3 q + 1, 3 q + 2}]
```

$\{9 q^2, 1 + 6 q + 9 q^2, 4 + 12 q + 9 q^2\}$

To obtain the congruence of each of these expressions, we apply modular arithmetic to obtain the desired answer. For example, if $n = 3q + 1$, then

$$1 + n + n^2 \equiv 1 + 6q + 9q^2 \equiv 1 \bmod 3 \tag{2.55}$$

since both $6q$ and $9q^2$ are divisible by 3. This proves $F_n \bmod 2 \equiv 1$. Another option is to have *Mathematica* perform the modular arithmetic:

```
Simplify[Mod[1 + 6 q + 9 q^2, 3], Element[q, Integers]]
```

1

The reader is encouraged to verify the other two cases by carrying out the same calculations.

b) Are the residues of the Fibonacci numbers periodic for other moduli? To answer this, we make a table of the residues $F_n \pmod m$, with n ranging from 1 to 20 and m ranging from 1 to 10:

Table 2.13:

F_n mod m (n=1,...,15)

	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10	n=11	n=12	n=13	n=14	n=15
m=1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
m=2	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0
m=3	1	1	2	0	2	2	1	0	1	1	2	0	2	2	1
m=4	1	1	2	3	1	0	1	1	2	3	1	0	1	1	2
m=5	1	1	2	3	0	3	3	1	4	0	4	4	3	2	0
m=6	1	1	2	3	5	2	1	3	4	1	5	0	5	5	4
m=7	1	1	2	3	5	1	6	0	6	6	5	4	2	6	1
m=8	1	1	2	3	5	0	5	5	2	7	1	0	1	1	2
m=9	1	1	2	3	5	8	4	3	7	1	8	0	8	8	7

F_n mod m (n=16,...,30)

	n= 16	n= 17	n= 18	n= 19	n= 20	n= 21	n= 22	n= 23	n= 24	n= 25	n= 26	n= 27	n= 28	n= 29	n= 30
m=1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
m=2	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0
m=3	0	1	1	2	0	2	2	1	0	1	1	2	0	2	2
m=4	3	1	0	1	1	2	3	1	0	1	1	2	3	1	0
m=5	2	2	4	1	0	1	1	2	3	0	3	3	1	4	0
m=6	3	1	4	5	3	2	5	1	0	1	1	2	3	5	2
m=7	0	1	1	2	3	5	1	6	0	6	6	5	4	2	6
m=8	3	5	0	5	5	2	7	1	0	1	1	2	3	5	0
m=9	6	4	1	5	6	2	8	1	0	1	1	2	3	5	8

Looking at the first several rows of this array seem to indicate that the residues are periodic for every modulus *m*. Indeed, this is true (can you prove why?). Here is a table listing the periods for *m* ranging from 1 to 20:

Table 2.14:

m	Period of F_n(mod m)	m	Period of F_n(mod m)
1	1	11	10
2	3	12	24
3	8	13	28
4	6	14	48
5	20	15	40
6	24	16	24
7	16	17	36
8	12	18	24
9	24	19	18
10	60	20	60

No formula in terms of *m* is known for these periods. However, there are other interesting patterns that can be extracted. Denote the period of F_n (mod m) by $P(m)$. Then observe that for $m > 2$, $P(m)$ is always even. Another pattern involves the sequence of residues {1, 0, 1}, which seems to appear with relative high frequency, especially in the second array above. Let's isolate this sequence in our table:

Table 2.15:

F_n mod m (n=1,...,15) – Residue Pattern {1,0,1}

	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10	n=11	n=12	n=13	n=14	n=15
m=1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
m=2	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0
m=3	1	–	–	–	–	–	1	0	1	–	–	–	–	–	1
m=4	1	–	–	–	1	0	1	–	–	–	1	0	1	–	–
m=5	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–
m=6	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–
m=7	1	–	–	–	–	–	–	–	–	–	–	–	–	–	1
m=8	1	–	–	–	–	–	–	–	–	–	1	0	1	–	–
m=9	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–

F_n mod m (n=16,...,30) – Residue Pattern {1,0,1}

	n= 16	n= 17	n= 18	n= 19	n= 20	n= 21	n= 22	n= 23	n= 24	n= 25	n= 26	n= 27	n= 28	n= 29	n= 30
m=1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
m=2	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0
m=3	0	1	–	–	–	–	–	1	0	1	–	–	–	–	–
m=4	–	1	0	1	–	–	–	1	0	1	–	–	–	1	0
m=5	–	–	–	1	0	1	–	–	–	–	–	–	–	–	–
m=6	–	–	–	–	–	–	–	1	0	1	–	–	–	–	–
m=7	0	1	–	–	–	–	–	–	–	–	–	–	–	–	–
m=8	–	–	–	–	–	–	–	1	0	1	–	–	–	–	–
m=9	–	–	–	–	–	–	–	1	0	1	–	–	–	–	–

A discernible pattern now emerges. The locations of {1, 0, 1} seem to be regularly spaced. Here's a table listing the positions of the residue 0 at these locations in comparison to the periods $P(m)$.

Table 2.16:

m	P(m)	Positions of residue 0 at occurrences of {1,0,1} (up to n=30)
2	3	{3, 6, 9, 12, 15, 18, 21, 24, 27}
3	8	{8, 16, 24}
4	6	{6, 12, 18, 24}
5	20	{20}
6	24	{24}
7	16	{16}
8	12	{12, 24}

The pattern is now clear. The locations of {1, 0, 1} occur at positions n that are multiples of $P(m)$. Thus, we've discovered the following result:

Theorem: Let $i = 0$ or 1. Then $F_{n\pm i} = i \ (\mathrm{mod}\ m)$ if and only if $P(m) \mid n$.

Can you prove this theorem?

2.2.5 Combinatorial Methods

2.2.5.1 Permutations

A permutation of a set of n elements is an ordering of its elements. For example, {1, 2, 3} and {2, 1, 3} are two different permutations of {1, 2, 3}. The *Mathematica* command **Permutations** will generate all permutations of a given set of elements:

```
? Permutations
```

Permutations[*list*] generates a list of all possible permutations of the elements in *list*.
Permutations[*list*, *n*] gives all permutations containing at most *n* elements.
Permutations[*list*, {*n*}] gives all permutations containing exactly *n* elements. ≫

For example, there are six permutations of {1, 2, 3}:

```
Permutations[{1, 2, 3}]
```

```
{{1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2}, {3, 2, 1}}
```

The set {1, 2, ..., n} has $n!$ permutations because of the Multiplication Principle.

2.2.5.2 Inversions

An inversion of a permutation σ is a pair of elements of σ that is 'out of order'. For example, the permutation {2, 3, 1} has two inversions: {2, 1} and {3, 1}. Denote by $i(\sigma)$ to be the total number of inversions of σ. The *Mathematica* command **Inversions** will calculate $i(\sigma)$:

```
? Inversions
```

Inversions[*p*] counts the number of inversions in permutation p. ≫

Let's use this command to can verify that $i(\{2, 3, 1\}) = 2$:

```
Inversions[{2, 3, 1}]
```

```
2
```

Example 2.40 - Distribution of Total Number of Inversions (see [Kn], p. 15)

In this example we analyze how the total number of inversions are distributed among all permutations on n elements. For example, the following table lists $i(\sigma)$ for all permutations of the set {1, 2, 3} corresponding to $n = 3$:

Total Number of Inversions

σ	I(σ)
{1, 2, 3}	0
{1, 3, 2}	1
{2, 1, 3}	1
{2, 3, 1}	2
{3, 1, 2}	2
{3, 2, 1}	3

Thus, we see that the total number of inversions are distributed as follows:

Distribution of i(p) for permutations on 3 elements

k	Number of Permutations with i(σ)=k
0	1
1	2
2	2
3	1

More generally, we define $I(n, k)$ to be the number of permutations σ on n elements having $i(\sigma) = k$. Below is a table of values for $I(n, k)$:

I(n,k)

I(n,k)	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10
n=1	1	0	0	0	0	0	0	0	0	0	0
n=2	1	1	0	0	0	0	0	0	0	0	0
n=3	1	2	2	1	0	0	0	0	0	0	0
n=4	1	3	5	6	5	3	1	0	0	0	0
n=5	1	4	9	15	20	22	20	15	9	4	1

Observe that each row in the table above is symmtric. Also, we find that the following recurrence holds:

$$I(n, k) = I(n-1, k) + I(n-1, k-1) \tag{2.56}$$

FURTHER EXPLORATION: Can you find other patterns in the table for $I(n, k)$?

2.2.5.3 Derangements

Derangements are permutations with no fixed points, i.e., no elements remains in place. For example, there are two derangements of the set {1, 2, 3}: {2, 3, 1} and {3, 1, 2}. Here is the *Mathematica* command for generating derangements:

```
? Derangements
```

Derangements[p] constructs all derangements of permutation p. ≫

For example, there are 9 derangements of {1, 2, 3, 4} (out of 24 permutations).

```
Derangements[4]
```

```
{{2, 1, 4, 3}, {2, 3, 4, 1}, {2, 4, 1, 3}, {3, 1, 4, 2},
 {3, 4, 1, 2}, {3, 4, 2, 1}, {4, 1, 2, 3}, {4, 3, 1, 2}, {4, 3, 2, 1}}
```

The number of derangements of the set {1, ..., n} is called the subfactorial of n and denoted by n_{i} (or $!\,n$). Thus, $3_{\text{i}} = 2$. Here is the corresponding *Mathematica* command:

```
? Subfactorial
```

Subfactorial[n] gives the number of permutations of n objects that leave no object fixed. ≫

```
Subfactorial[4]
```

```
9
```

Let's make a table of subfactorials:

```
nMax = 9;
datasubfactorials = Table[{n, Subfactorial[n]}, {n, 1, nMax}];
ColumnDataDisplay[datasubfactorials, 10, {"n", "nᵢ"}, "Subfactorial"]
```

Subfactorial

n	n_{i}
1	0
2	1
3	2
4	9
5	44
6	265
7	1854
8	14 833
9	133 496

Do you see a connection between any two consecutive terms? Let's look at their ratios, $n_{\text{i}} / (n-1)_{\text{i}}$.

```
nMax = 9;
datasubfactorialsratios2elements = Table[
   {n, Subfactorial[n], If[n < 3, "-", Row[{Subfactorial[n] / Subfactorial[n - 1],
       "≈", N[Subfactorial[n] / Subfactorial[n - 1], 5]}]]}, {n, 1, nMax}];
ColumnDataDisplay[datasubfactorialsratios2elements, 10,
 {"n", "n¡", "n¡/(n-1)¡"}, "Subfactorial"]
```

Subfactorial

n	$n_{¡}$	$n_{¡}/(n-1)_{¡}$
1	0	–
2	1	–
3	2	2≈2.0000
4	9	$\frac{9}{2}$≈4.5000
5	44	$\frac{44}{9}$≈4.8889
6	265	$\frac{265}{44}$≈6.0227
7	1854	$\frac{1854}{265}$≈6.9962
8	14 833	$\frac{14833}{1854}$≈8.0005
9	133 496	$\frac{133496}{14833}$≈8.9999

Observe that the ratios above are almost integers. If we round them off, then a simple linear progression emerges as a pattern. Denote by $[n_{¡}/(n-1)_{¡}]$ to be the round of $n_{¡}/(n-1)_{¡}$.

```
nMax = 9;
datasubfactorialsratios2elementsround = Table[{n, Subfactorial[n],
    If[n < 3, "-", Round[Subfactorial[n] / Subfactorial[n - 1]]]}, {n, 1, nMax}];
ColumnDataDisplay[datasubfactorialsratios2elementsround, 10,
 {"n", "n¡", "[n¡/(n-1)¡]"}, "Subfactorial"]
```

Subfactorial

n	$n_{¡}$	$[n_{¡}/(n-1)_{¡}]$
1	0	–
2	1	–
3	2	2
4	9	4
5	44	5
6	265	6
7	1854	7
8	14 833	8
9	133 496	9

Since it appears that $[n_{¡}/(n-1)_{¡}] = n$, we should consider scaling subfactorials by n, namely $n \cdot (n-1)_{¡}$.

```
nMax = 9;
datasubfactorialsscaled =
  Table[{n, Subfactorial[n], If[n < 3, "-", n * Subfactorial[n - 1]],
    If[n < 3, "-", Subfactorial[n] - n * Subfactorial[n - 1]]}, {n, 1, nMax}];
ColumnDataDisplay[datasubfactorialsscaled, 10,
 {"n", "n_i", "n(n-1)_i", "n_i-n(n-1)_i"}, "Subfactorial"]
```

Subfactorial

n	n_i	$n(n-1)_i$	$n_i - n(n-1)_i$
1	0	-	-
2	1	-	-
3	2	3	-1
4	9	8	1
5	44	45	-1
6	265	264	1
7	1854	1855	-1
8	14 833	14 832	1
9	133 496	133 497	-1

The pattern is now clear:

$$n_{\text{i}} = n(n-1)_{\text{i}} + (-1)^n \tag{2.57}$$

FURTHER EXPLORATION:

1. Formula (2.57) points to a connection between consecutive factorials, n_{i} and $(n-1)_{\text{i}}$. Find this connection and use it to prove formula (2.57).

2. Can you find a pattern involving the sum of any two consecutive subfactorials?

Example 2.41 - Even and Odd Derangements

A permutation is called an even (or odd) if it can be expressed as an even number of of even (or odd) number of transpositions, i.e., exchanges of two elements, respectively. An even (or odd) derangement is one that is also an even (or odd) permutation, respectively. Denote by $d_e(n)$ and $d_o(n)$ the number of even and odd derangements, respectively. Here's a table comparing $d_e(n)$ (A003221)and $d_o(n)$ (A145221):

Table 2.17:

n	$d_e(n)$	$d_o(n)$	$d_e(n)-d_o(n)$
1	0	0	0
2	0	1	-1
3	2	0	2
4	3	6	-3
5	24	20	4
6	130	135	-5
7	930	924	6

Thus, we find that the following identity holds:

$$d_e(n) - d_o(n) = (-1)^{n-1}\,(n-1) \tag{2.58}$$

For a combinatorial proof, see [BBN].

2.2.6 Two-Dimensional Sequences

We've already encountered two-dimensional sequences in earlier examples, i.e., sequences indexed by two parameters. A classic example is the sequence of binomial coefficients

$$a_{n,k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \tag{2.59}$$

These coefficients form Pascal's triangle (discussed further in Chapter 3).

Mathematica's **FindSequenceFunction** command does not recognize two-dimensional sequences. Thus, we demonstrate how a two-dimensional sequence can be studied by analyzing its one-dimensional subsequences.

Example 2.42 - Approximating $\sqrt{2}$

In this example we demonstrate how to approximate $\sqrt{2}$ using a two-dimensional sequence (see [LP]). To start, consider the sequence

$\{a_n\} = \left\{\left(\sqrt{2}\right)^n\right\} = \left\{1, \sqrt{2}, 2, 2\sqrt{2}, 4, 4\sqrt{2}, \ ...\right\}$

consisting of powers of $\sqrt{2}$. Since $\sqrt{2}$ is unknown, let's replace it with a crude approximation, say 1, to obtain a new sequence

$\{b_n\} = \{1, 1, 2, 2, 4, 4, \ ...\}$

whose formula is given by

```
FindSequenceFunction[{1, 2, 2, 4, 4, 8, 8}, n]
```

$$2^{-\frac{3}{2}+\frac{n}{2}} \left(1 - (-1)^n + \sqrt{2} + (-1)^n \sqrt{2}\right)$$

NOTE: A more concise formula is $2^{\lfloor n/2 \rfloor}$.

Now extend $\{b_n\}$ to a two-dimensional sequence $\{b_{n,k}\}$ defined by the recurrence

$$b_{n,k} = b_{n,k-1} + b_{n+1,k-1} \tag{2.60}$$

with initial row $b_{n,0} = b_n$. The array below displays the first 5 rows of $b_{n,k}$ (A117918):

Table 2.18:

	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9
k=0	1	1	2	2	4	4	8	8	16	16
k=1	2	3	4	6	8	12	16	24	32	48
k=2	5	7	10	14	20	28	40	56	80	112
k=3	12	17	24	34	48	68	96	136	192	272
k=4	29	41	58	82	116	164	232	328	464	656
k=5	70	99	140	198	280	396	560	792	1120	1584

a) Let's find a formula for $b_{n,k}$. Unfortunately, the **FindSequenceFunction** command cannot be used directly on two-dimensional sequences. Instead, we'll find formulas for the entries in each row and paste them together to get our two-dimensional formula. Define $f_k(n) = b_{n,k}$. The following table gives fomulas for $f_k(n)$ for k ranging from 1 to 10:

Table 2.19:

k	Formula for $f_k(n)$
0	$2^{\frac{1}{2}(-3+n)}\left(1-(-1)^n+\sqrt{2}+(-1)^n\sqrt{2}\right)$
1	$2^{\frac{1}{2}(-3+n)}\left(3-3(-1)^n+2\sqrt{2}+2(-1)^n\sqrt{2}\right)$
2	$2^{\frac{1}{2}(-3+n)}\left(7-7(-1)^n+5\sqrt{2}+5(-1)^n\sqrt{2}\right)$
3	$2^{\frac{1}{2}(-3+n)}\left(17-17(-1)^n+12\sqrt{2}+12(-1)^n\sqrt{2}\right)$
4	$2^{\frac{1}{2}(-3+n)}\left(41-41(-1)^n+29\sqrt{2}+29(-1)^n\sqrt{2}\right)$
5	$2^{\frac{1}{2}(-3+n)}\left(99-99(-1)^n+70\sqrt{2}+70(-1)^n\sqrt{2}\right)$
6	$2^{\frac{1}{2}(-3+n)}\left(239-239(-1)^n+169\sqrt{2}+169(-1)^n\sqrt{2}\right)$
7	$2^{\frac{1}{2}(-3+n)}\left(577-577(-1)^n+408\sqrt{2}+408(-1)^n\sqrt{2}\right)$
8	$2^{\frac{1}{2}(-3+n)}\left(1393-1393(-1)^n+985\sqrt{2}+985(-1)^n\sqrt{2}\right)$
9	$2^{\frac{1}{2}(-3+n)}\left(3363-3363(-1)^n+2378\sqrt{2}+2378(-1)^n\sqrt{2}\right)$

Observe that each formula involves a linear combination of $(-1)^n$ and $\sqrt{2}$. Thus, we'll need to find formulas for their corresponding coefficients (as functions of k):

{1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363} and {1, 2, 5, 12, 29, 70, 169, 408, 985, 2378}

Entering these coefficients into **FindSequenceFunction** yields

```
FindSequenceFunction[{3, 7, 17, 41, 99, 239, 577, 1393, 3363}, k]
```

$$\frac{-\left(1-\sqrt{2}\right)^k+3\left(1+\sqrt{2}\right)^k+2\sqrt{2}\left(1+\sqrt{2}\right)^k}{2\left(1+\sqrt{2}\right)}$$

```
FindSequenceFunction[{2, 5, 12, 29, 70, 169, 408, 985, 2378}, k]
```

$$\frac{\sqrt{2}\left(1-\sqrt{2}\right)^k+4\left(1+\sqrt{2}\right)^k+3\sqrt{2}\left(1+\sqrt{2}\right)^k}{4\left(1+\sqrt{2}\right)}$$

Substituting these formulas for the coefficients gives us the following formula for $b_{n,k}$:

b[n_, k_] =

$$\mathbf{Simplify}\Big[2^{\frac{1}{2}(-3+n)}\Big(\Big(\Big(-\left(1-\sqrt{2}\right)^k+3\left(1+\sqrt{2}\right)^k+2\sqrt{2}\left(1+\sqrt{2}\right)^k\Big)\Big/\Big(2\left(1+\sqrt{2}\right)\Big)\Big)-$$
$$\Big(\Big(-\left(1-\sqrt{2}\right)^k+3\left(1+\sqrt{2}\right)^k+2\sqrt{2}\left(1+\sqrt{2}\right)^k\Big)\Big/\Big(2\left(1+\sqrt{2}\right)\Big)\Big)(-1)^n+$$
$$\Big(\Big(\sqrt{2}\left(1-\sqrt{2}\right)^k+4\left(1+\sqrt{2}\right)^k+3\sqrt{2}\left(1+\sqrt{2}\right)^k\Big)\Big/\Big(4\left(1+\sqrt{2}\right)\Big)\Big)\sqrt{2}+$$
$$\Big(\Big(\sqrt{2}\left(1-\sqrt{2}\right)^k+4\left(1+\sqrt{2}\right)^k+3\sqrt{2}\left(1+\sqrt{2}\right)^k\Big)\Big/\Big(4\left(1+\sqrt{2}\right)\Big)\Big)(-1)^n\sqrt{2}\Big)\Big]$$

$$2^{\frac{1}{2}(-3+n)}\left(-1+\sqrt{2}\right)\left((-1)^n\left(1-\sqrt{2}\right)^k+\left(1+\sqrt{2}\right)^k\left(3+2\sqrt{2}\right)\right)$$

b) What do you notice about the ratios b_1^k/b_0^k as $k\to\infty$? Here's a table listing the first twenty ratios:

Table 2.20:

n	$b_{1,k}/b_{0,k}$
1	1.50000000000
2	1.40000000000
3	1.41666666667
4	1.41379310345
5	1.41428571429
6	1.41420118343
7	1.41421568627
8	1.41421319797
9	1.41421362489
10	1.41421355165

n	$b_{1,k}/b_{0,k}$
11	1.41421356421
12	1.41421356206
13	1.41421356243
14	1.41421356236
15	1.41421356237
16	1.41421356237
17	1.41421356237
18	1.41421356237
19	1.41421356237
20	1.41421356237

Mathematica shows that the limit of these ratios equals $\sqrt{2}$:

```
Limit[b[1, k] / b[0, k], k → Infinity]
```

$$\sqrt{2}$$

This is easy to derive from the formula for b_1^k / b_0^k:

```
Simplify[b[1, k] / b[0, k]]
```

$$\frac{\sqrt{2}\left(-\left(1-\sqrt{2}\right)^k+\left(1+\sqrt{2}\right)^k\left(3+2\sqrt{2}\right)\right)}{\left(1-\sqrt{2}\right)^k+\left(1+\sqrt{2}\right)^k\left(3+2\sqrt{2}\right)}$$

We rewrite this formula as

$$b_1^k / b_0^k = \frac{\sqrt{2}\left(-\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^k+\left(3+2\sqrt{2}\right)\right)}{\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^k+\left(3+2\sqrt{2}\right)} \tag{2.61}$$

Since $0 < \frac{1-\sqrt{2}}{1+\sqrt{2}} < 1$, it follows that $\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^k \to 0$ as $k \to \infty$. Thus,

$$b_1^k / b_0^k \to \frac{\sqrt{2}\left(3+2\sqrt{2}\right)}{\left(3+2\sqrt{2}\right)} = \sqrt{2} \tag{2.62}$$

Thus, the ratios b_1^k / b_0^k provide a rational approximation of $\sqrt{2}$. What about the ratios b_{n+1}^k / b_n^k as $k \to \infty$ for arbitrary n? See if you can determine a pattern for their limits.

```
Simplify[b[2, k] / b[1, k]]
```

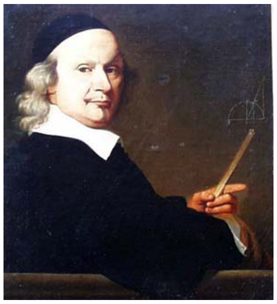
$$\frac{\sqrt{2}\left(1-\sqrt{2}\right)^k+\left(1+\sqrt{2}\right)^k\left(4+3\sqrt{2}\right)}{-\left(1-\sqrt{2}\right)^k+\left(1+\sqrt{2}\right)^k\left(3+2\sqrt{2}\right)}$$

NOTE: Observe that each row of $b_{n,k}$ satisfies the recurrence $b_{2n,k} = 2\,b_{n,k}$. A recursive formula for $b_{n,k}$ in terms of the elements in the first row is given by

$$b_n^k = \sum_{j=0}^{k} \binom{k}{j} b_{n+j}^0 \tag{2.63}$$

FURTHER EXPLORATION: Explore for powers of 3 instead of 2 as initial values for b_n^0.

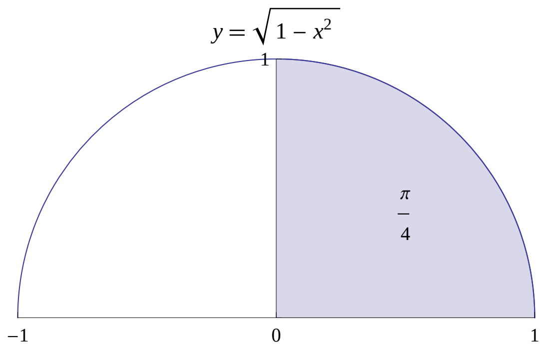
Wallis' Formula for π



John Wallis (1616 – 1703)

http://www-history.mcs.st-and.ac.uk/Mathematicians/Wallis.html

John Wallis, Savilian Professor of Geometry and founding member of the Royal Society, first honed his pattern detection abilities from his experience as a cryptologist during the English Civil War (1642-1651) in which he decoded Royalist messages for the victorious Parliamentarians. In his most important work, Arithmetica Infinitorum, published in 1656, Wallis demonstrated how he was able to derive his famous infinite product formula for π by interpolating number patterns obtained by quadrature (area enclosed by a curve). His journey begins with the area of a quarter unit circle, $\pi/4$, bounded by the graph of $y = \sqrt{1 - x^2}$ on the interval [0, 1].



In the language of calculus, this shaded area can be represented by a *definite integral* and denoted by

$$\int_0^1 \left(1 - x^2\right)^{1/2} dx = \frac{\pi}{4} \tag{2.64}$$

Unfortunately, the definite integral above allows Wallis little room for algebraic manipulation. Instead, he considers the definite integral $\int_0^1 \left(1 - x^{1/2}\right)^2 dx$ (where the exponents are replaced by their reciprocals) and more generally those of the form $\int_0^1 \left(1 - x^{1/p}\right)^q dx$ for arbitrary integers p and q. Now, Wallis' contemporaries, namely Robertval, Fermat and Pascal, had already determined that for non-negative integers k,

$$\int_0^a x^k \, dx = \frac{a^{k+1}}{k+1} \tag{2.65}$$

From here, Wallis boldly assumes that this formula continues to hold for all non-negative real values of k, and in particular for rational $k = \frac{p}{q}$ (p and q positive integers), so that

$$\int_0^1 x^{p/q} \, dx = \frac{1}{p/q + 1} \tag{2.66}$$

As a result, he is now able to calculate $\int_0^1 \left(1 - x^{1/p}\right)^q dx$ using binomial expansion and the formula above. For example, since

$$\left(1 - x^{1/2}\right)^2 = 1 - 2\,x^{1/2} + x \tag{2.67}$$

it follows that the area under $\left(1-x^{1/2}\right)^2$ can be calculated as the sum of the areas under the three curves 1, $2\,x^{1/2}$, and x, which Wallis already knew how to compute separately:

$$\int_0^1 \left(1-x^{1/2}\right)^2 dx = \int_0^1 \left(1-2\,x^{1/2}+x\right) dx = \int_0^1 1\,dx - \int_0^1 2\,x^{1/2}\,dx + \int_0^1 x\,dx = 1 - 2\cdot\frac{1}{1/2+1} + \frac{1}{1+1} = \frac{1}{6} \tag{2.68}$$

With this technique in mind, Wallis then makes a tabulation of the values of $\int_0^1 \left(1-x^{1/p}\right)^q dx$ using other positive integers for p and q. This generates a symmetrical table given below consisting of values that Wallis recognized to be reciprocals of figurate numbers, a generalization of the triangular numbers to higher-dimensions (discussed in detail in Chapter 3).

$\int_0^1 (1-x^p)^q dx$

	q=1	q=2	q=3	q=4	q=5
p=1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
p=2	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{21}$
p=3	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{35}$	$\frac{1}{56}$
p=4	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{1}{35}$	$\frac{1}{70}$	$\frac{1}{126}$
p=5	$\frac{1}{6}$	$\frac{1}{21}$	$\frac{1}{56}$	$\frac{1}{126}$	$\frac{1}{252}$

More precisely, if we define

$$g(p,\,q) = \frac{1}{\int_0^1 \left(1-x^{1/p}\right)^q dx} \tag{2.69}$$

then

$$g(p,\,q) = f_p^{q+1} = \frac{(q+1)\,(q+2)\cdots(q+p)}{1\cdot 2\cdots p} \tag{2.70}$$

are figurate numbers whenever p and q are positive integers. The values of $g(p,\,q)$ together form the Figurate Triangle, now commonly referred to as Pascal's triangle (see Chapter 3), with the exception that the entries equal to 1 are missing:

g(p,q)

	q=1	q=2	q=3	q=4	q=5
p=1	2	3	4	5	6
p=2	3	6	10	15	21
p=3	4	10	20	35	56
p=4	5	15	35	70	126
p=5	6	21	56	126	252

Next, to obtain a formula for $w = g(1/2,\,1/2)$, which corresponds to the reciprocal of the original definite integral $\int_0^1 \left(1-x^2\right)^{1/2} dx$, Wallis interpolates the values of $g(p,\,q)$ by assuming that the following formula for figurate numbers continues to hold even for arbitrary non-negative real values of p and q, including $q = -1/2$:

$$g(p,\,q) = \begin{cases} 1 & \text{if } p = 0 \\ \frac{(q+1)\,(q+2)\cdots(q+p)}{1\cdot 2\cdots p} & \text{if } p \neq 0 \end{cases} \tag{2.71}$$

Wallis it seems is not bothered by the fact that $g(0,\,q)$ is not formally defined according to definition 1.6; the scent of the figurate numbers must have been too strong for him not to follow. As a result of formula 1.8, Wallis is able to produce the following table containing column values for half-integer values of q:

g(p,q)

	$q=0$	$q=\frac{1}{2}$	$q=1$	$q=\frac{3}{2}$	$q=2$	$q=\frac{5}{2}$	$q=3$	$q=\frac{7}{2}$	$q=4$	$q=\frac{9}{2}$	$q=5$	
$p=1$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6
$p=2$	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15	$\frac{143}{8}$	21
$p=3$	$\frac{5}{16}$	1	$\frac{35}{16}$	4	$\frac{105}{16}$	10	$\frac{231}{16}$	20	$\frac{429}{16}$	35	$\frac{715}{16}$	56
$p=4$	$\frac{35}{128}$	1	$\frac{315}{128}$	5	$\frac{1155}{128}$	15	$\frac{3003}{128}$	35	$\frac{6435}{128}$	70	$12\,155/128$	126
$p=5$	$\frac{63}{256}$	1	$\frac{693}{256}$	6	$\frac{3003}{256}$	21	$\frac{9009}{256}$	56	$21\,879/256$	126	$46\,189/256$	252

To complete the table for half-integer values of p, he takes advantage of the following recurrence satisfied by figurate numbers and again assumes that it continues to hold for all non-negative values of p and q, including $p = -1/2$:

$$g(p+1,\, q) = g(p,\, q)\cdot\frac{q+p+1}{p+1} \tag{2.72}$$

This, coupled with the assumption that the values must be symmetric, i.e. $g(p,\, q) = g(q,\, p)$, allows him to fill in all the missing values by expressing them in terms of $w = g(1/2,\, 1/2)$:

g(p,q)

	$q=-\frac{1}{2}$	$q=0$	$q=\frac{1}{2}$	$q=1$	$q=\frac{3}{2}$	$q=2$	$q=\frac{5}{2}$	$q=3$	$q=\frac{7}{2}$	$q=4$
$p=-\frac{1}{2}$	∞	1	$\frac{w}{2}$	$\frac{1}{2}$	$\frac{w}{3}$	$\frac{3}{8}$	$\frac{4w}{15}$	$\frac{5}{16}$	$\frac{8w}{35}$	$\frac{35}{128}$
$p=0$	1	1	1	1	1	1	1	1	1	1
$p=\frac{1}{2}$	$\frac{w}{2}$	1	w	$\frac{3}{2}$	$\frac{4w}{3}$	$\frac{15}{8}$	$\frac{8w}{5}$	$\frac{35}{16}$	$\frac{64w}{35}$	$\frac{315}{128}$
$p=1$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5
$p=\frac{3}{2}$	$\frac{w}{3}$	1	$\frac{4w}{3}$	$\frac{5}{2}$	$\frac{8w}{3}$	$\frac{35}{8}$	$\frac{64w}{15}$	$\frac{105}{16}$	$(128w)/21$	$\frac{1155}{128}$
$p=2$	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15
$p=\frac{5}{2}$	$\frac{4w}{15}$	1	$\frac{8w}{5}$	$\frac{7}{2}$	$\frac{64w}{15}$	$\frac{63}{8}$	$(128w)/15$	$\frac{231}{16}$	$(512w)/35$	$\frac{3003}{128}$
$p=3$	$\frac{5}{16}$	1	$\frac{35}{16}$	4	$\frac{105}{16}$	10	$\frac{231}{16}$	20	$\frac{429}{16}$	35
$p=\frac{7}{2}$	$\frac{8w}{35}$	1	$\frac{64w}{35}$	$\frac{9}{2}$	$(128w)/21$	$\frac{99}{8}$	$(512w)/35$	$\frac{429}{16}$	$(1024w)/35$	$\frac{6435}{128}$
$p=4$	$\frac{35}{128}$	1	$\frac{315}{128}$	5	$\frac{1155}{128}$	15	$\frac{3003}{128}$	35	$\frac{6435}{128}$	70

Again we note that $g(-1/2,\, p)$ is undefined, but Wallis does not seem to care, as by now he has committed himself to following the trail marked by the pattern of figurate numbers.

With his table complete, Wallis focuses in on the row of values for $p = 1/2$ and factors them according to the pattern:

g(1/2,q)

	$q=-\frac{1}{2}$	$q=0$	$q=\frac{1}{2}$	$q=1$	$q=\frac{3}{2}$	$q=2$	$q=\frac{5}{2}$	$q=3$	$q=\frac{7}{2}$	$q=4$
$p=1/2$	$\frac{1}{2}w$	1	w	$\frac{3}{2}$	$\frac{4}{3}w$	$\frac{3\cdot 5}{2\cdot 4}$	$\frac{4\cdot 6}{3\cdot 5}w$	$(3\cdot 5\cdot 7)/(2\cdot 4\cdot 6)$	$(4\cdot 6\cdot 8)/(3\cdot 5\cdot 7)w$	$(3\cdot 5\cdot 7\cdot 9)/(2\cdot 4\cdot 6\cdot 8)$

Lastly, Wallis assumes that the ratios of consecutive terms in each row decrease montonically, i.e.,

$$\frac{g\left(\frac{p+1}{2},\, q\right)}{g\left(\frac{p}{2},\, q\right)} > \frac{g\left(\frac{p+2}{2},\, q\right)}{g\left(\frac{p+1}{2},\, q\right)} \tag{2.73}$$

In particular, for $p = 1/2$, we have

$$\frac{2}{w} > \frac{w}{1} > \frac{3}{2\,w} > \frac{2\cdot 4\, w}{3\cdot 3} > \frac{3\cdot 3\cdot 5}{2\cdot 4\cdot 4\, w} > \frac{2\cdot 4\cdot 4\cdot 6\, w}{3\cdot 3\cdot 5\cdot 5} > \ldots \tag{2.74}$$

It follows that the sequence of inequalities hold for *w*:

$$\sqrt{\frac{3}{2}} < w < \sqrt{2}\,,\ \frac{3\cdot 3}{2\cdot 4}\sqrt{\frac{5}{4}} < w < \frac{3\cdot 3}{2\cdot 4}\sqrt{\frac{4}{3}}\,,\ \frac{3\cdot 3\cdot 5\cdot 5}{2\cdot 4\cdot 4\cdot 6}\sqrt{\frac{7}{5}} < w < \frac{3\cdot 3\cdot 5\cdot 5}{2\cdot 4\cdot 4\cdot 6}\sqrt{\frac{5}{4}}\,,\ \ldots \tag{2.75}$$

In general, we have for $n \geq 1$

$$\frac{3\cdot 3\cdot 5\cdot 5\cdots(2\,n+1)\,(2\,n+1)}{2\cdot 4\cdot 4\cdot 6\cdots(2\,n)\,(2\,n+2)}\sqrt{\frac{2\,n+3}{2\,n+2}} < w < \frac{3\cdot 3\cdot 5\cdot 5\cdots(2\,n+1)\,(2\,n+1)}{2\cdot 4\cdot 4\cdot 6\cdots(2\,n)\,(2\,n+2)}\sqrt{\frac{2\,n+2}{2\,n+1}} \tag{2.76}$$

Since the factors $\sqrt{\frac{2\,n+3}{2\,n+2}}$ and $\sqrt{\frac{2\,n+2}{2\,n+1}}$ both converge to 1 in the limit as $n \to \infty$, we thus obtain Wallis' formula by setting $w = 4/\pi$:

$$\frac{4}{\pi} = \frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdots} \tag{2.77}$$

A brilliant mathematical tour-de-force!

Exercises

Readers can hunt on their own for patterns by solving the following exercises.

1. Products of Consecutive Integers

a. Find a formula for the product of four consecutive integers beginning with n, i.e., $n(n+1)(n+2)(n+3)$ in terms of perfect squares. For example, here are the products for n ranging from 1 to 5:

```
s = Table[n * (n + 1) (n + 2) (n + 3), {n, 1, 5}]
```

```
{24, 120, 360, 840, 1680}
```

b. Prove your formula algebraically.

2. Sums of Squares of Fibonacci Numbers (see [Ho])

a. Find a pattern for the sum of squares of two consecutive Fibonacci numbers:

Sums of Squares

n	F(n)^2+F(n+1)^2
0	1
1	2
2	5
3	13
4	34
5	89
6	233
7	610
8	1597
9	4181

Show Mathematica Answer

NOTE: *Mathematica*'s **FindSequenceFunction** gives a direct formula for the sequence above, but not a very interesting one. Find a more interesting formula involving the Fibonacci numbers.

b. Can you find patterns involving sums of squares of non-consecutive Fibonacci numbers, e.g., the even Fibonacci numbers?

c. What about sums of square of three consecutive Fibonacci numbers?

Show Mathematica Answer

d. What about sums/differences of cubes of consecutive Fibonacci numbers? (see [Me])

Show Mathematica Answer

3. Concordia Function ([LLHC])

Consider the Concordia function $c(n)$ which counts the number of partitions of n consisting only of prime numbers. For example, the table below shows that there are seven partitions of 5, of which two contain only prime numbers, namely the partitions {5} and {3, 2}. Thus, $c(5) = 2$.

n	Partions of n
1	{{1}}
2	{{2}, {1, 1}}
3	{{3}, {2, 1}, {1, 1, 1}}
4	{{4}, {3, 1}, {2, 2}, {2, 1, 1}, {1, 1, 1, 1}}
5	{{5}, {4, 1}, {3, 2}, {3, 1, 1}, {2, 2, 1}, {2, 1, 1, 1}, {1, 1, 1, 1, 1}}

Here is a table listing the first thirty values of $c(n)$:

Number of Partitions
Consisting of Primes

n	c(n)
1	0
2	1
3	1
4	1
5	2
6	2
7	3
8	3
9	4
10	5

n	c(n)
11	6
12	7
13	9
14	10
15	12
16	14
17	17
18	19
19	23
20	26

n	c(n)
21	30
22	35
23	40
24	46
25	52
26	60
27	67
28	77
29	87
30	98

Now define $b(n)$ to be the number of partitions of n that consist only of 2's or 3's.

Number of Partitions
Consisting of 2's or 3's

n	b(n)
1	0
2	1
3	1
4	1
5	1
6	2
7	1
8	2
9	2
10	2

n	b(n)
11	2
12	3
13	2
14	3
15	3
16	3
17	3
18	4
19	3
20	4

n	b(n)
21	4
22	4
23	4
24	5
25	4
26	5
27	5
28	5
29	5
30	6

Find a formula for $b(n)$.

4. Revisited: Dishonest Men, Coconuts, and a Monkey (see [Ga], Chapter 1, p. 3)

Recall the problem in Example 1.2 involving the division of coconuts among five sailors and a monkey. Suppose in the final division each received an equal share with no coconuts remaining.

a. Find the number of coconuts that the five sailors had gathered.

b. Generalize the problem to n sailors and solve it to find a formula for the number of coconuts.

5. Rational solutions of quadratics with coefficients in arithmetic progression (see [LoHe])

Consider the following quadratic equation whose coefficients are in arithmetic progression:

$$x^2 + (n+1)\,x - (n+2) = 0$$

a. For which integer values of n does this equation yield rational solutions for x? Hint: Use the **Solve** command to obtain the solution formula for x and then apply the **FindInstance** command to find particular solutions for n.

b. Find formulas for the two rational solutions for x. Then prove your formulas.

6. Divisibility of the Perrin sequence by primes (see [PS]

Define a sequence a_n by $a_0 = 3$, $a_1 = 0$, $a_2 = 2$, and $a_{n+3} = a_{n+1} + a_n$. Find a divisibility pattern involving a_n.

Answer: a_n is divisible by prime integers n.

```
a[0] = 3; a[1] = 0; a[2] = 2;
a[n_] := a[n - 2] + a[n - 3]

Table[a[n], {n, 0, 5}]

{3, 0, 2, 3, 2, 5}

Table[Mod[a[n], n], {n, 3, 15}]

{0, 2, 0, 5, 0, 2, 3, 7, 0, 5, 0, 9, 8}
```

NOTE: This result generalizes to sequences $a_0 = k$, $a_1 = a_2 =. .. a_{k-2} = 0$, $a_{k-1} = k - 1$, and $a_{n+k} = a_{n+1} + a_n$.

7. Lacunary Recurrences (see [Yo-P])

a. Consider the Fibonacci sequence $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$.

i. Find a recurrence for the subsequence $\{F_{2m}\}$. Prove your answer.

ii. Find a recurrence for the subsequence $\{F_{a m}\}$ where a is a positive integer:

iii. Find a recurrence for the subsequence $\{F_{a m+b}\}$ where a and b are positive integers.

b. Next consider the sequence $x_0 = 0$, $x_1 = 1$, $x_{n+1} = 2 x_n + x_{n-1}$.

i. Find a recurrence for the subsequence $\{x_{a m}\}$ where a is a positive integer:

ii. Find a recurrence for the subsequence $\{x_{a m+b}\}$ where a is a positive integer:

c. Next consider the tribonacci sequence $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_{n+2} = x_{n+1} + x_n + x_{n-1}$. Find a recurrence for the subsequence $\{x_{a m}\}$ where a is a positive integer:

8. Summing a Sequence (see [ES2])

Consider the rational sequence $\{a_n\} = \{1, 1/2, 1/4, 3/8, 3/16, 8/32, 7/64, 21/128, 15/256, 55/512, 31/1024, 144/2048, 63/4096, \ldots\}$:

a. Find a formula for a_n:

Show Mathematica Answer

b. Find the sum of the series $\sum_{n=0}^{\infty} a(n)$.

9. Sums of trinomials of roots of a cubic.

a. The quadratic equation $x^2 - x - 1 = 0$ has two roots: $a = \frac{1-\sqrt{5}}{2}$ and $b = \frac{1+\sqrt{5}}{2}$. Define $u(n)$ to be the binomial sum

$$u(n) = \sum_{i=0}^{n} a^i\, b^{n-i}$$

Find an explict formula and a recurrence for $u(n)$.

b. Denote the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$ by a, b, c. Find an explicit formula and a recurrence for the trinomial sum

$$u(n) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a^i\, b^j\, c^{n-i-j}$$

c. Denote the roots of the quartic equation $x^3 - x^2 - x - 1 = 0$ by a, b, c, d. Conjecture a recurrence for the trinomial sum

$$u(n) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} a^i\, b^j\, c^k\, d^{n-i-j-k}$$

and experimentally verify your conjecture. NOTE: You may find that *Mathematica* will have difficulty evaluating $u(n)$ for large values of n using the trinomial sum formula above. This shows that recurrences can be more effective in generating sequences than direct formulas.

10. Fibonacci to Lucas (see [ES3])

Find a recurrence for the ratios of Fibonacci to Lucas numbers:

```
Table[Fibonacci[n] / LucasL[n], {n, 1, 10}]
```

$$\left\{1, \frac{1}{3}, \frac{1}{2}, \frac{3}{7}, \frac{5}{11}, \frac{4}{9}, \frac{13}{29}, \frac{21}{47}, \frac{17}{38}, \frac{55}{123}\right\}$$

11. Non-Totients (see [Pu])

A non-totient is an integer n for which there is no solution to the equation $\varphi(x) = n$, where $\varphi(x)$ is Euler's totient function. Find a number pattern among the set of non-totients and prove that it is true.

12. Continued Fractions

Find a formula for the convergents of the continued fraction (1,1,1,...):

```
Convergents[PadRight[{}, 10, 1]]
```

$$\left\{1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}\right\}$$

Give a proof of your formula.

13. Subsets with no adjacent elements

Let s_n denote the number of subsets of $\{1, 2, \ldots, n\}$ such that no two elements are adjacent. For example, if $n = 4$, then the subsets with no adjacent elements are: {{1,3},{2,4}, {1,4}}

For $n = 5$, we have: {{1,3},{1,4},{1,5},{2,4},{2,5}}

Recursion: $F_n = F_{n-1} + F_{n-2}$ (non-adjacent subsets of {1,2,3,4,6} = non-adjacent subsets of {1,2,3,4,5} + non-adjacent subsets of {1,2,3,4} (with 6 added as an element since any non-adjacent subset containing 6 will be a non-adjacent subset of {1,2,3,4} with 6 deleted))

14. Counting Triangles in a Square (see [BoKo])

Let $T(n)$ denote the number of lattice triangles lying inside the region $[0, n] \times [0, n]$ of the Cartesian plane whose sides lie on lines of slope 0, ∞, 1, or -1. Determine a closed formula for $T(n)$.

15. Diophantine Triplets (see [DeBr])

A Diophantine triplet is a set of three positive integers (a, b, c) such that $a < b < c$ and $a\,b + 1$, $b\,c + 1$, and $a\,c + 1$ are all perfect squares. Find patterns describing Diophantine triplets.

3
Classical Number Patterns

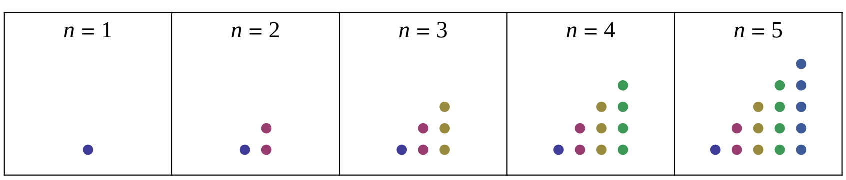
In this chapter we describe some classical numerical experiments involving integer sequences to reveal the myriad of number patterns that can arise. We make two disclaimers about these experiments. First they focus more on analysis of data and formulation of mathematical conjectures as opposed to rigorous proofs of the results obtained, although proofs of certain conjectures are given if they are short and elementary; otherwise, more complicated proofs are given in the appendix or referenced elsewhere in the mathematics literature. Second we provide litte historical background behind the numerical experiments discussed,, some of which have a rich history and date as far back as the Greeks, and instead provide references where appropriate.

3.1 Figurate Numbers

The figurate numbers date back to Pythagoras (c. 570–c. 495 BC) and other ancient Greek philosophers who were the first to study properties of real numbers and to study them for knowledge sake. Figurate numbers are generated from arrangements of points into regular geometrical shapes. If the shape is a polygon, such as a triangle, square, or pentagons, then figurate numbers are also called polygonal numbers, which we shall explore first.

Part 1 - Triangular Numbers

The triangular numbers t_n count the number of points in the following sequence of triangles which represent their area:



Thus, we have by construction

$t_1 = 1$
$t_2 = 1 + 2 = 3$
$t_3 = 1 + 2 + 3 = 6$
...

More generally, we have

$$t_n = 1 + ... + n = t_{n-1} + n \tag{3.1}$$

To begin our exploration of triangular numbers, we first make a table listing the first ten values.

Triangular Numbers

n	$t_n = \sum_{k=1}^{n} k$
1	1 = 1
2	3 = 1+2
3	6 = 1+2+3
4	10 = 1+2+3+4
5	15 = 1+2+3+4+5
6	21 = 1+2+3+4+5+6
7	28 = 1+2+3+4+5+6+7
8	36 = 1+2+3+4+5+6+7+8
9	45 = 1+2+3+4+5+6+7+8+9
10	55 = 1+2+3+4+5+6+7+8+9+10

STEP 1

The first goal in understanding any sequence of values defined by summation is to find an efficient formula for calculating it (besides by brute force). For example, can you compute t_{100} without summing all 100 terms? Since no obvious multiplicative formula appears in the table above, we shall demonstrate four different methods for extracting such a formula for t_n:

Method 1:

This classic technique is a variation of the "summing in pairs" trick commonly told as an anecdote involving Carl Friedrich Gauss, who at the age of 10, was able to quickly sum the first 100 positive integers to the surprise of his schoolmaster (see [Hay]).

Suppose we sum two copies of t_n by adding corresponding terms in pairs, but reverse the order of the terms in the second copy:

Triangular Numbers

n	$t_n + t_n = \sum_{k=1}^{n} k + \sum_{k=n}^{1} k$
1	1 = 1 1 = 1 ------------- 2 = 2
2	3 = 1+2 3 = 2+1 ------------- 6 = 3+3
3	6 = 1+2+3 6 = 3+2+1 ------------- 12 = 4+4+4
4	10 = 1+2+3+4 10 = 4+3+2+1 ------------- 20 = 5+5+5+5
5	15 = 1+2+3+4+5 15 = 5+4+3+2+1 ------------- 30 = 6+6+6+6+6

n	$t_n + t_n = \sum_{k=1}^{n} k + \sum_{k=n}^{1} k$
6	21 = 1+2+3+4+5+6 21 = 6+5+4+3+2+1 ------------- 42 = 7+7+7+7+7+7
7	28 = 1+2+3+4+5+6+7 28 = 7+6+5+4+3+2+1 ------------- 56 = 8+8+8+8+8+8+8
8	36 = 1+2+3+4+5+6+7+8 36 = 8+7+6+5+4+3+2+1 ------------- 72 = 9+9+9+9+9+9+9+9
9	45 = 1+2+3+4+5+6+7+8+9 45 = 9+8+7+6+5+4+3+2+1 ------------- 90 = 10+10+10+10+10+10+10+10+10
10	55 = 1+2+3+4+5+6+7+8+9+10 55 = 10+9+8+7+6+5+4+3+2+1 ------------- 110 = 11+11+11+11+11+11+11+11+11+11

Aha! It is now clear from the table above that $2\,t_n$ equals the sum of n copies of $n+1$, or equivalently, $n(n+1)$. Assuming this, we conclude that

$$t_n = \frac{n(n+1)}{2} \tag{3.2}$$

Method 2:

Let's examine the divisors of each of the triangular numbers:

n	t_n	Divisors of t_n
1	1	{1}
2	4	{1, 3}
3	10	{1, 2, 3, 6}
4	20	{1, 2, 5, 10}
5	35	{1, 3, 5, 15}
6	56	{1, 3, 7, 21}
7	84	{1, 2, 4, 7, 14, 28}
8	120	{1, 2, 3, 4, 6, 9, 12, 18, 36}
9	165	{1, 3, 5, 9, 15, 45}
10	220	{1, 5, 11, 55}

It appears that for n is a divisor of t_n for n odd. This suggests that we should examine the values of t_n/n:

n	t_n/n
1	1
2	$\frac{3}{2}$
3	2
4	$\frac{5}{2}$
5	3
6	$\frac{7}{2}$
7	4
8	$\frac{9}{2}$
9	5
10	$\frac{11}{2}$

Converting all the values for t_n/n into half-integers shows that $t_n/n = \frac{n+1}{2}$, which results in the same formula as (3.2).

<u>Method 3:</u>

Suppose we summed the even and odd integers appearing in the sum $t_n = 1 + 2 + ... + n$ separately. Denote by o_n and e_n to be the sum of the odd terms and even terms of $\{1, 2, ..., n\}$, respectively. Here is a table listing the first ten values of o_n and e_n:

Triangular Numbers

n	$t_n=\sum_{k=1}^{n} k$	o_n	e_n
1	1 = 1	1 = 1	0 = {}
2	3 = 1+2	1 = 1	2 = 2
3	6 = 1+2+3	4 = 1+3	2 = 2
4	10 = 1+2+3+4	4 = 1+3	6 = 2+4
5	15 = 1+2+3+4+5	9 = 1+3+5	6 = 2+4
6	21 = 1+2+3+4+5+6	9 = 1+3+5	12 = 2+4+6
7	28 = 1+2+3+4+5+6+7	16 = 1+3+5+7	12 = 2+4+6
8	36 = 1+2+3+4+5+6+7+8	16 = 1+3+5+7	20 = 2+4+6+8
9	45 = 1+2+3+4+5+6+7+8+9	25 = 1+3+5+7+9	20 = 2+4+6+8
10	55 = 1+2+3+4+5+6+7+8+9+10	25 = 1+3+5+7+9	30 = 2+4+6+8+10

Clear patterns now emerge. Depending on whether n is odd or even, which we re-index as $2n-1$ and $2n$, respectively, we find that

$$o_{2n-1} = o_{2n} = n^2$$

As for e_n, observe that we can re-express it as a sum of odd integers by subtracting 1 from each even integer. For example,

$$e_5 = 2+4 = (1+1)+(3+1) = (1+3)+2 = o_4+2$$
$$e_6 = 2+4+6 = (1+1)+(3+1)+(5+1) = (1+3+5)+3 = o_6+3$$

More generally, we have

$$e_{2n} = e_{2n+1} = o_{2n} + n = n^2 + n = n(n+1)$$

It follows that

$$t_{2n-1} = o_{2n-1} + e_{2n-1} = n^2 + (n-1)\,n = n(2n-1) = \tfrac{2n(2n-1)}{2}$$
$$t_{2n} = o_{2n} + e_{2n} = n^2 + n(n+1) = n(2n+1) = \tfrac{2n(2n+1)}{2}$$

These formulas are equivalent to (3.2).

NOTE: Factoring 2 from each term in e_n yields the relation

$$e_{2n} = e_{2n+1} = 2\,t_n$$

and leads to the following recurrences for the triangular numbers:

$$\begin{aligned} t_{2n-1} &= o_{2n-1} + e_{2n-1} = n^2 + 2\,t_{n-1} \\ t_{2n} &= o_{2n} + e_{2n} = n^2 + 2\,t_n \end{aligned} \tag{3.3}$$

<u>Method 4:</u>

We calculate successive differences up to order 2 (assume $t_0 = 0$):

d	$\Delta^d t_n$
0	{0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55}
1	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
2	{1, 1, 1, 1, 1, 1, 1, 1, 1}
3	{0, 0, 0, 0, 0, 0, 0, 0}

Since $\Delta^0\,t_0 = 1$, $\Delta^1\,t_0 = 1$, $\Delta^2\,t_0 = 1$, and $\Delta^d\,t_0 = 0$ for all $d \geq 3$, it follows that

$$t_n = \sum_{m=0}^{n} \binom{n}{m} \Delta^m\,a_0 = \binom{n}{0}\Delta^0\,a_0 + \binom{n}{1}\Delta^1\,a_0 + \binom{n}{2}\Delta^2\,a_0 = 1\cdot 0 + n\cdot 1 + \frac{n(n-1)}{2}\cdot 1 = \frac{n^2+n}{2} = \frac{n(n+1)}{2} \tag{3.4}$$

which agrees with (3.2).

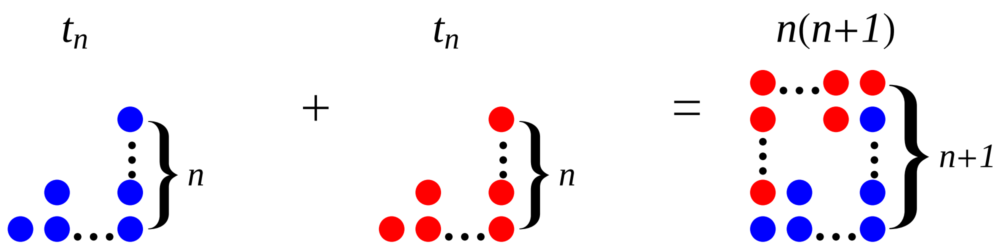
NOTES:

1. We can of course use *Mathematica* to verify formula (3.2) as follows:

```
Sum[k, {k, 1, n}]
```

$$\frac{1}{2} n (1 + n)$$

2. A visual “proof without words” of (3.2) is given below ([]):



STEP 2

Consider the following table of sums of two consecutive triangular numbers. Do you observe a pattern?

n	t_n	$t_n + t_{n+1}$
1	1	4
2	3	9
3	6	16
4	10	25
5	15	36
6	21	49
7	28	64
8	36	81
9	45	100
10	55	121

Yes, the pattern here is quite clear:

$$t_n + t_{n+1} = (n + 1)^2 \tag{3.5}$$

Of course, we can obtain the same answer by using *Mathematica* to substitute the formula for t_n (obtained in Step 1) into $t_n + t_{n+1}$ and then simplify the result:

```
Simplify[Sum[k, {k, 1, n}] + Sum[k, {k, 1, n + 1}]]
```

$$(1 + n)^2$$

Can you sketch a similar “proof without words” to demonstrate this formula?

STEP 3

What about a formula for $T_n = t_1 + t_2 + ... + t_n$, the sum of the first n triangular numbers?

n	t_n	$t_1 + ... + t_{n+1}$
1	1	1
2	3	4
3	6	10
4	10	20
5	15	35
6	21	56
7	28	84
8	36	120
9	45	165
10	55	220

The pattern is not so simple here. Let's try the same trick as before by examining the values of T_n/n:

n	t_n	T_n/n
1	1	1
2	3	2
3	6	$\frac{10}{3}$
4	10	5
5	15	7
6	21	$\frac{28}{3}$
7	28	12
8	36	15
9	45	$\frac{55}{3}$
10	55	22

To convert the values for T_n/n to integers, we multiply each by 3:

n	t_n	$3T_n/n$
1	1	3
2	3	6
3	6	10
4	10	15
5	15	21
6	21	28
7	28	36
8	36	45
9	45	55
10	55	66

This shows that values for $3\,T_n/n$ are the same as the triangular numbers, except shifted by one position. Thus, $3\,T_n/n = t_{n+1} = (n+1)(n+2)/2$, or equivalently,

$$T_n = \frac{n(n+1)(n+2)}{6} \tag{3.6}$$

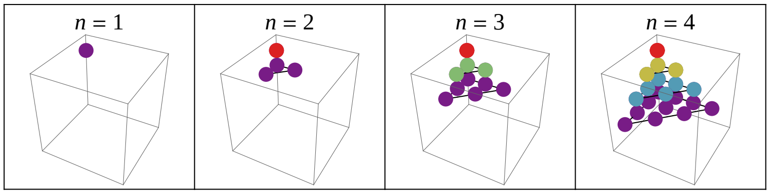
Again, we can obtain the same answer using *Mathematica*:

```
Sum[TriangularNumber[k], {k, 1, n}]
```

$$\frac{1}{6} n (1+n) (2+n)$$

NOTE: The numbers T_n are also referred to as *tetrahedral (or pyramidal) numbers*. This can be seen geometrically by stacking the triangular numbers $t_1, t_2, \ldots, t_n$ on top of each other as layers to form the tetrahedron corresponding to T_n:

Tetrahedral Numbers



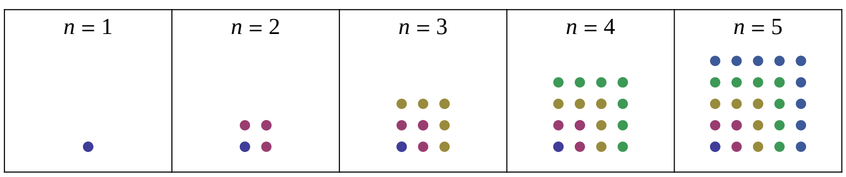
FURTHER EXPLORATION:

1. Can you detect any patterns for the weighted sum $\sum_{k=1}^{n} k\, t_{n+1} = t_1 + 2\, t_2 + \ldots + n\, t_n$? Use *Mathematica* to verify your conjectures.

2. Leonard Euler observed that if n is a triangular number, then $9\,n+1$, $25\,n+3$, $49\,n+6$, $81\,n+10$, and so forth, are also triangular numbers involving odd square multiples of n. Are there other triangular numbers that involve odd square multiples of n?

Part 2 - Square Numbers

Triangular numbers can be generalized to *square* numbers s_n, whose values are given by the number of points inside a square representing its area as illustrated below:



Square numbers can be defined recursively:

$s_1 = 1$

$s_2 = 1 + 3 = 4$

$s_3 = 1 + 3 + 5 = 9$

...

More generally, we have

$$s_n = 1 + 3 + 5 + \ldots + (2\,n - 1) = s_{n-1} + (2\,n - 1) \tag{3.7}$$

Of course, square numbers can also be defined more explicitly by the formula

$$s_n = \underbrace{n + \ldots + n}_{(n \text{ summands})} = n^2 \tag{3.8}$$

This leads to the classic identity

$$1 + 3 + 5 + \ldots + (2\,n - 1) = n^2 \tag{3.9}$$

A more interesting pattern involving square numbers has to do with their connection to triangular numbers. In particular, are there triangular numbers that are also square numbers? Examining the first ten triangular numbers show that there is indeed one: $t_8 = 36$. Let's call such numbers square-triangular numbers and denote them by r_n. Further investigation shows that there are relatively few of these numbers and that they are dispersed among the triangular numbers. The following *Mathematica* module will search by brute-force for the n-th square-triangular number.

```
TriangularSquareNumbers[n_] := Module[{list = {}, k = 1, i = 0}, While[n > i,
    If[Sqrt[TriangularNumber[k]] == IntegerPart[Sqrt[TriangularNumber[k]]],
     AppendTo[list, {k, TriangularNumber[k]}]; i++]; k++]; list];
```

Here's a table showing the first six square-triangular numbers:

n	r_n
1	1
2	36
3	1225
4	41 616
5	1 413 721

A further search for larger square-triangular numbers becomes too exhaustive; thus, it is desirable to find a more efficient formula for generating them.

Unfortunately, **FindSequenceFunction** fails to determine an explicit formula for square-triangular numbers:

```
Simplify[FindSequenceFunction[datasquaretriangularnumbers[[All, 2]], n]]
```

```
FindSequenceFunction[{1, 36, 1225, 41 616, 1 413 721}, n]
```

Similarly, the command **FindLinearRecurrence** fails to find a recurrence:

```
FindLinearRecurrence[datasquaretriangularnumbers[[All, 2]]]
```

```
FindLinearRecurrence[{1, 36, 1225, 41 616, 1 413 721}]
```

However, since these values are integer squares, perhaps we should employ their radicals. Here's a table listing the square roots of the six square-triangular numbers given above:

n	r_n
1	1
2	6
3	35
4	204
5	1189

Indeed we now find that the values for $\sqrt{r_n}$ do satisfy a recurrence:

```
FindLinearRecurrence[datasquaretriangularnumberssquareroots[[All, 2]]]
```

```
{6, -1}
```

On the other hand, **FindSequenceFunction** fails to find a formula for $\sqrt{r_n}$:

```
FindSequenceFunction[datasquaretriangularnumberssquareroots[[All, 2]]]
```

```
FindSequenceFunction[{1, 6, 35, 204, 1189}]
```

Perhaps more terms are needed. Fortunately, we now have a recurrence formula, which we can use to quickly generate additional terms:

```
datasquaretriangularnumbersmoreterms = LinearRecurrence[{6, -1}, {1, 6}, 10]
```

```
{1, 6, 35, 204, 1189, 6930, 40 391, 235 416, 1 372 105, 7 997 214}
```

Feeding this longer list of terms into **FindSequenceFunction** yields the following formula for $\sqrt{t_n}$:

```
Simplify[FindSequenceFunction[datasquaretriangularnumbersmoreterms, n]]
```

$$\frac{-\left(3-2\sqrt{2}\right)^{n}+\left(3+2\sqrt{2}\right)^{n}}{4\sqrt{2}}$$

Squaring this formula yields a corresponding formula for t_n, originally discovered by Leonard Euler in 1788:

$$r_n = \left(\frac{\left(3+2\sqrt{2}\right)^{n}-\left(3-2\sqrt{2}\right)^{n}}{4\sqrt{2}}\right)^{2} \tag{3.10}$$

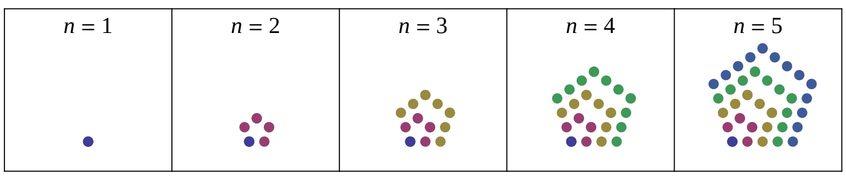
NOTE: A more novel approach to studying square-triangular numbers is to describe them algebraically, i.e., $r_n = s^2 = \frac{t(t+1)}{2}$ for some positive integers s and t. Multiplying both sides of this equation by 8 and completing the square on the left-hand side yields $(2t+1)^2 - 1 = 8s^2$. This is equivalent to Pell's equation $x^2 - 2y^2 = 1$ with $x = 2t+1$ and $y = 2s$, which we discussed in the previous chapter.

In fact, the values for $\sqrt{r_n}$ correspond to the product $x \cdot y$, where (x, y) represent solutions to the more general Pell equation $x^2 - 2y^2 = \pm 1$.

Part 3 - Pentagonal Numbers

A more interesting generalization of the triangular numbers are the pentagonal numbers P_n, which represent the number of points arranged inside a pentagon consisting of edges n dots in length as illustration below:

```
GraphicsRow[Table[PentagonalNumberPlot[n0], {n0, 0, 4}],
 Alignment → Bottom, Frame → All, ImageSize → 400]
```



Thus the pentagonal numbers are defined by

$p_1 = 1$
$p_2 = 1 + 4 = 5$
$p_3 = 1 + 4 + 7 = 12$
...

More generally, we have

$$p_n = 1 + 4 + 7 + 10 \ldots + (3n-2) = p_{n-1} + (3n-2) \tag{3.11}$$

Let's proceed as before by making a table listing the values of the first ten pentagonal numbers.

n	$p_n=\sum_{k=1}^{n}(3k-2)$
1	1 = 1
2	5 = 1+4
3	12 = 1+4+7
4	22 = 1+4+7+10
5	35 = 1+4+7+10+13
6	51 = 1+4+7+10+13+16
7	70 = 1+4+7+10+13+16+19
8	92 = 1+4+7+10+13+16+19+22
9	117 = 1+4+7+10+13+16+19+22+25
10	145 = 1+4+7+10+13+16+19+22+25+28

STEP 1

Conjecture a formula for the pentagonal numbers p_n in terms of n. HINT: Try either Gauss's technique of summing in pairs or else examine their divisors. Does your formula match *Mathematica*'s formula?

STEP 2

What connection do pentagonal numbers have with triangular numbers. HINT: Partition the pentagon that corresponds to each pentagonal number into an appropriate number of triangles. Prove your conjecture algebraically.

Part 3.1.4 - Polygonal Numbers

Let's make our notation for polygonal numbers more uniform by writing P_n^d to denote the n-th polygonal number corresponding to a polygon with d sides. The following table summarizes what we know so far about the values of triangular ($d = 3$), square ($d = 4$), and pentagonal ($d = 5$) numbers.

Polygonal Numbers P_n^d

P_n^d	n=1	n=2	n=3	n=4	n=5
d=3 (Triangular)	1	3	6	10	15
d=4 (Square)	1	4	9	16	25
d=5 (Pentagonal)	1	5	12	22	35

"

Step 1

Based on the pattern, can you guess what the first five values for some higher-order polygonal numbers, say hexagonal, heptagonal, and octagonal?

Polygonal Numbers P_n^d

P_n^d	n=1	n=2	n=3	n=4	n=5
d=3	1	3	6	10	15
d=4	1	4	9	16	25
d=5	1	5	12	22	35
d=6	1	6	15	28	45
d=7	1	7	18	34	55
d=8	1	8	21	40	65

Conjecture a formula for the n-th polygonal number $P_d(n)$.

Step 2

Below is a more extensive table of $P_d(n)$ for $3 \le d \le 9$ and $1 \le n \le 9$.

Polygonal Numbers P_n^d

P_n^d	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9
d=3	1	3	6	10	15	21	28	36	45
d=4	1	4	9	16	25	36	49	64	81
d=5	1	5	12	22	35	51	70	92	117
d=6	1	6	15	28	45	66	91	120	153
d=7	1	7	18	34	55	81	112	148	189
d=8	1	8	21	40	65	96	133	176	225
d=9	1	9	24	46	75	111	154	204	261

What other patterns do you observe about polygonal numbers?

Part 5 - Higher-Dimensional Figurate Numbers

We saw in the first part how sums of natural numbers generate triangular numbers and in turn how sums of triangular numbers generate tetrahedral numbers (pyramids with triangular base). Of course, we can continue this pattern by considering sums of tetrehedral numbers, which can be visualized as a four-dimensional pyramid. If we refer to these sums as figurate numbers f_n^d, then this process can be continued indefinitely to an arbitrary number of dimensions as follows:

$$f_n^0 = 1 \tag{3.12}$$

$$f_n^1 = f_1^0 + f_2^0 + \dots + f_n^0 = 1 + 1 + \dots + 1 = n \tag{3.13}$$

$$f_n^2 = f_1^1 + f_2^1 + \dots + f_n^1 = 1 + 2 + \dots + n = t_n = \frac{n(n+1)}{2} \tag{3.14}$$

$$f_n^3 = f_1^2 + f_2^2 + \dots + f_n^2 = t_1 + t_2 + \dots + t_n = T_n = \frac{n(n+1)\,(n+2)}{6} \tag{3.15}$$

...

$$f_n^d = f_1^{d-1} + f_2^{d-1} \dots + f_n^{d-1} \tag{3.16}$$

Based on known formulas for f_n^1, f_n^2, and f_n^3, can you conjecture a formula for f_n^d?

The following is a tabulation of f_n^d:

Figurate Numbers f_n^d

f_n^d	n=1	n=2	n=3	n=4	n=5
d=0	1	1	1	1	1
d=1	1	2	3	4	5
d=2	1	3	6	10	15
d=3	1	4	10	20	35
d=4	1	5	15	35	70

It is clear from this table that the following recursive identity holds:

$$f_n^d = f_n^{d-1} + f_{n-1}^d \tag{3.17}$$

This recurrence is the basis for many other fascinating patterns in the table, which in the context of figurate numbers is referred to as the Figurate Triangle. However, it is more well known as Pascal's triangle, the next topic in this chapter.

3.2 Pascal's Triangle

Pascal’s triangle is one of the most recognized number patterns in the world, being an array of coefficients that appear in the binomial expansion of $(x+y)^n$ for all non-negative integers n. For example, below are the first 5 expansions (up to $n = 4$):

$(x+y)^0 = 1$
$(x+y)^1 = 1 \cdot x + 1 \cdot y$
$(x+y)^2 = 1 \cdot x^2 + 2\,x\,y + 1 \cdot y^2$
$(x+y)^3 = 1 \cdot x^3 + 3\,x^2\,y + 3\,x\,y^2 + 1 \cdot y^3$
$(x+y)^4 = 1 \cdot x^4 + 4\,x^3\,y + 6\,x^2\,y^2 + 4\,x\,y^3 + 1 \cdot y^4$

These coefficients, called *binomial coefficients* (in blue), are typically arranged in the form of an equilateral triangle and form Pascal's triangle:

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
```

More precisely, we define the binomial coefficient $\binom{n}{k}$ to be the coefficient of $x^k\,y^{n-k}$ in the expansion

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}\,y + \binom{n}{2} x^{n-2}\,y^2 + \ldots + \binom{n}{n} y^n \tag{3.18}$$

It is clear that $\binom{n}{0} = \binom{n}{n} = 1$.

Part 1 - Formula for Pascal’s Triangle

To find a formula for the binomial coefficients, we observe that Pascal’s Triangle is essentially a reconfiguration of the Figurate Triangle (discussed at the end of the previous section) where the binomial coefficient $\binom{n}{k}$ corresponds to the figurate number f_{n-k+1}^{k}. Since figurate numbers are given by formula (3.XXX), it suffices to adapt this formula to describe binomial coefficients. First, we express figurate numbers in terms of factorials as follows:

$$f_n^d = \frac{n(n+1)\,(n+2)\cdots(n+d-1)}{1\cdot 2\cdot 3\cdots d} = \frac{(1\cdot 2\cdot 3\cdots(n-1))\,n(n+1)\,(n+2)\cdots(n+d-1)}{(1\cdot 2\cdot 3\cdots(n-1))\,(1\cdot 2\cdot 3\cdots d)} = \frac{(n+d-1)!}{(n-1)!\,d!} \tag{3.19}$$

It follows that

$$\binom{n}{k} = f_{n-k+1}^{k} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{k!\,(n-k)!} \tag{3.20}$$

For example, $\binom{5}{2} = \frac{5!}{2!\cdot 3!} = 10$. Can you prove formula (3.20)? A proof is given in Appendix A.1.

The most easily recognized patttern involving Pascal’s triangle is the following: every interior entry is the sum of the two entries above it (e.g. the entry 3 in the third row is the sum of entries 1 and 2 in the second row). This fundamental relationship is expressed mathematically by the identity (called Pascal’s identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{3.21}$$

Observe that if we merely input this identity into *Mathematica*, then *Mathematica* is not able to verify it:

```
Binomial[n, k] == Binomial[n - 1, k - 1] + Binomial[n - 1, k]
```

```
Binomial[n, k] == Binomial[-1 + n, -1 + k] + Binomial[-1 + n, k]
```

However, if we use the **FullSimplify** command and assume n and k to be non-negative integers, then *Mathematica* is now able to confirm it:

```
FullSimplify[Binomial[n, k] == Binomial[n - 1, k - 1] + Binomial[n - 1, k],
 Element[n, Integers] && n >= 0 && Element[k, Integers] && k >= 0]
```

```
True
```

NOTE: The equilateral configuration of Pascal's triangle is quite arbitrary and is not the only one that is useful. A more natural configuration which aligns coefficients corresponding to the monomials $x^k\, y^{n-k}$ (see 1.1-1.5) is that of a right triangle (column justified), a form used originally by Michael Stifel and other when it made it appearance in Western mathematical texts in the 1500's:

Binomial Coefficents $\binom{n}{k}$

$\binom{n}{k}$	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9
n=0	1									
n=1	1	1								
n=2	1	2	1							
n=3	1	3	3	1						
n=4	1	4	6	4	1					
n=5	1	5	10	10	5	1				
n=6	1	6	15	20	15	6	1			
n=7	1	7	21	35	35	21	7	1		
n=8	1	8	28	56	70	56	28	8	1	
n=9	1	9	36	84	126	126	84	36	9	1

We note that Pascal's triangle can also be displayed in array form by including those values of $\binom{n}{k}$ for $1 \le n < k$, a form that is referred to as Pascal's matrix (or square).

Binomial Coefficents $\binom{n}{k}$

	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9
n=0	1	0	0	0	0	0	0	0	0	0
n=1	1	1	0	0	0	0	0	0	0	0
n=2	1	2	1	0	0	0	0	0	0	0
n=3	1	3	3	1	0	0	0	0	0	0
n=4	1	4	6	4	1	0	0	0	0	0
n=5	1	5	10	10	5	1	0	0	0	0
n=6	1	6	15	20	15	6	1	0	0	0
n=7	1	7	21	35	35	21	7	1	0	0
n=8	1	8	28	56	70	56	28	8	1	0
n=9	1	9	36	84	126	126	84	36	9	1

Part 2 - Rows, Columns, Diagonals

Pascal's triangle contain many fascinating number pattterns involving its rows, columns, and diagonals.

Step 1

Let's consider the sum of the terms in each row of Pascal's triangle. For example, the sum of the entries in row $n = 3$ equals 8 (entires in this row are shown in red below):

Binomial Coefficents $\binom{n}{k}$

	k=0	k=1	k=2	k=3	k=4	k=5
n=0	1					
n=1	1	1				
n=2	1	2	1			
n=3	1	3	3	1		
n=4	1	4	6	4	1	
n=5	1	5	10	10	5	1

The following table lists the sums of the first 6 rows ($n = 0$ corresponds to the first row):

n	$\sum_{k=0}^{n}\binom{n}{k}$
0	1 = 1
1	2 = 1+1
2	4 = 1+2+1
3	8 = 1+3+3+1
4	16 = 1+4+6+4+1
5	32 = 1+5+10+10+5+1

The table reveals that each row sums to a power of 2. Thus we have discovered the another well known fundamental identity involving Pascal's triangle:

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n \tag{3.22}$$

Can you prove this formula?

FURTHER EXPLORATION: What do you observe about the sum of the terms at even positions in each row (starting at $n = 0$)? Odd positions?

Step 2

Next, let's sum the terms in the first column of Pascal's triangle. Since this column (and every other column) is infinite in length, we'll keep a running total of its entries by calculating its partial sums, i.e., the sum of the entries up to the n-th row. Moreover, since each entry is equal to 1, this is easily calculated by the sum

$$\sum_{k=0}^{n}\binom{n}{0} = 1 + 1 + \ldots + 1 = n \tag{3.23}$$

For example, the partial sum of the first four entries in the first colum (corresponding to $n = 3$) equals 4. Here is a table of partial sums for n ranging from 1 to 5:

n	$\sum_{k=0}^{n}\binom{n}{k}$
0	1 = 1
1	2 = 1+1
2	3 = 1+1+1
3	4 = 1+1+1+1
4	5 = 1+1+1+1+1
5	6 = 1+1+1+1+1+1

The interesting observation is that these partial sums are recorded precisely in the second column of Pascal's triangle.

Binomial Coefficents $\binom{n}{k}$

	k=0	k=1	k=2	k=3	k=4	k=5
n=0	1					
n=1	1	1				
n=2	1	2	1			
n=3	1	3	3	1		
n=4	1	4	6	4	1	
n=5	1	5	10	10	5	1

To test if this is a coincidence, we compute the partial sums of the second column, given by

$$\sum_{k=1}^{n}\binom{n}{1}=\sum_{k=1}^{n}n=1+2+3+\ldots+n \tag{3.24}$$

to see if a similar pattern holds (the entries corresponding case $n=4$ is displayed in red):

```
Grid[Prepend[databinomialsecondcolumnpartialsums, {"n", "∑_{k=0}^n (n k) "}],
  Frame → All, Alignment → {{Left, Left}, Automatic},
  Background → {None, {{{Lighter[LightGray], White}}, {1 → LightBrown}}}]
```

n	$\sum_{k=0}^{n}\binom{n}{k}$
1	1 = 1
2	3 = 1+2
3	6 = 1+2+3
4	10 = 1+2+3+4
5	15 = 1+2+3+4+5

Indeed, we find that the partial sums {1, 3, 6, 10, 15, ...} are again given precisely by the third column in Pascal's triangle, which the reader should recognize as the triangular numbers t_n discussed in the previous section.

Binomial Coefficents $\binom{n}{k}$

	k=0	k=1	k=2	k=3	k=4	k=5
n=0	1					
n=1	1	1				
n=2	1	2	1			
n=3	1	3	3	1		
n=4	1	4	6	4	1	
n=5	1	5	10	10	5	1

Verify on your own that this relationship continues to hold for any column so that the sum of the first n entries in the m-th column of Pascal's triangle is given by the $(n+1)$-th entry in the $(m+1)$-th column:

Thus, we've discovered classic binomial formula:

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1} \tag{3.25}$$

Observe that *Mathematica* also recognizes this identity:

```
Simplify[Sum[Binomial[k, m], {k, m, n}]]
Binomial[1 + n, 1 + m]
```

Can you prove this identity?

FURTHER EXPLORATION: Can you find other figurate and polygonal numbers lurking inside Pascal's triangle?

Step 3

Observe that the diagonals (top left to bottom right) of Pascal's triangle are the same as its columns due to its orientation. Let's consider the opposite diagonals (bottom left to top right) and sum its entries (for example the first six such diagonals are colored below). Do you recognize a pattern?

Diagonals of Binomial Coefficents $\binom{n}{k}$

	k=0	k=1	k=2	k=3	k=4	k=5
n=0	1	0	0	0	0	0
n=1	1	1	0	0	0	0
n=2	1	2	1	0	0	0
n=3	1	3	3	1	0	0
n=4	1	4	6	4	1	0
n=5	1	5	10	10	5	1

n	Sum of n-th Diagonal (Pascal's Triangle)
0	1 = 1
1	1 = 1+0
2	2 = 1+1+0
3	3 = 1+2+0+0
4	5 = 1+3+1+0+0
5	8 = 1+4+3+0+0+0

It appears that the diagonals sum to the Fibonacci numbers $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$. Thus we have discovered the identity

$$\sum_{m=0}^{n} \binom{n-m}{m} = F_{n+1} \qquad (3.26)$$

Observe that *Mathematica* also recognizes this identity:

```
Sum[Binomial[n - m, m], {m, 0, n}]
Fibonacci[1 + n]
```

Can you prove this? HINT: Use Pascal's Identity to prove that the left hand side of (3.26) satisfies the same recurrence as the Fibonacci numbers.

Part 3 - Pascal's Triangle in Reverse

Note that we can extend Pascal's triangle in the reverse direction, i.e., for negative integer values of n, by rewriting Pascal's identity (3.21) as

$$\binom{n-1}{k} = \binom{n}{k} - \binom{n-1}{k-1} \qquad (3.27)$$

Here is table show Pascal's triangle in reverse:

Pascal's Triangle in Reverse

	k=0	k=1	k=2	k=3	k=4	k=5
n=−5	1	−5	15	−35	70	−126
n=−4	1	−4	10	−20	35	−56
n=−3	1	−3	6	−10	15	−21
n=−2	1	−2	3	−4	5	−6
n=−1	1	−1	1	−1	1	−1
n=0	1	0	0	0	0	0
n=1	1	1	0	0	0	0
n=2	1	2	1	0	0	0
n=3	1	3	3	1	0	0
n=4	1	4	6	4	1	0
n=5	1	5	10	10	5	1

What patterns do you observe? Can you find a formula for $\binom{-n}{k}$ where $n > 0$ and $k > 0$?

Part 4 - Pascal's Triangle (Mod *n*)

Let's consider the congruence $\binom{n}{k}$ (mod n):

```
nMax = 11;
databinomialmodulon =
  Table[If[n == 0, "-", Mod[Binomial[n, k], n]], {n, 0, nMax}, {k, 0, nMax}];
Column[{"(n k) (mod n)",
  Grid[Prepend[(Prepend[databinomialmodulon[[#]], "n=" ~~ ToString[# - 1]]) & /@
     Range[1, Length[databinomialmodulon]],
    Prepend[("k=" ~~ ToString[#]) & /@ Range[0, nMax], " "]], Frame → All,
   Alignment → Center, Background → {{1 → LightBrown}, {1 → LightBrown}}]}, Center]
```

$\binom{n}{k} \pmod{n}$

	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10	k=11
n=0	-	-	-	-	-	-	-	-	-	-	-	-
n=1	0	0	0	0	0	0	0	0	0	0	0	0
n=2	1	0	1	0	0	0	0	0	0	0	0	0
n=3	1	0	0	1	0	0	0	0	0	0	0	0
n=4	1	0	2	0	1	0	0	0	0	0	0	0
n=5	1	0	0	0	0	1	0	0	0	0	0	0
n=6	1	0	3	2	3	0	1	0	0	0	0	0
n=7	1	0	0	0	0	0	0	1	0	0	0	0
n=8	1	0	4	0	6	0	4	0	1	0	0	0
n=9	1	0	0	3	0	0	3	0	0	1	0	0
n=10	1	0	5	0	0	2	0	0	5	0	1	0
n=11	1	0	0	0	0	0	0	0	0	0	0	1

Do you see any patterns involving the rows? Columns? HINT: Consider the rows and columns at prime positions.

FURTHER EXPLORATION: Binomials to Binomials (see [Os])

Recall that binomial coefficients describe the expansion of $(1 + x)^n$. Let's fix $x = \sqrt{2}$ and consider the expansion of the binomial

$$\left(1 + \sqrt{2}\right)^n = a_n + b_n \sqrt{2} \tag{3.28}$$

which results in another binomial. For example,

$$\left(1 + \sqrt{2}\right)^0 = 1$$

$$\left(1 + \sqrt{2}\right)^1 = 1 + \sqrt{2}$$

$$\left(1 + \sqrt{2}\right)^2 = 1 + 2\sqrt{2} + \left(\sqrt{2}\right)^2 = 3 + 2\sqrt{2}$$

$$\left(1 + \sqrt{2}\right)^3 = 1 + 3\sqrt{2} + 3\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^3 = 7 + 5\sqrt{2}$$

Here is a table listing the first ten values of a_n and b_n:

```
nMax = 9; databinomialroot2 = Table[{n, Simplify[(1 + Sqrt[2])^n + (1 - Sqrt[2])^n] / 2,
   Simplify[(1 + Sqrt[2])^n - (1 - Sqrt[2])^n] / (2 Sqrt[2])}, {n, 0, nMax}];
ColumnDataDisplay[databinomialroot2, 10, {"n", "a_n", "b_n"},
 "Expansion of (1+√2)^n = a_n+b_n√2", {{Left, Left, Left}, Automatic}]
```

Expansion of $(1+\sqrt{2})^n = a_n+b_n\sqrt{2}$

n	a_n	b_n
0	1	0
1	1	1
2	3	2
3	7	5
4	17	12
5	41	29
6	99	70
7	239	169
8	577	408
9	1393	985

a) Can you find both recursive and explicit formulas for a_n and b_n?

Show Mathematica Answer

b) Find the limiting value of a_n / b_n as $n \to \infty$.

Show Mathematica Answer

c) Find recursive formulas for a_n and b_n (in terms of c and d) for the binomial expansion $\left(c + \sqrt{d}\right)^n = a_n + b_n \sqrt{2}$.

3.3 Pythagorean Triples

Part 1 - Pythagorean Triples Whose Sides are Consecutive Integers

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Pythag/pythag.html

Note 75.23 Pythagorean triples Chris Evans Mathematical Gazette 75 (1991), page 317

A Pythagorean triple $\{a, b, c\}$ is a set of three positive integers satisfying the Pythagorean Theorem: $a^2 + b^2 = c^2$. Thus, $\{a, b, c\}$ represent the sides of a right triangle. For example, the smallest Pythagorean triple is {3, 4, 5}. Another triple is {5, 12, 13}. There are many algorithms for generating Pythagorean triples and their subsets. In this section, we'll explore some of these algorithms.

Step 1

The Pythagorean triple {3, 4, 5} stands out because of the property that it consists of consecutive integers. Are there other such triples? To find out, we begin by substituting $b = a + 1$ and $c = a + 2$ into the equation $a^2 + b^2 = c^2$.

```
eq = Simplify[a^2 + b^2 == c^2 /. {b → a + 1, c → a + 2}]
```

$a^2 == 3 + 2\,a$

This yields a quadratic equation in a that can easily be solved:

```
Solve[eq, a]
```

```
{{a → -1}, {a → 3}}
```

We rule out the negative solution $a = -1$; thus, the only solution is $a = 3$, corresponding to the triple {3, 4, 5}.

Step 2

Let's relax the restriction that all three sides must be consecutive integers by considering triples $\{a, b, c\}$ where say, the longer leg and hypotenuse, are consecutive integers, i.e., $c = b + 1$; one example, besides {3, 4, 5}, is the triple {5, 12, 13}. Now, not all Pythagorean triples have this property, for example, {8, 15, 17}. On the other hand, are there more such triples? Are there infinitely many such triples? If so, is there a formula for generating them?

To answer these questions, again it suffices to substitute $c = b + 1$ into the equation $a^2 + b^2 = c^2$ and solve for b in terms of a:

```
Clear[a, b, eq];
eq = Simplify[a^2 + b^2 == (b + 1)^2]
Solve[eq, b]
```

$a^2 == 1 + 2\,b$

$\left\{\left\{b \to \frac{1}{2}\left(-1 + a^2\right)\right\}\right\}$

We now argue as follows: if b is required to a positive integer, then $a^2 - 1$ must be even (divisible by 2). It follows that a^2 must be an odd integer greater than 1 and hence a must be odd integer greater than 1. This allows us to index the solutions as

```
Clear[a, b , c];
a[n_] := 2 n + 1;
b[n_] := (a[n]^2 - 1) / 2;
c[n_] := b[n] + 1;
{a[n], Simplify[b[n]], Simplify[c[n], Assumptions → Element[n, Integers] && n > 0]}
```

$\left\{1 + 2\,n,\ 2\,n\,(1 + n),\ 1 + 2\,n + 2\,n^2\right\}$

where n ranges over the positive integers. Here is a table listing the first ten solutions:

Pythagorean Triples {a,b,c} with c=b+1

n	$a_n=2n+1$	$b_n=2n(n+1)$	$c_n=2n^2+2n+1$
1	3	4	5
2	5	12	13
3	7	24	25
4	9	40	41
5	11	60	61
6	13	84	85
7	15	112	113
8	17	144	145
9	19	180	181
10	21	220	221

FURTHER EXPLORATION: Can you find formulas to describe Pythagorean triples where the hypotenuse and longer leg differ by 2?

Step 3

A more challenging problem is to find Pythagorean triples where both legs are consecutive integers, i.e., triples $\{a, b, c\}$ with $b = a + 1$. Substituting this restriction into $a^2 + b^2 = c^2$ yields

```
Simplify[a^2 + b^2 == c^2 /. {b → a + 1}]
```

$a^2 + (1 + a)^2 == c^2$

This equation is not as useful as in previous cases. All we can conclude here is that c must be odd. This follows from the fact if a^2 is even (or odd), then $(a + 1)^2$ is odd (or even, respectively); thus, $c^2 = a^2 + b^2$ must be odd, or equivalently, c must be odd. Here is a table of the first seven solutions obtained through a brute force search:

Pythagorean Triples {a,b,c} with b=a+1

n	a_n	$b_n = a_n + 1$	$c_n = \sqrt{a^2 + (a+1)^2}$
1	3	4	5
2	20	21	29
3	119	120	169
4	696	697	985
5	4059	4060	5741
6	23 660	23 661	33 461
7	137 903	137 904	195 025

These are enough solutions for *Mathematica* to find a formula and recurrence for a_n:

```
Simplify[FindSequenceFunction[dataconsecutivelegs[[All, 1]], n]]
```

$$\frac{1}{12 + 8\sqrt{2}} \left(-6 - 4\sqrt{2} - \left(3 - 2\sqrt{2}\right)^n \left(1 + \sqrt{2}\right) + \left(3 + 2\sqrt{2}\right)^n \left(7 + 5\sqrt{2}\right)\right)$$

```
FindLinearRecurrence[dataconsecutivelegs[[All, 1]]]
```

{7, -7, 1}

As for c_n, we find

```
Simplify[FindSequenceFunction[dataconsecutivelegs[[All, 3]], n]]
```

$$\frac{1}{12 + 8\sqrt{2}} \left(\left(3 - 2\sqrt{2}\right)^n \left(2 + \sqrt{2}\right) + \left(3 + 2\sqrt{2}\right)^n \left(10 + 7\sqrt{2}\right)\right)$$

```
FindLinearRecurrence[dataconsecutivelegs[[All, 3]]]
```

{6, -1}

NOTE: *Mathematica* doesn't seem to recognize that a_n satisfies an even more simple, although non-homogeneous, recurrence: $a_{n+1} = 6\,a_n - a_{n-1} + 2$.

```
RecurrenceTable[{a[n + 1] == 6 a[n] - a[n - 1] + 2, a[1] == 3, a[2] == 20}, a, {n, 1, 7}]
```

{3, 20, 119, 696, 4059, 23 660, 137 903}

Part 2 - Describing Pythagorean Triples by Height

As a generalization to the FURTHER EXPLORATION in Step 2 above, we describe a recursive method for parametrizing all Pythagorean Triples by their height. Let $\{a, b, c\}$ be a Pythagorean triple. We define its *height* to be $H = c - b$. We saw earlier that for $H = 1$,

$a = 2\,n + 1$
$b = 2\,n(n + 1)$
$c = 2\,n^2 + 2\,n + 1$

Hopefully, the reader was able to show that for $H = 2$,

$a = 2\,n$
$b = n^2 - 1$
$c = n^2 + 1$

Step 1

Let's complete the analysis for all higher values of H. Substituting $c = b + H$ into $a^2 + b^2 = c^2$ yields

```
Clear[a, b, c];
sol = Solve[Simplify[a^2 + b^2 == c^2 /. {c → b + H}], b]
```

$$\left\{\left\{b \to \frac{a^2 - H^2}{2\,H}\right\}\right\}$$

We now use the formula above for b to tabulate values of Pythagorean triples for H ranging from 1 to 10:

H=c−b=1

n	a_n	b_n	c_n
1	3	4	5
2	5	12	13
3	7	24	25
4	9	40	41
5	11	60	61

H=c−b=2

n	a_n	b_n	c_n
1	4	3	5
2	6	8	10
3	8	15	17
4	10	24	26
5	12	35	37

H=c−b=3

n	a_n	b_n	c_n
1	9	12	15
2	15	36	39
3	21	72	75
4	27	120	123
5	33	180	183

H=c−b=4

n	a_n	b_n	c_n
1	8	6	10
2	12	16	20
3	16	30	34
4	20	48	52
5	24	70	74

H=c−b=5

n	a_n	b_n	c_n
1	15	20	25
2	25	60	65
3	35	120	125
4	45	200	205
5	55	300	305

H=c−b=6

n	a_n	b_n	c_n
1	12	9	15
2	18	24	30
3	24	45	51
4	30	72	78
5	36	105	111

H=c−b=7

n	a_n	b_n	c_n
1	21	28	35
2	35	84	91
3	49	168	175
4	63	280	287
5	77	420	427

H=c−b=8

n	a_n	b_n	c_n
1	12	5	13
2	16	12	20
3	20	21	29
4	24	32	40
5	28	45	53

H=c−b=9

n	a_n	b_n	c_n
1	15	8	17
2	21	20	29
3	27	36	45
4	33	56	65
5	39	80	89

H=c−b=10

n	a_n	b_n	c_n
1	20	15	25
2	30	40	50
3	40	75	85
4	50	120	130
5	60	175	185

An analysis of the differences between consecutive values of a_n in the first few tables seems to suggest the following:

$$a_{n+1} - a_n = \begin{cases} H & \text{if } H \text{ even} \\ 2\,H & \text{if } H \text{ odd} \end{cases}$$

If true, this would imply

$$a_n = \begin{cases} H\,(n+1) & \text{if } H \text{ even} \\ H(2\,n+1) & \text{if } H \text{ odd} \end{cases}$$

However, this pattern fails for $H = 8$ and $H = 9$. A check of the higher values of H (up to $H = 50$) shows that this pattern also fails for the following values: 16, 18, 24, 25, 27, 32, 36, 40, 45, 48, and 50 as shown in the table below:

Formula for a_n

H	a_n
1	$1+2n$
2	$2(1+n)$
3	$3(1+2n)$
4	$4(1+n)$
5	$5(1+2n)$
6	$6(1+n)$
7	$7(1+2n)$
8	$4(2+n)$
9	$3(3+2n)$
10	$10(1+n)$

H	a_n
11	$11(1+2n)$
12	$12(1+n)$
13	$13(1+2n)$
14	$14(1+n)$
15	$15(1+2n)$
16	$8(2+n)$
17	$17(1+2n)$
18	$6(3+n)$
19	$19(1+2n)$
20	$20(1+n)$

H	a_n
21	$21(1+2n)$
22	$22(1+n)$
23	$23(1+2n)$
24	$12(2+n)$
25	$5(5+2n)$
26	$26(1+n)$
27	$9(3+2n)$
28	$28(1+n)$
29	$29(1+2n)$
30	$30(1+n)$

H	a_n
31	$31(1+2n)$
32	$8(4+n)$
33	$33(1+2n)$
34	$34(1+n)$
35	$35(1+2n)$
36	$12(3+n)$
37	$37(1+2n)$
38	$38(1+n)$
39	$39(1+2n)$
40	$20(2+n)$

H	a_n
41	$41(1+2n)$
42	$42(1+n)$
43	$43(1+2n)$
44	$44(1+n)$
45	$15(3+2n)$
46	$46(1+n)$
47	$47(1+2n)$
48	$24(2+n)$
49	$7(7+2n)$
50	$10(5+n)$

Observe that this list includes the odd perfect squares 9, 25, and 49. Let's focus then on just these types of values:

Formula for a_n

$\sqrt{H}$	a_n
1	$1+2n$
3	$3(3+2n)$
5	$5(5+2n)$
7	$7(7+2n)$
9	$9(9+2n)$
11	$11(11+2n)$
13	$13(13+2n)$
15	$15(15+2n)$
17	$17(17+2n)$
19	$19(19+2n)$

$\sqrt{H}$	a_n
21	$21(21+2n)$
23	$23(23+2n)$
25	$25(25+2n)$
27	$27(27+2n)$
29	$29(29+2n)$
31	$31(31+2n)$
33	$33(33+2n)$
35	$35(35+2n)$
37	$37(37+2n)$
39	$39(39+2n)$

Aha! We find that if $H = (2h+1)^2$, then

$$a_n = \sqrt{H}\left(\sqrt{H}+2n\right) = (2h+1)((2h+1)+2n)$$

Let's now check to see if this pattern holds for the even perfect squares:

Formula for a_n

$\sqrt{H}$	a_n
2	$4(1+n)$
4	$8(2+n)$
6	$12(3+n)$
8	$16(4+n)$
10	$20(5+n)$
12	$24(6+n)$
14	$28(7+n)$
16	$32(8+n)$
18	$36(9+n)$
20	$40(10+n)$

$\sqrt{H}$	a_n
22	$44(11+n)$
24	$48(12+n)$
26	$52(13+n)$
28	$56(14+n)$
30	$60(15+n)$
32	$64(16+n)$
34	$68(17+n)$
36	$72(18+n)$
38	$76(19+n)$
40	$80(20+n)$

If we distribute a factor of 2 through each formula for a_n, then indeed the same pattern holds in the case $H = (2h)^2$.

What about those other exceptional values of H that are not perfect squares such as 8, 18, 24, 27, 32, etc. It appears that these values are multiples of perfect squares. As a start, let's focus on double perfect squares, i.e., $H = 2h^2$:

Formula for a_n

$h=\sqrt{H/2}$	a_n
1	$2(1+n)$
2	$4(2+n)$
3	$6(3+n)$
4	$8(4+n)$
5	$10(5+n)$
6	$12(6+n)$
7	$14(7+n)$
8	$16(8+n)$
9	$18(9+n)$
10	$20(10+n)$

$h=\sqrt{H/2}$	a_n
11	$22(11+n)$
12	$24(12+n)$
13	$26(13+n)$
14	$28(14+n)$
15	$30(15+n)$
16	$32(16+n)$
17	$34(17+n)$
18	$36(18+n)$
19	$38(19+n)$
20	$40(20+n)$

A nice pattern emerges for a_n:

$$a_n = \sqrt{2H}\left(\sqrt{H/2} + 2n\right) = 2h(h + 2n)$$

What about triple perfect squares, i.e., $H = 3h^2$?

Formula for a_n

$h=\sqrt{H/3}$	a_n
1	$3(1+2n)$
2	$12(1+n)$
3	$9(3+2n)$
4	$24(2+n)$
5	$15(5+2n)$
6	$36(3+n)$
7	$21(7+2n)$
8	$48(4+n)$
9	$27(9+2n)$
10	$60(5+n)$

$h=\sqrt{H/3}$	a_n
11	$33(11+2n)$
12	$72(6+n)$
13	$39(13+2n)$
14	$84(7+n)$
15	$45(15+2n)$
16	$96(8+n)$
17	$51(17+2n)$
18	$108(9+n)$
19	$57(19+2n)$
20	$120(10+n)$

If we manipulate each formula so that the factor $(h+2\,n)$ appears, then we find that a similar pattern holds:

$$a_n = \sqrt{3\,H}\left(\sqrt{H/2}+2\,n\right) = 2\,h(h+2\,n)$$

The reader is encouraged to verify this pattern for higher multiples of perfect squres. Thus, we conclude with the following theorem:

Theorem: The following formulas generate Pythagorean triples $\{a, b, c\}$ of height $H=c-b$:
a) If H is NOT a multiple of a perfect square, then

$$\begin{aligned}
a_n &= \begin{cases} 2\,h(n+1) & \text{if } H = 2\,h \text{ even} \\ (2\,h+1)(2\,n+1) & \text{if } H = 2\,h+1 \text{ odd} \end{cases} \\
b_n &= \frac{a_n^2 - H^2}{2\,H} = \begin{cases} h\,n(n+2) & \text{if } H = 2\,h \text{ even} \\ 2\,(2\,h+1)\,n(n+1) & \text{if } H = 2\,h+1 \text{ odd} \end{cases} \\
c_n &= b_n + H = \begin{cases} h\left(n^2+2\,n+2\right) & \text{if } H = 2\,h \text{ even} \\ (2\,h+1)\left(2\,n^2+2\,n+1\right) & \text{if } H = 2\,h+1 \text{ odd} \end{cases}
\end{aligned} \tag{3.29}$$

b) If H multiple of a perfect square, i.e., $H = mh^2$ with m square-free, then

$$\begin{aligned}
a_n &= m\,h(h+2\,n) \\
b_n &= \frac{a_n^2 - H^2}{2\,H} = 2\,m\,n(h+n) \\
c_n &= b_n + H = m\left(h^2+2\,h\,n+2\,n^2\right)
\end{aligned} \tag{3.30}$$

Step 2

Let's investigate whether Pythagorean triples satisfy any recurrences. It suffices to consider the different cases as in Step 1 depending on whether the height H equals a multiple of a perfect square. Let's begin with H an even integer, but not a multiple of a perfect square:

H=c−b=2

n	a_n	b_n	c_n
1	4	3	5
2	6	8	10
3	8	15	17
4	10	24	26
5	12	35	37

H=c−b=6

n	a_n	b_n	c_n
1	12	9	15
2	18	24	30
3	24	45	51
4	30	72	78
5	36	105	111

H=c−b=10

n	a_n	b_n	c_n
1	20	15	25
2	30	40	50
3	40	75	85
4	50	120	130
5	60	175	185

H=c−b=14

n	a_n	b_n	c_n
1	28	21	35
2	42	56	70
3	56	105	119
4	70	168	182
5	84	245	259

H=c−b=22

n	a_n	b_n	c_n
1	44	33	55
2	66	88	110
3	88	165	187
4	110	264	286
5	132	385	407

By examining the sums $a_n + b_n$, it appears that the values for b_n in the tables above satisfy the following recurrence:

$$b_{n+1} = a_n + b_n + H/2$$

Does a similar recurrence hold for H an odd integer, but not a multiple of a perfect square? Let’s analyze the tables below:

```
nMax = 500;
Hmax = 5;
dataPythagoreantriplesheightsumab =
  Map[Take[DeleteCases[Table[{#, a, sol[[1, 1, 2]] /. H → #,
        Simplify[sol[[1, 1, 2]] + H] /. H → #}, {a, 1, nMax}],
      x__ /; (! (IntegerQ[x[[3]]] && x[[3]] > 0))], 5] &, {3, 5, 7, 11, 13}];
Row[Table[ColumnDataDisplay[Table[ReplacePart[
      dataPythagoreantriplesheightsumab[[H]][[n]], 1 -> n],
     {n, 1, Length[dataPythagoreantriplesheightsumab[[H]]]}],
    10, {"n", "an", "bn", "cn"}, "
 H=c-b=" <> ToString[dataPythagoreantriplesheightsumab[[H]][[1, 1]]] <> "",
    {{Left, Left, Left, Left}, Automatic}],
  {H, 1, Length[dataPythagoreantriplesheightsumab]}], " "]
```

H=c−b=3

n	a_n	b_n	c_n
1	9	12	15
2	15	36	39
3	21	72	75
4	27	120	123
5	33	180	183

H=c−b=5

n	a_n	b_n	c_n
1	15	20	25
2	25	60	65
3	35	120	125
4	45	200	205
5	55	300	305

H=c−b=7

n	a_n	b_n	c_n
1	21	28	35
2	35	84	91
3	49	168	175
4	63	280	287
5	77	420	427

H=c−b=11

n	a_n	b_n	c_n
1	33	44	55
2	55	132	143
3	77	264	275
4	99	440	451
5	121	660	671

H=c−b=13

n	a_n	b_n	c_n
1	39	52	65
2	65	156	169
3	91	312	325
4	117	520	533
5	143	780	793

This time we find that

$$b_{n+1} = 2\,a_n + b_n + 2\,H$$

We now move on to the case where H is a multiple of a perfect square. Let’s begin with $H = h^2$:

H=c−b=1

n	a_n	b_n	c_n
1	3	4	5
2	5	12	13
3	7	24	25
4	9	40	41
5	11	60	61
6	13	84	85
7	15	112	113
8	17	144	145
9	19	180	181
10	21	220	221

H=c−b=4

n	a_n	b_n	c_n
1	8	6	10
2	12	16	20
3	16	30	34
4	20	48	52
5	24	70	74
6	28	96	100
7	32	126	130
8	36	160	164
9	40	198	202
10	44	240	244

H=c−b=9

n	a_n	b_n	c_n
1	15	8	17
2	21	20	29
3	27	36	45
4	33	56	65
5	39	80	89
6	45	108	117
7	51	140	149
8	57	176	185
9	63	216	225
10	69	260	269

H=c−b=16

n	a_n	b_n	c_n
1	24	10	26
2	32	24	40
3	40	42	58
4	48	64	80
5	56	90	106
6	64	120	136
7	72	154	170
8	80	192	208
9	88	234	250
10	96	280	296

H=c−b=25

n	a_n	b_n	c_n
1	35	12	37
2	45	28	53
3	55	48	73
4	65	72	97
5	75	100	125
6	85	132	157
7	95	168	193
8	105	208	233
9	115	252	277
10	125	300	325

H=c−b=36

n	a_n	b_n	c_n
1	48	14	50
2	60	32	68
3	72	54	90
4	84	80	116
5	96	110	146
6	108	144	180
7	120	182	218
8	132	224	260
9	144	270	306
10	156	320	356

H=c−b=49

n	a_n	b_n	c_n
1	63	16	65
2	77	36	85
3	91	60	109
4	105	88	137
5	119	120	169
6	133	156	205
7	147	196	245
8	161	240	289
9	175	288	337
10	189	340	389

H=c−b=64

n	a_n	b_n	c_n
1	80	18	82
2	96	40	104
3	112	66	130
4	128	96	160
5	144	130	194
6	160	168	232
7	176	210	274
8	192	256	320
9	208	306	370
10	224	360	424

H=c−b=81

n	a_n	b_n	c_n
1	99	20	101
2	117	44	125
3	135	72	153
4	153	104	185
5	171	140	221
6	189	180	261
7	207	224	305
8	225	272	353
9	243	324	405
10	261	380	461

H=c−b=100

n	a_n	b_n	c_n
1	120	22	122
2	140	48	148
3	160	78	178
4	180	112	212
5	200	150	250
6	220	192	292
7	240	238	338
8	260	288	388
9	280	342	442
10	300	400	500

It appears that for $H = 1$, we have $b_{n+1} = 2\,a_n + b_n + 2$, but for $H = 4$, we have $b_{n+1} = a_n + b_n + 2$. No recurrence seems to exist for $H = 9$, 16, 25. However, in these cases, if we restrict to triples that are spaced $\sqrt{H}$ rows apart, then $a_{n+k} - a_n$ is a multiple of H and the pattern holds. For example, if $H = 9$, then every third triple satisfies the recurrence $b_{n+3} = 2\,a_n + b_n + 18$, for example,

$\{a_1, b_1, c_1\}, \{a_4, b_4, c_4\}, \{a_7, b_7, c_7\}, \ldots$

On the other hand, if $H = 16$, then every fourth triple satisfies the recurrence $b_{n+4} = a_n + b_n + 8$, for example,

$\{a_1, b_1, c_1\}, \{a_5, b_5, c_5\}, \{a_9, b_9, c_9\}, \ldots$

This leads to the recurrence

$$b_{n+h} = \begin{cases} a_n + b_n + H/2 & \text{if } H = h^2 \text{ even} \\ 2\,a_n + b_n + 2\,H & \text{if } H = h^2 \text{ odd} \end{cases} \tag{3.31}$$

when H is a perfect square.

FURTHER EXPLORATION:

1. Determine recurrences for b_n where H is a multiple of a perfect square, i.e., $H = m\,h^2$ with h square-free.

2. Find similar recurrence formulas for c_n by assuming the various cases for H discussed in the steps above.

NOTE: For a more detailed treatment and a proof that these recurrences generate all Pythagorean triples of height H, see [WW] and [MW].

3.4 Permutations

Recall that a permutation of a set is a just an ordering of the set. The following experiments reveal the wide range of properties of permutations.

3.4.1 Catching Your Graduation Cap

Concrete Mathematics, Section 5.3, p. 193

Suppose upon receiving their high school diplomas at graduation, all students celebrate by throwing this graduation caps into the air. Assuming that the caps randomly fall back down and each student catches exactly one cap, what is the probability that a quarter of the students will catch their own caps in a graduating class of 20 students?

Let's solve this problem more generally by considering a graduating class of n students and denoting by $c(n, k)$ the number of ways in which k students catch their own caps. The following table gives the first several rows of values of $c(n, k)$.

```
perm = Permutations[{A, B, C, D}]
```

```
{{A, B, C, D}, {A, B, D, C}, {A, C, B, D}, {A, C, D, B}, {A, D, B, C}, {A, D, C, B},
 {B, A, C, D}, {B, A, D, C}, {B, C, A, D}, {B, C, D, A}, {B, D, A, C}, {B, D, C, A},
 {C, A, B, D}, {C, A, D, B}, {C, B, A, D}, {C, B, D, A}, {C, D, A, B}, {C, D, B, A},
 {D, A, B, C}, {D, A, C, B}, {D, B, A, C}, {D, B, C, A}, {D, C, A, B}, {D, C, B, A}}
```

```
temp = 4 - HammingDistance[#, {A, B, C, D}] & /@ perm
```

```
{4, 2, 2, 1, 1, 2, 2, 0, 1, 0, 0, 1, 1, 0, 2, 1, 0, 0, 0, 1, 1, 2, 0, 0}
```

```
c[n_, k_] :=
 Count[n - HammingDistance[#, Range[1, n]] & /@ Permutations[Range[1, n]], k]
```

```
nMax = 5;
Table[c[n, k], {n, 0, nMax}, {k, 0, n}] // Grid
```

```
 1
 0  1
 1  0  1
 2  3  0  1
 9  8  6  0 1
44 45 20 10 0 1
```

We immediately observe that $c(n, n-1) = 0$ and $c(n, n) = 1$. The latter identity is clear: there is only one permutation, namely the identity permutation, in which every student catches his or her own cap. The former identity can be explained as follows: if $(n-1)$ students catch their own caps, then the remaining student must also catch his or her own cap since it is the only cap remaining; thus, $c(n, n-1) = 0$.

Observe that since the number of permutations equals $n!$ and grows rapidly, a brute force computation of $c(n, k)$ is impractical here for large n, as demonstrated by the amount of time (in seconds) required to compute $c(n, 0)$ for $n = 1, 2, \ldots 10$:

```
temp = Table[{n, Timing[c[n, 0]]}, {n, 1, 10}] // Grid
```

```
1       {0., 0}
2       {0., 1}
3       {0., 2}
4       {0., 9}
5       {0., 44}
6    {0.015, 265}
7     {0.11, 1854}
8    {0.343, 14 833}
9    {3.016, 133 496}
10  {30.969, 1 334 961}
```

Thus it is useful to find a more efficient formula for $c(n, k)$. Let's start by trying to find a pattern for the values $c(n, 0)$ in the first column:

```
datahn0 = Table[c[n, 0], {n, 1, 8}]
```

```
{0, 1, 2, 9, 44, 265, 1854, 14 833}
```

```
FindSequenceFunction[datahn0, n]
```

```
Subfactorial[n]
```

This shows that $c(n, 0)$ represents the number of permutations where no student retrieves his or her own cap, i.e., the number of derangements (discussed in Chapter 2). Thus,

$$c(n, 0) = n_{\mathrm{i}} \tag{3.32}$$

where n_{i} is the subfactorial function. This leads us to suspect that $c(n, k)$ should involve the subfactorial $(n-k)_{\mathrm{i}}$ since in this case $(n-k)$ students will NOT have retrieved his or her own cap. Thus, we consider values of $c(n, k)$ divided by $(n-k)_{\mathrm{i}}$:

```
Quiet[Table[c[n, k] / Subfactorial[n - k], {n, 0, 5}, {k, 0, n}] // Grid]
```

1					
Indeterminate	1				
1	Indeterminate	1			
1	3	Indeterminate	1		
1	4	6	Indeterminate	1	
1	5	10	10	Indeterminate	1

We quickly recognize this table as Pascal's triangle consisting of binomial coefficients (the Indeterminate values resulted from division of $c(n, n-1)$ by $(n-(n-1))_{\mathrm{i}} = 1_{\mathrm{i}} = 0$). Thus, the formula for $c(n, k)$ in terms of the factorial function becomes clear:

$$c(n, k) = \binom{n}{k} c(n-k, 0) = \binom{n}{k} c(n-k, 0) \tag{3.33}$$

```
FindSequenceFunction[Table[Subfactorial[k], {k, 1, 5}], n]
```

```
Subfactorial[n]
```

NOTE: We can of course have reasoned further by viewing $c(n, k)$ as the number of ways of choosing k students who end up retrieving their own cap multiplied with the number of ways in which the remaining $(n-k)$ students who do NOT retrieve their own cap, i.e., $c(n-k, 0)$. Thus,

$$c(n, k) = \binom{n}{k} (n-k)_{\mathrm{i}} \tag{3.34}$$

This formula now allows us to quickly compute $c(n, k)$:

```
Subfactorial[10]
```

```
1 334 961
```

Thus, the probability that k students will catch their own caps equals $c(n, k)/n!$.

$$\frac{c(n, k)}{n!} = \binom{n}{k} \frac{(n-k)_{¡}}{n!} = \frac{n!\,(n-k)_{¡}}{k!\,(n-k)!\,n!} = \frac{(n-k)_{¡}}{k!\,(n-k)!} \tag{3.35}$$

To answer the original problem, the probability that exactly 5 out of 20 students will catch their own caps equals

$$\frac{c(20, 5)}{20!} = \frac{(15)_{¡}}{5!\cdot 15!} = \frac{48\,066\,515\,734}{120\cdot 1\,307\,674\,368\,000} \approx 0.003,$$

which is quite remote.

NOTE: Much more likely is when no student catches his or her own cap. The probability of this occuring out of 20 students equals

$$\frac{c(20, 0)}{20!} = \frac{(20)_{¡}}{20!} = \frac{895\,014\,631\,192\,902\,121}{2\,432\,902\,008\,176\,640\,000} \approx 0.368$$

or more than a third chance.

FURTHER EXPLORATION: Can you determine the limiting value of $c(n, k)/n!$ as $n \to \infty$? If necessary, use the in-house *Mathematica* command ISC or go directly to the Inverse Symbolic Calculator (ISC) website.

3.4.2 Runs

ACP Vol. 3, p.34

An ascending run of a permutation is an increasing contiguous subsequence of the permutation that cannot be extended at either end. For example, the permutation {2, 4, 1, 3, 5} contains the run {1, 3, 5}. To find all runs, we use the *Mathematica* command **Runs**.

```
Runs[{2, 4, 1, 3, 5}]
```

```
{{2, 4}, {1, 3, 5}}
```

Thus, {2, 4, 1, 3, 5} has two runs, {2, 4} and {1, 3, 5}. Note that runs specify a partition for the permutation.

Denote by $\left\langle {n \atop k} \right\rangle$ to be the number of permutations of length n that have k runs each. For example, here is a table listing the runs for each permutation of length 3:

Runs for Permutations of Length 3

σ	Runs of σ
{1, 2, 3}	{{1, 2, 3}}
{1, 3, 2}	{{1, 3}, {2}}
{2, 1, 3}	{{2}, {1, 3}}
{2, 3, 1}	{{2, 3}, {1}}
{3, 1, 2}	{{3}, {1, 2}}
{3, 2, 1}	{{3}, {2}, {1}}

We now use the table to calculate $\left\langle {3 \atop k} \right\rangle$:

```
Runs[#] & /@ Permutations[{1, 2, 3}]
```

```
{{{1, 2, 3}}, {{1, 3}, {2}}, {{2}, {1, 3}},
 {{2, 3}, {1}}, {{3}, {1, 2}}, {{3}, {2}, {1}}}
```

```
Count[Runs[#] & /@ Permutations[{1, 2, 3}], t_ /; Length[t] == 3]
```

```
1
```

```
datanumberofruns = Table[
   {k, Count[Runs[#] & /@ Permutations[{1, 2, 3}], t_ /; Length[t] == k]}, {k, 1, 3}];
ColumnDataDisplay[datanumberofruns, 10, {"k", "⟨3 k⟩"}, "Distribution of Runs"]
```

Distribution of Runs

k	$\left\langle {3 \atop k} \right\rangle$
1	1
2	4
3	1

Let's now make an array of values for $\left\langle {n \atop k} \right\rangle$:

$\left\langle {n \atop k} \right\rangle$	k=1	k=2	k=3	k=4	k=5	k=6	k=7
n=1	1	0	0	0	0	0	0
n=2	1	1	0	0	0	0	0
n=3	1	4	1	0	0	0	0
n=4	1	11	11	1	0	0	0
n=5	1	26	66	26	1	0	0
n=6	1	57	302	302	57	1	0
n=7	1	120	1191	2416	1191	120	1

Using *Mathematica*, we find that the second column satisfies the formula

```
FindSequenceFunction[{0, 1, 4, 11, 26, 57, 120}, n]
```

$-1 + 2^n - n$

Can you find other patterns in the table above?

NOTE: The values $\left\langle {n \atop k} \right\rangle$ are referred to as the Eulerian numbers.

3.4.3 Alternating Runs

ACP Vol. 3, p. 46

In the previous subsection, we considered purely ascending runs in partitioning a permutation. Of course, we couuld have replaced ascending runs by descending runs. However, it is also possible to consider ascending and descending runs that alternate in a given permutation as follows. Let $\sigma = \{a_1, a_2, \ldots, a_n\}$ be a permutation of $\{1, 2, \ldots, n\}$. We define the first run to begin at a_1 and prescribe it to be ascending or descending based on whether $a_1 < a_2$ or $a_1 > a_2$, respectively. Suppose the first run ends at a_i before changing its climb, i.e., changes from ascending to descending or vice versa. Then a_i becomes the start of the second run and its ascension or descension is opposite that of the first run. By continuing this process down the last element of σ, we obtain a sequence of alternating runs. For example, the permutation {2, 5, 3, 1, 4} has three alternating runs: {2, 5}, {5, 3, 1}, and

{1, 4}. The in-house *Mathematica* command **AlternatingRuns** will generate alternating runs of a permutation.

```
AlternatingRuns[{2, 5, 3, 1, 4}]
{{2, 5}, {5, 3, 1}, {1, 4}}
```

Let's now generate a table of alternating runs for all permutations of length 3.

Alternating Runs for Permutations of Length 3

σ	Alternating Runs of σ	Number of Alternating Runs
{1, 2, 3}	{{1, 2, 3}}	1
{1, 3, 2}	{{1, 3}, {3, 2}}	2
{2, 1, 3}	{{2, 1}, {1, 3}}	2
{2, 3, 1}	{{2, 3}, {3, 1}}	2
{3, 1, 2}	{{3, 1}, {1, 2}}	2
{3, 2, 1}	{{3, 2, 1}}	1

To investigate further, denote by $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ to be the number of permutations of length n which have exactly k alternating runs. For example, we see from the table above that $\left\langle\!\left\langle {3 \atop 1} \right\rangle\!\right\rangle = 2$ and $\left\langle\!\left\langle {3 \atop 2} \right\rangle\!\right\rangle = 4$. Here is an array of values for $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$.

$\langle {n \atop k} \rangle$	k=1	k=2	k=3	k=4	k=5	k=6	k=7
n=1	1	0	0	0	0	0	0
n=2	2	0	0	0	0	0	0
n=3	2	4	0	0	0	0	0
n=4	2	12	10	0	0	0	0
n=5	2	28	58	32	0	0	0
n=6	2	60	236	300	122	0	0
n=7	2	124	836	1852	1682	544	0

Can you find any patterns in the array above for $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$?

3.4.4 Involutions

ACP Vol. 3, p. 65

A permutation σ of length n can also be thought of as a mapping of the set $\{1, 2, \ldots, n\}$. For example, if $\sigma = \{2, 3, 1\}$, then we can view it as the function

$\sigma(1) = 2$
$\sigma(2) = 3$
$\sigma(3) = 1$

More generally, if $\sigma = \{k_1, k_2, \ldots, k_n\}$, then $\sigma(i) = k_i$, i.e., the integer i is mapped to k_i. Every permutation has an inverse, denoted by σ^{-1}, which maps k_i back to i. For example, if $\sigma = \{2, 3, 1\}$, then $\sigma^{-1} = \{3, 1, 2\}$ since we require

$\sigma^{-1}(1) = 3$
$\sigma^{-1}(2) = 1$
$\sigma^{-1}(3) = 2$

Thus, σ^{-1} satisfies the properties $\sigma^{-1}(\sigma(i)) = i$ and $\sigma(\sigma^{-1}(i)) = i$.

An involution is a permutation which is equal to its inverse, namely $\sigma = \sigma^{-1}$. For example, $\sigma = \{3, 2, 1\}$ is an involution. We confirm this using the *Mathematica* command **InversePermutation**.

```
InversePermutation[{3, 2, 1}]
```

```
{3, 2, 1}
```

Let’s investigate the numer of involutions of a given length. Towards this end, denote by $I(n)$ to be the number of involutions of length n. Here is a table of values of $I(n)$:

Number of Involutions of Length n

n	I(n)
1	1
2	2
3	4
4	10
5	26
6	76
7	232
8	764

FURTHER EXPLORATION: Can you find a recurrence relation for $I(n)$?

3.5 Partitions

3.5.1 Partitions of Integers

Recall that a partition of a positive integer n is a way of writing n as a sum of positive integers. For example, $n = 5$ has seven partitions. We can verify this using Mathematica’s **IntegerPartitions** command:

```
IntegerPartitions[5]
```

```
{{5}, {4, 1}, {3, 2}, {3, 1, 1}, {2, 2, 1}, {2, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

3.5.1.1 Partition Congruences

Consider the partition function $p(n)$ which counts the number of ways that a positive integer n can be expressed as a sum of positive integers. Since $n = 5$ has seven partitions, we have $p(5) = 7$. The partition function in *Mathematica* is defined by the command **PartitionsP**.

```
PartitionsP[5]
```

```
7
```

Let’s consider the partition congruences, i.e., the congruence of $p(n) \bmod q$, where q is a positive integer. The table below gives congruences for the first thirty values of $p(n) \bmod 2$.

Partition Congruences

n	p(n)	p(n) mod 2
1	1	1
2	2	0
3	3	1
4	5	1
5	7	1
6	11	1
7	15	1
8	22	0
9	30	0
10	42	0

n	p(n)	p(n) mod 2
11	56	0
12	77	1
13	101	1
14	135	1
15	176	0
16	231	1
17	297	1
18	385	1
19	490	0
20	627	1

n	p(n)	p(n) mod 2
21	792	0
22	1002	0
23	1255	1
24	1575	1
25	1958	0
26	2436	0
27	3010	0
28	3718	0
29	4565	1
30	5604	0

Unfortunately, no pattern is evident. A similar dead-end arises if we consider congruences of $p(n) \bmod q$ for $q = 3$ and $q = 4$. However, when $q = 5$, we find an interesting pattern:

Partition Congruences

n	p(n)	p(n) mod 2
1	1	1
2	2	2
3	3	3
4	5	0
5	7	2
6	11	1
7	15	0
8	22	2
9	30	0
10	42	2
11	56	1
12	77	2
13	101	1
14	135	0
15	176	1
16	231	1
17	297	2
18	385	0
19	490	0
20	627	2
21	792	2
22	1002	2
23	1255	0
24	1575	0
25	1958	3
26	2436	1
27	3010	0
28	3718	3
29	4565	0
30	5604	4

If we isolate those congruences where $p(n) \equiv 0 \bmod 5$, then a pattern emerges:

Partition Congruences

n	p(n) mod 2
4	0
7	0
9	0
14	0
18	0
19	0
23	0
24	0
27	0
29	0
34	0
38	0
39	0
44	0
49	0
54	0
58	0
59	0
61	0
64	0

Observe that the table above includes those integers n ending in 4 or 9, i.e., integers of the form $5n + 4$. Thus, we've discovered the first of Ramanujan's congruences for the partition function.

$$p(5n + 4) \equiv 0 \bmod 5 \tag{3.36}$$

FURTHER INVESTIGATION: There are two other Ramanujan congruences. See if you can discover them by testing different values of the modulus q.

3.5.1.2 Number of Smallest Parts

http://www.math.psu.edu/vstein/alg/antheory/preprint/andrews/17.pdf

The total number of smallest parts appearing in all the partitions of a positive integer n is defined to be $spt(n)$. For example, the partitions of $n = 4$ are:

```
Replace[#, #[[Length[#]]] → Style[#[[Length[#]]], Underlined], 2] & /@
 IntegerPartitions[4]
```

```
{{4}, {3, 1}, {2, 2}, {2, 1, 1}, {1, 1, 1, 1}}
```

The smallest parts in each partition are underlined in the above output. The table below lists the number of smallest parts for each partition:

Partition	Number of Smallest Parts
{4}	1
{3, 1}	1
{2, 2}	2
{2, 1, 1}	2
{1, 1, 1, 1}	4

Thus, we have *spt*(4) = 10. Let's define spt as the command **spt** in *Mathematica*:

```
Clear[spt]
spt[n_] := Total[Count[#, #[[Length[#]]]] & /@ IntegerPartitions[n]]
```

The following table lists the first 30 values of spt.

n	spt(n)
1	1
2	3
3	5
4	10
5	14
6	26
7	35
8	57
9	80
10	119

n	spt(n)
11	161
12	238
13	315
14	440
15	589
16	801
17	1048
18	1407
19	1820
20	2399

n	spt(n)
21	3087
22	3998
23	5092
24	6545
25	8263
26	10 486
27	13 165
28	16 562
29	20 630
30	25 773

We follow the trail blazed in the previous part by considering congruences of spt(n) mod 5.

n	spt(n) mod 5
1	1
2	3
3	0
4	0
5	4
6	1
7	0
8	2
9	0
10	4

n	spt(n) mod 5
11	1
12	3
13	0
14	0
15	4
16	1
17	3
18	2
19	0
20	4

n	spt(n) mod 5
21	2
22	3
23	2
24	0
25	3
26	1
27	0
28	2
29	0
30	3

We find that the same congruence holds for spt(n) as it does for the partition function, namely

$$\text{spt}(5\,n+4) \equiv 0 \bmod 5 \tag{3.37}$$

FURTHER EXPLORATION: Find two other congruence relations for spt(n).

3.5.1.3 Palindromic Compositions

http://www.fq.math.ca/Scanned/41-3/heubach.pdf

A composition of n is an ordered sequence of positive integers whose sum is n. Thus, a composition is an ordered partition where the order of the terms is taken into account. For example, there are five partitions of $n = 4$.

```
IntegerPartitions[4]
```

```
{{4}, {3, 1}, {2, 2}, {2, 1, 1}, {1, 1, 1, 1}}
```

On the other hand, $n = 4$ has 16 compositions.

```
Permutations[#] & /@ IntegerPartitions[4]
```

```
{{{4}}, {{3, 1}, {1, 3}}, {{2, 2}}, {{2, 1, 1}, {1, 2, 1}, {1, 1, 2}}, {{1, 1, 1, 1}}}
```

A *palindromic composition* is a composition that reads the same forwards and backwards (palindrome). For example, $n = 4$ has three palindromic compositions:

4
1 + 2 + 1
1 + 1 + 1 + 1

Let's investigate the number of palindromic compositions of an arbitrary positive integer n.

3.5.2 Partitions of Sets

A partition of a set A is a collection of disjoint subsets $\{A_1, A_2, ..., A_k\}$ whose union equals A. For example, the collection {{1, 3}, {2}} is a partition of {1, 2, 3}. There are five different partitions of {1, 2, 3}. Let's generate them using the *Mathematica* command **SetPartitions**:

```
SetPartitions[{1, 2, 3}]
```

```
{{{1, 2, 3}}, {{1}, {2, 3}}, {{1, 2}, {3}}, {{1, 3}, {2}}, {{1}, {2}, {3}}}
```

3.5.2.1 Congruences of Bell Numbers

http://oeis.org/A000110

The Bell numbers B_n are defined to be the number of partitions of a set consisting of n elements. We've already seen that the set {1, 2, 3} has five partitions. Thus, $B_3 = 5$. The Bell numbers can be generated in *Mathematica* using the command **BellB**. Here is a list of the first 10 Bell numbers:

```
Table[BellB[n], {n, 1, 10}]
```

```
{1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975}
```

Let's consider congruences of Bell numbers $B_n \bmod 2$.

http://oeis.org/A054767

n	B_n mod 2
1	1
2	0
3	1
4	1
5	0
6	1
7	1
8	0
9	1
10	1

n	B_n mod 2
11	0
12	1
13	1
14	0
15	1
16	1
17	0
18	1
19	1
20	0

The pattern is clear: the residues above are periodic with period 3, i.e.,

$$B(n+3) \equiv B(n) \bmod 2 \tag{3.38}$$

Let's continue this trail by considering congruences of $B_n \bmod 3$.

n	B_n mod 2
1	1
2	2
3	2
4	0
5	1
6	2
7	1
8	0
9	0
10	1

n	B_n mod 2
11	0
12	1
13	1
14	1
15	2
16	2
17	0
18	1
19	2
20	1

n	B_n mod 2
21	0
22	0
23	1
24	0
25	1
26	1
27	1
28	2
29	2
30	0

Here, the residues seem to have period 13 over a range of 2 periods. We should experimentally verify this over a wider range. Towards this end, we use our in-house *Mathematica* command **SequencePeriod** to experimentally determine whether this period holds for the first 100 terms:

```
? SequencePeriod
```

SequencePeriod[*data*] experimentally determines the period p of a finite sequence *data*
by detecting a subsequence of p consecutive terms that repeats, i.e., runs for at least two periods;
returns p if such a subsequence exists, else returns 0.

```
SequencePeriod[Table[Mod[BellB[n], 3], {n, 1, 100}]]
```

```
13
```

The reader can confirm this over an even wider range, say the first 1000 terms. Therefore, we've discovered the congruence relation

$$B(n+13) \equiv B(n) \bmod 3 \tag{3.39}$$

What about other values for q? Is the sequence of congruences $B_n \bmod 3$ always periodic for every q? Define π_q to be the period of $B_n \bmod 3\,q$. Below is a table of values for π_q for q ranging from 1 to 6:

q	π_q
1	1
2	3
3	13
4	12
5	781
6	39

Observe that the periods in the table above are relatively small except for $q = 5$, which jumps to a period 781.

```
dataperiodBellnumbers = {1, 3, 13, 12, 781, 39, 137 257, 24,
  39, 2343, 28 531 167 061, 156, 25 239 592 216 021, 411 771, 10 153, 48,
  51 702 516 367 896 047 761, 117, 109 912 203 092 239 643 840 221, 9372, 1 784 341,
  85 593 501 183, 949 112 181 811 268 728 834 319 677 753, 312, 3905, 117}
```

```
{1, 3, 13, 12, 781, 39, 137 257, 24, 39, 2343, 28 531 167 061, 156, 25 239 592 216 021,
 411 771, 10 153, 48, 51 702 516 367 896 047 761, 117, 109 912 203 092 239 643 840 221,
 9372, 1 784 341, 85 593 501 183, 949 112 181 811 268 728 834 319 677 753, 312, 3905, 117}
```

FURTHER EXPLORATION:

1. Make a table of periods for B_n mod 3 for q ranging from 1 to 20. NOTE: Some of these periods will be extremely large.

2. Do you see a pattern for those values of q where the period jumps to a relatively large value (in comparison to its immediate neighbors)?

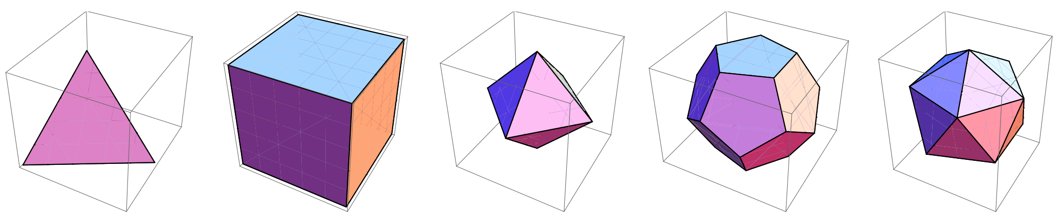
3. Can you find a formula for the relatively large periods mentioned in part 2.

3.6 Hyper-Polyhedra

In this section we investigate patterns involving the number of vertices, edges, and faces of higher-dimensional polyhedra.

3.6.1 Regular Polyhedra

It is well known that there exists only five regular polyhedra: tetrahedron, cube, octahedron, dodecahedron, and icosahedron.



Vertices, Edges, and Faces

Each regular polygon has a certain number of vertices, edges, and faces. Below is a table listing this information for all five regular polygons:

Regular Polyhedra

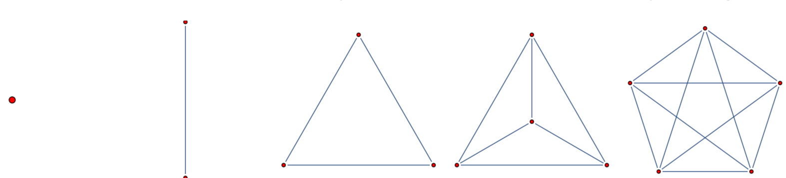
Regular Polyhedron	# Vertices	# Edges	# Faces
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Euler's polyhedron formula describes how V, E, and F are related for all convex polyhedra:

$$V - E + F = 2 \tag{3.40}$$

3.6.2 Hypertetrahedron

A *hypertetrahedron* (also called a simplex or pentatope) is generalization of a tetrahedron to four dimensions (4-D). More generally, an n-tetrahedron is a generalization of a tetrahedron to n dimensions. An n-dimensional unit hypertrahedron is defined to be the object obtained by inserting a vertex along the n-th dimension and forming edges of length 1 between it and all vertices of an $(n-1)$-dimensional unit hypertetrahedron. Essentially an n-dimensional hypertetrahedron can be represented by the complete graph K_n on $(n+1)$ vertices where any two vertices are connected by an edge.



Step 1

Let's determine the number of vertices and edges for a hypertetrahedron:

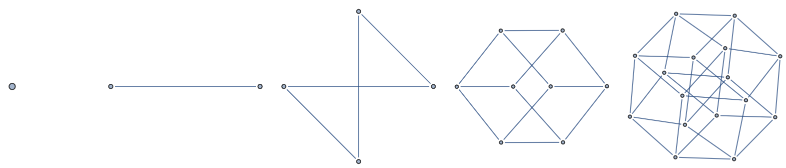
Hypertetrahedra

Dimension	# Vertices	# Edges
0	1	0
1	2	1
2	3	3
3	4	6
4	5	10

The pattern for the number of vertices is clear. Do you recognize the values for number of edges?

3.6.3 Hypercube

A *hypercube* is generalization of a cube to four dimensions (4-D). More generally, an n-cube is a generalization of a cube to n dimensions. An n-dimensional unit hypercube is defined to be the object obtained by taking two copies of an $(n-1)$-dimensional unit hypercube, parallel to each other along the n-th dimension, and forming edges of length 1 between corresponding vertices of the two copies.



Step 2

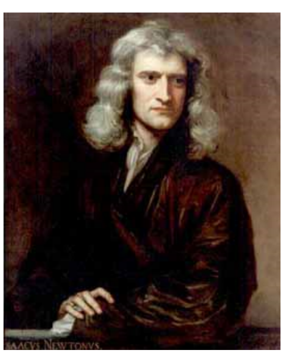
The following table lists the number of vertices and edges of an n-dimensional hypercube for $n = 1, 2, 3, 4$:

Hypercube

Dimension	# Vertices	# Edges
0	1	0
1	2	1
2	4	4
3	8	12
4	16	32
5	32	80

Do you recognize the pattern for the number of vertices and edges? What about faces of a n-cube? See if you can find a formula for the number of faces.

Isaac Newton: The Generalized Binomial Theorem



Sir Isaac Newton (1643 – 1727)

http://www-history.mcs.st-and.ac.uk/Mathematicians/Newton.html

Binomial Theorem:

Newton's Discovery of the General Binomial Theorem
D. T. Whiteside
The Mathematical Gazette
Vol. 45, No. 353 (Oct., 1961), pp. 175-180
(article consists of 6 pages)
Published by: The Mathematical Association
Stable URL: http://www.jstor.org/stable/3612767

Newton, having studied Wallis' *Arithmetica Infinitorum*, knew that Wallis had interpolated the definite integral $\int_0^1 \left(1 - x^{1/2}\right)^2 \, dx$ that led to his infinite product representation of π. By following the same trail, Newton was able to generalize the Binomial Theorem to non-integer exponents. He originally succeeded in obtaining infinite series expressions for $\int_0^x \left(a^2 - x^2\right)^{1/2} \, dx$, $\int_0^x \left(a^2 + x^2\right)^{1/2} \, dx$, and $\int_0^x a^2 (b + x)^{-1} \, dx$, representing the areas the circle $y^2 = a^2 - x^2$ and hyperbolas $y(b + x) = a^2$, respectively. As he explains it in [], Newton in each case reduced the problem to interpolating the integrand, say $a^2 (b + x)^{-1}$, from the sequence of expansions $a^2 (b + x)^n$ for non-negative integers n, whose coefficients were already known to be given by Pascal's triangle:

Binomial Expansion of $a^2(b+x)^n$

n	$a^2(b+x)^n$
0	a^2
1	$a^2 b + a^2 x$
2	$a^2 b^2 + 2 a^2 b x + a^2 x^2$
3	$a^2 b^3 + 3 a^2 b^2 x + 3 a^2 b x^2 + a^2 x^3$
4	$a^2 b^4 + 4 a^2 b^3 x + 6 a^2 b^2 x^2 + 4 a^2 b x^3 + a^2 x^4$
5	$a^2 b^5 + 5 a^2 b^4 x + 10 a^2 b^3 x^2 + 10 a^2 b^2 x^3 + 5 a^2 b x^4 + a^2 x^5$

Now, by creating a table of these coefficients, called binomial coefficients (see Chapter 2) and denoted by $\binom{n}{k}$ to refer to the coefficient of $a^2 b^{n-k} x^k$ in the expansion of $a^2(b + x)^n$, Newton was able to extend the recursive pattern for binomial coefficients to negative integer exponents:

Binomial Coefficients $\binom{n}{k}$

	k=0	k=1	k=2	k=3	k=4	k=5
n=−5	1	−5	15	−35	70	−126
n=−4	1	−4	10	−20	35	−56
n=−3	1	−3	6	−10	15	−21
n=−2	1	−2	3	−4	5	−6
n=−1	1	−1	1	−1	1	−1
n=0	1	0	0	0	0	0
n=1	1	1	0	0	0	0
n=2	1	2	1	0	0	0
n=3	1	3	3	1	0	0
n=4	1	4	6	4	1	0
n=5	1	5	10	10	5	1

As a result, this led him to the series expansion for $a^2(b + x)^{-1}$ corresponding to $n = -1$:

$$a^2(b + x)^{-1} = \frac{a^2}{b} - \frac{a^2}{b^2} x + \frac{a^2}{b^3} x^2 - \frac{a^2}{b^4} x^3 + \ldots \tag{3.41}$$

Of course, the more difficult problem was to interpolate the binomial coefficients of $a^2(b+x)^n$ for fractional exponents n. To achieve this, Newton made the spectacular observation that the values in each column followed a linear progression of the form

Coefficients for x^k in expansion of $a^2(b+x)^n$

	k=0	k=1	k=2	k=3	k=4	k=5
n=−5	a	b − 5 c	d − 5 e + 15 f	g − 5 h + 15 i − 35 j	k − 5 l + 15 m − 35 n + 70 o	p − 5 q + 15 r − 35 s + 70 t − 126 u
n=−4	a	b − 4 c	d − 4 e + 10 f	g − 4 h + 10 i − 20 j	k − 4 l + 10 m − 20 n + 35 o	p − 4 q + 10 r − 20 s + 35 t − 56 u
n=−3	a	b − 3 c	d − 3 e + 6 f	g − 3 h + 6 i − 10 j	k − 3 l + 6 m − 10 n + 15 o	p − 3 q + 6 r − 10 s + 15 t − 21 u
n=−2	a	b − 2 c	d − 2 e + 3 f	g − 2 h + 3 i − 4 j	k − 2 l + 3 m − 4 n + 5 o	p − 2 q + 3 r − 4 s + 5 t − 6 u
n=−1	a	b − c	d − e + f	g − h + i − j	k − l + m − n + o	p − q + r − s + t − u
n=0	a	b	d	g	k	p
n=1	a	b + c	d + e	g + h	k + l	p + q
n=2	a	b + 2 c	d + 2 e + f	g + 2 h + i	k + 2 l + m	p + 2 q + r
n=3	a	b + 3 c	d + 3 e + 3 f	g + 3 h + 3 i + j	k + 3 l + 3 m + n	p + 3 q + 3 r + s
n=4	a	b + 4 c	d + 4 e + 6 f	g + 4 h + 6 i + 4 j	k + 4 l + 6 m + 4 n + o	p + 4 q + 6 r + 4 s + t
n=5	a	b + 5 c	d + 5 e + 10 f	g + 5 h + 10 i + 10 j	k + 5 l + 10 m + 10 n + 5 o	p + 5 q + 10 r + 10 s + 5 t + u

where $a = 1$, $b = 0$, $c = 1$, $d = 0$, $e = 0$, $f = 1$, etc.

Newton then boldly assumed that a similar linear progression would continue to hold when values for half-integer exponents, i.e., $n = m/2$, were inserted into the Table? above (indicated by * since they are unknown for the moment):

Coefficients for x^k in expansion of $a^2(b+x)^n$

	k=0	k=1	k=2	k=3	k=4
n=−4	a	b − 8 c	d − 8 e + 36 f	g − 8 h + 36 i − 120 j	k − 8 l + 36 m − 120 n + 330 o
n=−7/2	a	b − 7 c	d − 7 e + 28 f	g − 7 h + 28 i − 84 j	k − 7 l + 28 m − 84 n + 210 o
n=−3	a	b − 6 c	d − 6 e + 21 f	g − 6 h + 21 i − 56 j	k − 6 l + 21 m − 56 n + 126 o
n=−5/2	a	b − 5 c	d − 5 e + 15 f	g − 5 h + 15 i − 35 j	k − 5 l + 15 m − 35 n + 70 o
n=−2	a	b − 4 c	d − 4 e + 10 f	g − 4 h + 10 i − 20 j	k − 4 l + 10 m − 20 n + 35 o
n=−3/2	a	b − 3 c	d − 3 e + 6 f	g − 3 h + 6 i − 10 j	k − 3 l + 6 m − 10 n + 15 o
n=−1	a	b − 2 c	d − 2 e + 3 f	g − 2 h + 3 i − 4 j	k − 2 l + 3 m − 4 n + 5 o
n=−1/2	a	b − c	d − e + f	g − h + i − j	k − l + m − n + o
n=0	a	b	d	g	k
n=1/2	a	b + c	d + e	g + h	k + l
n=1	a	b + 2 c	d + 2 e + f	g + 2 h + i	k + 2 l + m
n=3/2	a	b + 3 c	d + 3 e + 3 f	g + 3 h + 3 i + j	k + 3 l + 3 m + n
n=2	a	b + 4 c	d + 4 e + 6 f	g + 4 h + 6 i + 4 j	k + 4 l + 6 m + 4 n + o
n=5/2	a	b + 5 c	d + 5 e + 10 f	g + 5 h + 10 i + 10 j	k + 5 l + 10 m + 10 n + 5 o
n=3	a	b + 6 c	d + 6 e + 15 f	g + 6 h + 15 i + 20 j	k + 6 l + 15 m + 20 n + 15 o
n=7/2	a	b + 7 c	d + 7 e + 21 f	g + 7 h + 21 i + 35 j	k + 7 l + 21 m + 35 n + 35 o
n=4	a	b + 8 c	d + 8 e + 28 f	g + 8 h + 28 i + 56 j	k + 8 l + 28 m + 56 n + 70 o

Since these values are known for those rows where n is an integer exponent, it suffices to solve the infinite family of equations (assuming they are all consistent) for the variables $a, b, c, d,$ .... For example, the equations derived from the third column are

$$\ldots \tag{3.42}$$

$$d - 4e + 10f = 3$$
$$d - 2e + 3f = 1$$

$d = 0$
$d + 2e + f = 0$
$d + 4e + 6f = 1$
...

It follows that $d = 0$, $e = -1/8$, and $f = 1/4$. By applying this to every column, Newton was able to fill in his table for half-integer exponents:

Coefficients for x^k in expansion of $a^2(b+x)^n$

	k=0	k=1	k=2	k=3	k=4
n=−4	1	−4	10	−20	35
n=−7/2	1	$-\frac{7}{2}$	$\frac{63}{8}$	$-\frac{231}{16}$	$\frac{3003}{128}$
n=−3	1	−3	6	−10	15
n=−5/2	1	$-\frac{5}{2}$	$\frac{35}{8}$	$-\frac{105}{16}$	$\frac{1155}{128}$
n=−2	1	−2	3	−4	5
n=−3/2	1	$-\frac{3}{2}$	$\frac{15}{8}$	$-\frac{35}{16}$	$\frac{315}{128}$
n=−1	1	−1	1	−1	1
n=−1/2	1	$-\frac{1}{2}$	$\frac{3}{8}$	$-\frac{5}{16}$	$\frac{35}{128}$
n=0	1	0	0	0	0
n=1/2	1	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{16}$	$-\frac{5}{128}$
n=1	1	1	0	0	0
n=3/2	1	$\frac{3}{2}$	$\frac{3}{8}$	$-\frac{1}{16}$	$\frac{3}{128}$
n=2	1	2	1	0	0
n=5/2	1	$\frac{5}{2}$	$\frac{15}{8}$	$\frac{5}{16}$	$-\frac{5}{128}$
n=3	1	3	3	1	0
n=7/2	1	$\frac{7}{2}$	$\frac{35}{8}$	$\frac{35}{16}$	$\frac{35}{128}$
n=4	1	4	6	4	1

In particular,

$$\left(a^2 - x^2\right)^{1/2} = a\left(1 - \frac{x^2}{a^2}\right)^{1/2} = a\left[1 + \frac{1}{2}\left(\frac{x^2}{a^2}\right) - \frac{1}{8}\left(\frac{x^2}{a^2}\right)^2 + \frac{1}{16}\left(\frac{x^2}{a^2}\right)^3 - \frac{5}{128}\left(\frac{x^2}{a^2}\right)^4 + \ldots\right] \tag{3.44}$$

Lastly, it remains to find a general formula for generating binomial coefficients for all fractional exponents without having to interpolate each row individually. This is where Newton must have been influenced by Wallis' use of infinite products for he wrote down the formula

$$\binom{p/q}{k} = \frac{1 \times p \times (p-q) \times (p-2q) \times (p-3q) \times \cdots}{1 \times k \times 2k \times 3k \times 4k \times \cdots} \tag{3.45}$$

leading to the modern version of the Binomial Theorem, which holds for all real exponents n:

Theorem: If n is real-valued and $|x|<1$, then

$$(1+x)^n = 1 + \frac{n}{1}\, x + \frac{n(n-1)}{1\cdot 2}\, x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\, x^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\, x^4 + \ldots \tag{3.46}$$

Appendix

Beginner’s Guide to Mathematica

Bibliography

[Ba] Edward J. Barbeau, *Pell's Equation*, Problem Books in Mathematics, Springer-Verlag, 2003.

[BB] Jonathan Borwein and David Bailey, *Mathematics by Experiment: Plausible Reasoning for the 21st Century*, A K Peters, Wellesley, MA, 2008.

[BBC] David Bailey, Jonathan M. Borwein, Neil J. Calkin, Roland Girgensohn, D. Russell Luke, and Victor H. Moll, *Experimental Mathematics in Action*, A K Peters, Wellesley, MA, 2007.

[BoCu] John P. Bonomo and Carolyn K. Cuff, How Do You Stack Up?, *The College Mathematics Journal*, Vol. 35, No. 5 (Nov., 2004), pp. 351-361: http://www.jstor.org/stable/4146844.

[BBG] Jonathan Borwein, David Bailey and Roland Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, A K Peters, Natick, MA, 2004.

[BBN] Arthur T. Benjamin, Curtis T. Bennett, Florence Newberger, *Recounting the Odds of an Even Derangement*, Mathematics Magazine, Vol. 78, No. 5, Dec., 2005, pp. 387-390.

[BoKo] Douglas Bowman and Nasha Komanda, E3450, *The American Mathematical Monthly*, Vol. 100, No. 3 (Mar., 1993), p. 300: http://www.jstor.org/stable/2324473.

[Ch] Hongwei Chen, *Excursions in Classical Analysis*, Mathematical Association of America, 2010.

[DB] Keith Devlin and Jonathan Borwein, *The Computer as Crucible: An Introduction to Experimental Mathematics*, A K Peters, Wellesley, MA, 2008.

[DeBr] M. N. Deshpande and Ezra Brown, Diophantine Triplets and the Pell Sequence, *The Fibonacci Quarterly*, Vol. 39, No. 3, 2001: http://www.fq.math.ca/Scanned/39-3/deshpande.pdf

[El] Niles Eldridge, *The Pattern of Evolution*, W. H. Freeman, 1998.

[Eu] Leonard Euler, Decouverte d'une loi tout extraordinaire des nombres par rapport a la somme de leurs diviseurs (Discovery of a most extraordinary law of the numbers concerning the sum of their divisors), *Bibliotheque impartiale* 3, 1751, pp. 10-31 (*Opera Omnia*: Series 1, Volume 2, pp. 241 - 253), Eneström index: E175. Available on the Euler Archive: http://www.math.dartmouth.edu/~euler/.

[ES] Russ Euler and Jawad Sadek, Elementary Problems and Solutions (B951), *The Fibonacci Quarterly*, Vol. 41, No. 5, 2003. http://www.fq.math.ca/Scanned/41-5/elementary41-5.pdf

[ES2] Russ Euler and Jawad Sadek, Elementary Problems and Solutions (B959), *The Fibonacci Quarterly*, Vol. 41, No. 2, 2003: http://www.fq.math.ca/Scanned/41-2/elementary41-2.pdf

[ES3] Russ Euler and Jawad Sadek, Elementary Problems and Solutions (B959), *The Fibonacci Quarterly*, Vol. 42, No. 2, 2004: http://www.fq.math.ca/Papers1/42-2/May2004elementary.pdf

[EwSc] J. Ewing and G. Schober, The area of the Mandelbrot set, *Numerische Mathematik*, Vol. 61 (1992), 59 - 72.

[Fi] Fibonacci, *Liber Abaci. A Translation into Modern English of Leonardo Pisano's Book of Calculuation*, L. E. Sigler (translator), Springer-Verlag, NY 2002.

[Ga] Martin Gardner, *The Book of Colossal Mathematics*, W. W. Norton, New York, 2001.

[Go] L. J. Goldstein, A History of the Prime Number Theorem, *The American Mathematical Monthly*, Vol. 80, No. 6 (Jun. - Jul., 1973), pp. 599-615: http://www.jstor.org/stable/2319162

[Gr] Andrew Granville, Zaphod Beeblebrox's Brian and the Fifty-ninth Row of Pascal's Triangle, The American Mathematical Monthly

Vol. 99, No. 4 (Apr., 1992), pp. 318-331: http://www.jstor.org/stable/2324898.

[Gu] Richard Guy, Divisors and Desires, *The American Mathematical Monthly*, Vol. 104, No. 4 (Apr., 1997), pp. 359-360: http://www.jstor.org/stable/2974586

[GKP] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.

[Ha] Paul Halmos, *I Want to be a Mathematician: An Automathography*, Mathematical Association of America, 1985.

[Hay] Brian Hayes, *Gauss's Day of Reckoning*, American Scientist, Volume 94, Number 3 (May-June 2006), p. 200.

[Ho] F. T. Howard, The Sum of the Squares of Two Generalized Fibonacci Numbers, *The Fibonacci Quarterly*, Vol. 41, No. 1, 2003: http://www.fq.math.ca/Scanned/41-1/howard.pdf.

[I] Inverse Symbolic Calculator: Wikipedia - the free encyclopedia: http://en.wikipedia.org/wiki/Inverse_symbolic_calculator

[Ki] Clark Kimberling, Fractal Sequences, http://faculty.evansville.edu/ck6/integer/fractals.html.

[Kn] Donald E. Knuth, The Art of Computer Programming: Volume 3 - Sorting and Searching, Addison-Wesley, 1973.

[La] L. J. Lange, An Elegant Continued Fraction for π, *The American Mathematical Monthly*, Vol. 106, No. 5 (May, 1999), pp. 456-458: http://www.jstor.org/stable/2589152

[LP] Thomas M. Liggett and Peter Petersen, ,The Law of Large Numbers and $\sqrt 2$, *The American Mathematical Monthly*, Vol. 102, No. 1 (Jan., 1995), pp. 31-35: http://www.jstor.org/stable/2974854

[LLHC] John F. Loase, David Lansing, Cassie Hryczaniuk and Jamie Cahoon, A Variant of the Partition Function, *The College Mathematics Journal*, Vol. 36, No. 4 (Sep., 2005), pp. 320-321: http://www.jstor.org/stable/30044876

[LoHe] Calvin T. Long and Boyd Henry, Linear Relations Between Powers of Terms in Arithmetic Progression, *The College Mathematics Journal*, Vol. 32, No. 2 (Mar., 2001), pp. 135-137: http://www.jstor.org/stable/2687121

[MT] Colin L. Mallows and John W. Tukey, An overview of techniques of data analysis, in *Some Recent Advances in Statistics*, de Oliviera and Epstein (editors), Academic Press, London, 1982.

[Me] R. S. Melham, A Three-Variable Identity Involving Cubes of Fibonacci Numbers, *The Fibonacci Quarterly*, Vol. 41, No. 3, 2003: http://www.fq.math.ca/Scanned/41-3/melham.pdf.

[MP] G. Myerson and A. J. van der Poorten, Some Problems Concerning Recurrence Sequences, *The American Mathematical Monthly*, Vol. 102, No. 8 (Oct., 1995), pp. 698-705: http://www.jstor.org/stable/2974639

[MW] Darryl McCullough and Elizabeth Wade, Recursive Enumeration of Pythagorean Triples, The College Mathematics Journal Vol. 34, No. 2 (Mar., 2003), pp. 107-111: http://www.jstor.org/stable/3595782

[O] On-Line Encyclopedia of Integer Sequences: Wikipedia - the free encyclopedia: http://en.wikipedia.org/wiki/On-Line_Encyclopedia _of _Integer _Sequences

[Od] Odlyzko, A. M. (1993), Iterated absolute values of differences of consecutive primes, *Mathematics of Computation* 61: 373–380, http://www.dtc.umn.edu/~odlyzko/doc/arch/gilbreath.conj.ps

[Os] Thomas J. Osler, Binomials to Binomials, *The College Mathematics Journal*, Vol. 31, No. 3 (May, 2000), pp. 211-212: http://www.jstor.org/stable/2687492

[Pa] Poo-Sung Park, Ramanujan's Continued Fraction for a Puzzle, *The College Mathematics Journal*, Vol. 36, No. 5 (Nov., 2005), pp. 363-365: http://www.jstor.org/stable/30044886

[PS] Problems and Solutions 653, *The College Mathematics Journal*, Vol. 31, No. 3 (May, 2000), pp. 219-226: http://www.jstor.org/stable/2687495.

[Pu] Jan-Christoph Puchta, On the Distribution of Totients, *The Fibonacci Quarterly*, Vol. 40, No. 1, 2002: http://www.fq.math.ca/Scanned/40-1/puchta2.pdf

[Wi-Fi] Fibonacci number: Wikipedia - the free encyclopedia: http://en.wikipedia.org/wiki/Fibonacci_number

[Wi-Gi] Gilbreath's Conjecture: Wikipedia - the free encyclopedia: http://en.wikipedia.org/wiki/Gilbreath%27 s_conjecture

[Wi-Pe] Perfect Number: Wikipedia - the free encyclopedia: http://en.wikipedia.org/wiki/Perfect_number

[Po] George Polya, *Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics*, Princeton University Press, 1954.

[ShSh] Harris S. Shultz and Ray C. Shiflett, M&m Sequences, *The College Mathematics Journal*, Vol. 36, No. 3 (May, 2005), pp. 191-198: http://www.jstor.org/stable/30044851.

[Sp] Michael Z. Spivey, The Humble Sum of Remainders Function, *Mathematics Magazine*, Vol. 78, No. 4, Oct., 2005, pp. 300-305: http://www.jstor.org/stable/30044176

[St1] Richard P. Stanley, Enumerative Combinatorics: Volume 1,

[St2] Richard P. Stanley, Enumerative Combinatorics: Volume 2,

[WW] Peter W. Wade and William R. Wade, Recursions That Produce Pythagorean Triples, *The College Mathematics Journal*, Vol. 31, No. 2 (Mar., 2000), pp. 98-101: http://www.jstor.org/stable/2687578

[Yo-P] Paul T. Young, On Lacunary Recurrences, *The Fibonacci Quarterly*, Volume 41 (2003), No. 1, pp.41-47: http://www.fq.-math.ca/Scanned/41-1/young.pdf

[Yo-T] Robert M. Young, *Excursions in Calculus: An Interplay of the Continuous and the Discrete*, The Mathematical Association of America, 1992.